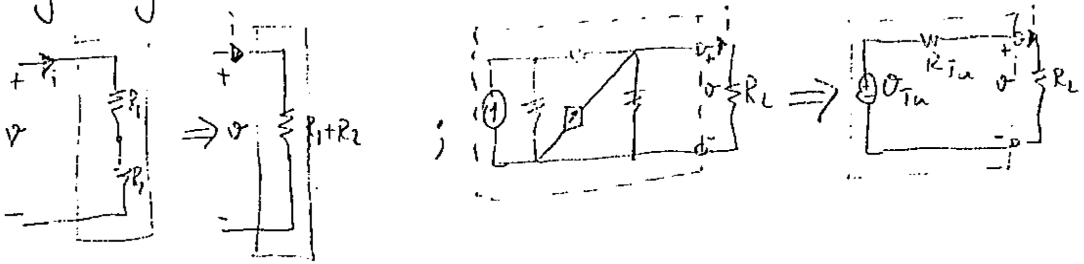
Two Ports:

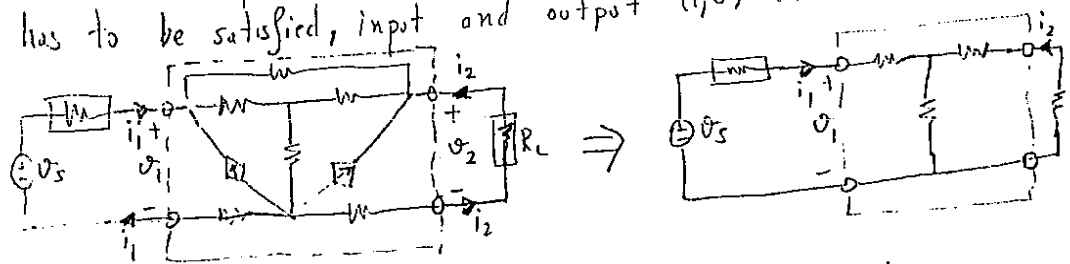
* A circuit simplification technique more general than resistor combining, source transformation, Thevenin-Norton equivalents.

* The simplification methods previously discussed is based on finding a simpler circuit having the exact (i, v) characteristic of a given circuit.



One port simplification

* The two port circuits have two (i, v) characteristics that has to be satisfied, input and output (i, v) characteristics.

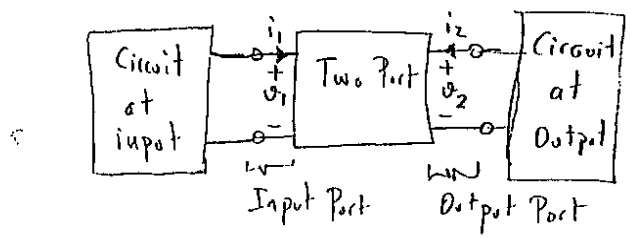


A two port simplified to a "T" connection.

Definition:

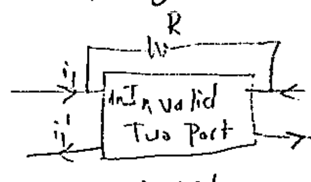
A two port should partition a circuit into three.

1. Block to be simplified
2. Components that are connected to the input port (components can be indirectly connected)
3. Components connected to the output port.



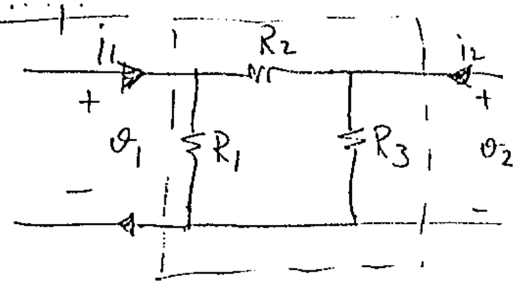
A valid two port has

1. Currents entering and leaving a port should be identical. In other words, the connections between the ports should completely reside in the two port.



$i_1 \neq i_1'$

2. No independent sources in the two port. (This condition can be removed if necessary)
3. The dependent sources should have their controlling variable in the two-port. (well defined dependent sources)



A valid two port

From the diagram, it is clear that V_1 is a function of i_1 and i_2;

$$V_1 = r_{11} I_1 + r_{12} I_2$$
$$V_2 = r_{21} I_1 + r_{22} I_2$$

$\{r_{11}, r_{12}, r_{21}, r_{22}\}$ are the parameters to be determined from the circuit.

Similarly, we can express i_1 and i_2 in terms of v_1 and v_2

or i_1 and v_1 in term of i_2 and v_2,

or any pair of input-output of circuit variable in terms of the other pair.

$$\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \underbrace{\begin{bmatrix} r_{11} & r_{12} \\ r_{21} & r_{22} \end{bmatrix}}_{\boxed{\text{Resistance}}} \begin{bmatrix} I_1 \\ I_2 \end{bmatrix} ; \quad \begin{bmatrix} I_1 \\ I_2 \end{bmatrix} = \underbrace{\begin{bmatrix} y_{11} & y_{12} \\ y_{21} & y_{22} \end{bmatrix}}_{\boxed{\text{Conductance}}} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} ;$$

$$\begin{bmatrix} v_1 \\ I_1 \end{bmatrix} = \begin{bmatrix} a_{11} & -a_{12} \\ a_{21} & -a_{22} \end{bmatrix} \begin{bmatrix} v_2 \\ I_2 \end{bmatrix} ; \quad \begin{bmatrix} v_2 \\ I_2 \end{bmatrix} = \begin{bmatrix} b_{11} & -b_{12} \\ -b_{21} & b_{22} \end{bmatrix} \begin{bmatrix} v_1 \\ I_1 \end{bmatrix} ;$$

$\boxed{\text{Transmission}}$

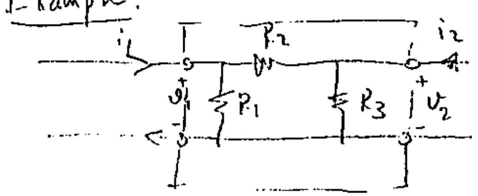
$$\begin{bmatrix} v_1 \\ I_2 \end{bmatrix} = \begin{bmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{bmatrix} \begin{bmatrix} I_1 \\ v_2 \end{bmatrix} ; \quad \begin{bmatrix} I_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{bmatrix} \begin{bmatrix} v_1 \\ I_2 \end{bmatrix}$$

$\boxed{\text{Hybrid}}$

Different Two Port Representations

Notes: r parameters have the unit of Ω's
y parameters of $\mho$'s.

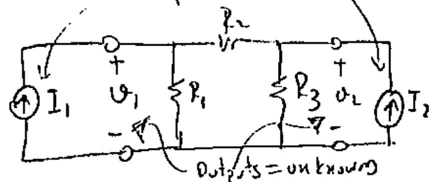
Example:



Calculate the r parameters

Method 1:

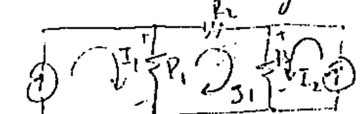
$$\underbrace{\begin{bmatrix} v_1 \\ v_2 \end{bmatrix}}_{\text{Output Argument}} = \begin{bmatrix} r_{11} & r_{12} \\ r_{21} & r_{22} \end{bmatrix} \underbrace{\begin{bmatrix} I_1 \\ I_2 \end{bmatrix}}_{\text{Input Argument}} \Longrightarrow$$



Find v_1 and v_2

$\Downarrow$

Mesh Analysis



$$R_1(J_1 - I_1) + R_2 J_1 + R_3(J_1 + I_2) = 0$$

$$J_1 = \frac{R_1 I_1 - R_3 I_2}{R_1 + R_2 + R_3}$$

$\Longleftarrow$

$$v_1 = (I_1 - J_1) R_1$$
$$v_2 = (I_2 + J_1) R_2$$

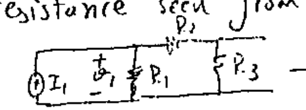
$\Downarrow$

$$\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \frac{1}{R_1 + R_2 + R_3} \begin{bmatrix} R_1R_2 + R_1R_3 & R_1R_3 \\ R_1R_3 & R_1R_3 + R_2R_3 \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \end{bmatrix}$$

Method 2:

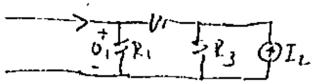
$$r_{11} = \frac{v_1}{I_1}\Bigg|_{I_2 = 0} \qquad (v_1 = r_{11} I_1 + r_{12} I_2)$$

Let $I_2 = 0$, open-circuit at port-2, and calculate the equivalent resistance seen from port-1.



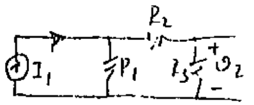
$$\longrightarrow \quad v_1 = (R_1 \,//\, (R_2 + R_3)) I_1 = \underbrace{\frac{R_1R_2 + R_1R_3}{R_1 + R_2 + R_3}}_{r_{11}} I_1$$

$r_{12} = \frac{v_1}{I_2}\Big|_{I_1=0} \longrightarrow$



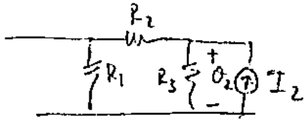
Find $v_1 \longrightarrow v_1 = R_1\underbrace{\left(\frac{R_3}{R_1+R_2+R_3}\right)}_{r_{12}} I_2$

$r_{21} = \frac{v_2}{I_1}\Big|_{I_2=0} \Longrightarrow$



Find $v_2 \longrightarrow v_2 = \underbrace{R_3\left(\frac{R_1}{R_1+R_2+R_3}\right)}_{r_{21}} I_2$

$r_{22} = \frac{v_2}{I_2}\Big|_{I_1=0} \Longrightarrow$



$\longrightarrow v_2 = I_2\underbrace{\left(R_3 // (R_1+R_2)\right)}_{r_{22}}$

<u>Note:</u> If we had chosen to apply superposition method instead of mesh analysis method in the first solution, the second solution would be identical to the first one.

Other parameters (hybrid, conductance etc.) can be derived similarly.

Relation Between Different Parameters:

There are 6 possible ways of expressing input-output relation of two port circuit. It is possible to derive one set of representation from another set by simple linear algebra.

<u>Example:</u> Assume resistance parameters are given

$$\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} r_{11} & r_{12} \\ r_{21} & r_{22} \end{bmatrix}\begin{bmatrix} i_1 \\ i_2 \end{bmatrix}$$

The conductance parameters are $\begin{bmatrix} i_1 \\ i_2 \end{bmatrix} = \begin{bmatrix} r_{11} & r_{12} \\ r_{21} & r_{22} \end{bmatrix}^{-1}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$

The transmission parameters:

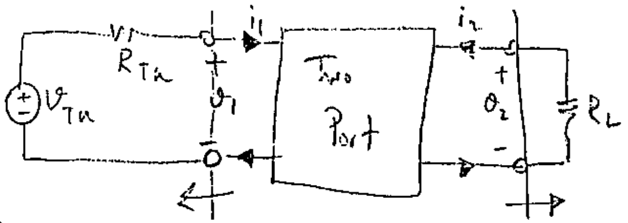
$$\begin{bmatrix} v_1 \\ I_1 \end{bmatrix} = \begin{bmatrix} \cdots & \cdots \\ \cdots & \cdots \end{bmatrix}\begin{bmatrix} v_2 \\ I_2 \end{bmatrix} \; : \Rightarrow \quad \begin{aligned} v_1 &= r_{11} i_1 + r_{12} i_2 \\ i_1 &= \frac{+1}{r_{21}} v_2 - \frac{r_{22}}{r_{21}} i_2 \end{aligned}$$

$$\Downarrow$$

$$\begin{bmatrix} v_1 \\ I_1 \end{bmatrix} = \begin{bmatrix} \frac{r_{11}}{r_{21}} & r_{12} - \frac{r_{11} r_{22}}{r_{21}} \\ \frac{1}{r_{21}} & -\frac{r_{22}}{r_{21}} \end{bmatrix}\begin{bmatrix} v_2 \\ I_2 \end{bmatrix} \Leftarrow \quad \begin{aligned} v_1 &= r_{11}\left(\frac{1}{r_{21}} v_2 - \frac{r_{22}}{r_{21}}\right) i_2 + r_{12} i_2 \\ v_2 &= \frac{1}{r_{21}} v_2 - \frac{r_{22}}{r_{21}} i_2 \end{aligned}$$

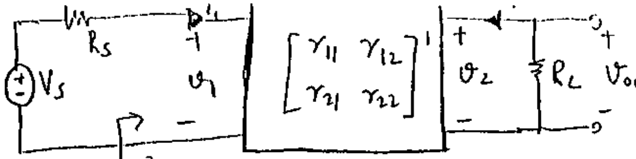
Application of Two Ports:

1) Terminated Two Ports:



The two has 4 circuit variables $\{i_1, v_1, i_2, v_2\}$ and two representation equations. By writing two more equations for the circuits at the input-output of the two-port circuit, we complete the equation system. 4 equations, 4 unknowns.

Tramp



a) Find R_{in}
b) Find V_{out}

a) Two Port Relations:

(1) $v_1 = r_{11} i_1 + r_{12} i_2$

(2) $v_2 = r_{21} i_1 + r_{22} i_2$

Load Side

(3) $v_2 = -i_2 R_L$

Source Side

(4) $v_s = R_S i_1 + v_1$

4 equations, 4 unknowns, solve for the variables

(1)-(3): $0 = r_{21} i_1 + (r_{22}+R_L) i_2 \Longrightarrow i_2 = -\dfrac{r_{21}}{r_{22}+R_L} i_1$ $(*)$

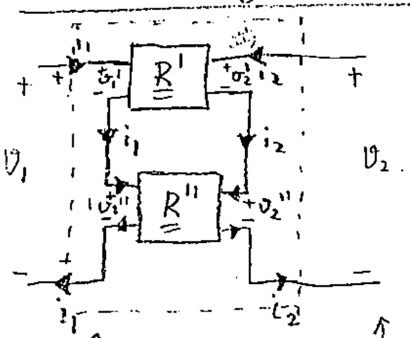
Insert last eqn to (1): $v_1 = i_1 \left(r_{11} - \dfrac{r_{12} r_{21}}{r_{22}+R_L} \right)$ $(**)$

$$R_{in} = \frac{v_1}{i_1} = \left(r_{11} - \frac{r_{12} r_{21}}{r_{22}+R_L} \right)$$

b) $V_{out} = V_2 \overset{(3)}{=} -i_2 R_L$ $(***)$

$$v_s \overset{(4)}{=} R_S i_1 + v_1 \overset{(1)}{=} (R_S + r_{11}) i_1 + r_{12} i_2 \overset{(*)}{=} \left[\frac{-(R_S + r_{11})(r_{22}+R_L)}{r_{21}} + r_{12} \right] i_2 \quad (****)$$

$$V_{out} \overset{(***)}{=} -i_2 R_L \overset{(****)}{=} \underbrace{\frac{R_L}{\dfrac{(r_{11}+R_S)(r_{22}+R_L)}{r_{21}} - r_{12}}}_{\text{Amplification Factor.}} V_s$$

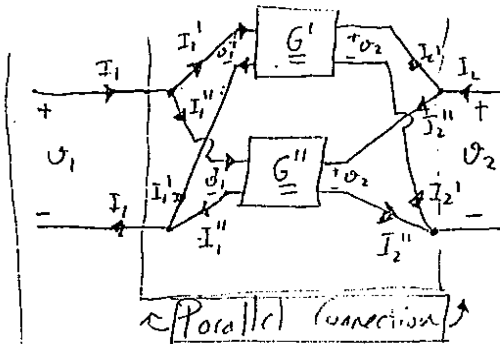


Series Connection

$$\begin{bmatrix} v_1' \\ v_2' \end{bmatrix} = \begin{bmatrix} \underline{\underline{R}}' \end{bmatrix} \begin{bmatrix} i_1 \\ i_2 \end{bmatrix} ; \begin{bmatrix} v_1'' \\ v_2'' \end{bmatrix} = \begin{bmatrix} \underline{\underline{R}}'' \end{bmatrix} \begin{bmatrix} i_1 \\ i_2 \end{bmatrix}$$

$$\underbrace{\begin{bmatrix} v_1' + v_1'' \\ v_2' + v_2'' \end{bmatrix}}_{\text{output par. of the combined two-port}} = \begin{bmatrix} \underline{\underline{R}}' + \underline{\underline{R}}'' \end{bmatrix} \underbrace{\begin{bmatrix} i_1 \\ i_2 \end{bmatrix}}_{\text{input par. of the combined two port}}$$

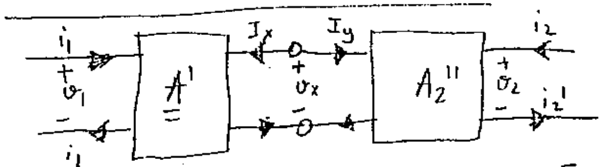
→ Resistance Parameters of the combined two port $= R' + R''$



Parallel Connection

The conductance parameters of the combined block is

$\underline{\underline{G}}' + \underline{\underline{G}}''$



Cascade Connection

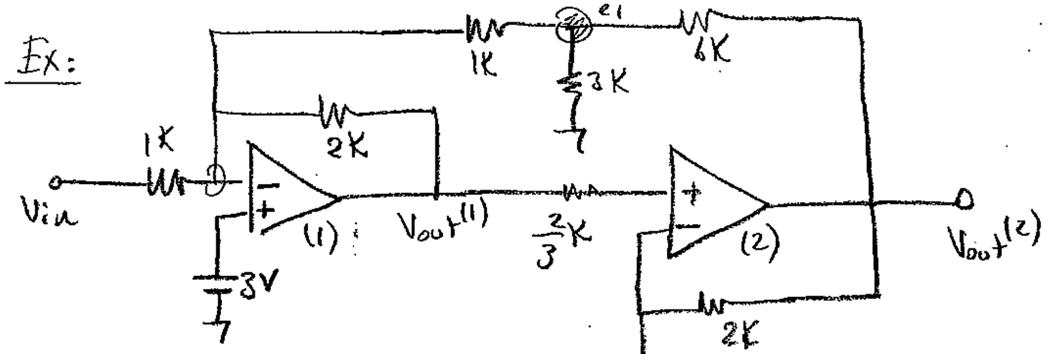
$$\begin{bmatrix} v_1 \\ i_1 \end{bmatrix} = \underline{\underline{A}}' \begin{bmatrix} v_x \\ -i_x \end{bmatrix} ; \begin{bmatrix} V_x \\ I_y \end{bmatrix} = \underline{\underline{A}}'' \begin{bmatrix} v_2 \\ -i_2 \end{bmatrix}$$

$I_y = -I_x$

$$\begin{bmatrix} v_1 \\ i_1 \end{bmatrix} = \underline{\underline{A}}' \underline{\underline{A}}'' \begin{bmatrix} v_2 \\ -i_2 \end{bmatrix} \Longrightarrow$$

Transmission Rep. of the cascade block is multiplication of $\underline{\underline{A}}'$ and $\underline{\underline{A}}''$ matrices.

Ex:



$E_{sat} = 18$

$\boxed{\text{Equations Valid for all op-amp regions:}}$

$$\frac{V_-^{(1)} - V_{in}}{1} + \frac{V_-^{(1)} - V_{out}^{(1)}}{2} + \frac{V_-^{(1)} - e_1}{1} = 0 \rightarrow 5V_-^{(1)} = 2V_{in} + V_{out}^{(1)} + 2e_1 \quad (1)$$

$$\frac{e_1 - V_-^{(1)}}{1} + \frac{e_1}{3} + \frac{e_1 - V_{out}^{(2)}}{6} = 0 \rightarrow 9e_1 = 6V_-^{(1)} + V_{out}^{(2)} \quad (2).$$

Replace e_1 in (1): $\boxed{\underbrace{\left(5 - \frac{4}{3}\right)}_{11/3} V_-^{(1)} = 2V_{in} + V_{out}^{(1)} + \frac{2}{9} V_{out}^{(2)}}$ (I)

$\boxed{V_+^{(2)} = V_{out}^{(1)}}$ (II) $\boxed{V_-^{(2)} = \frac{V_{out}^{(2)}}{3}}$ (III)

(A) Op-Amp

(1)	(2)
linear	linear

$V_-^{(1)} = V_+^{(1)} = 3V$

$V_-^{(2)} = V_+^{(2)} \Rightarrow \frac{V_{out}^{(2)}}{3} = V_{out}^{(1)}$

Insert in (I):

$$11 = 2V_{in} + V_{out}^{(1)} + \frac{2}{9} \underbrace{V_{out}^{(2)}}_{3V_{out}^{(1)}}$$

$\boxed{V_{out}^{(1)} = \frac{3}{5}(11 - 2V_{in})}$

$\boxed{V_{out}^{(2)} = \frac{9}{5}(11 - 2V_{in})}$

$|V_{out}^{(1)}| < 18 \rightarrow |2V_{in} - 11| < 30 \rightarrow \frac{-19}{2} < V_{in} < \frac{41}{2}$

$|V_{out}^{(2)}| < 18 \rightarrow |2V_{in} - 11| < 10 \rightarrow \frac{1}{2} < V_{in} < \frac{21}{2}$

$|V_{out}^{(1)}| < 18$ and $|V_{out}^{(2)}| < 18 \rightarrow \boxed{\frac{1}{2} < V_{in} < \frac{21}{2}}$

(B) Op-Amp

(1)	(2)
linear	+Sat

$V_-^{(1)} = V_+^{(1)} = 3V$; $\boxed{V_{out}^{(2)} = 18\ V}$

$V_{out}^{1} > \frac{V_{out}^{(2)}}{3} = 6$

From (I): $11 = 2V_{in} + V_{out}^{(1)} + 4$

$\boxed{V_{out}^{(1)} = 7 - 2V_{in}}$

$|V_{out}^{(1)}| < 18 \rightarrow \frac{-11}{2} < V_{in} < \frac{25}{2}$

$V_d^{(2)} > 0 \rightarrow V_{out}^{(1)} > 6 \rightarrow V_{in} < 1/2$

$\boxed{\frac{-11}{2} < V_{in} < 1/2}$

C

Op-Amp (I)	Op-Amp (II)
lined	-Sat

$V_-^{(1)} = V_+^{(1)} = 3V$; $\boxed{V_{out}^{(2)} = -18\ V}$

$V_{out}^{1} < \frac{V_{out}^{(2)}}{3} = -6$

From (I): $11 = 2V_{in} + V_{out}^{(1)} - 4$

$\boxed{V_{out}^{(1)} = 15 - 2V_{in}}$

$|V_{out}^{(1)}| < 18 \longrightarrow -\frac{3}{2} < V_{in} < \frac{33}{2}$

$V_d^{(2)} < 0 \longrightarrow V_{out}^{(1)} < -6 \longrightarrow V_{in} > \frac{21}{2}$

Both satisfied when $\boxed{\frac{21}{2} < V_{in} < \frac{33}{2}}$

D

Op-Amp	
+ Sat	?
+ Sat	+ Sat

$V_{out}^{(1)} = 18V$

$V_-^{(2)} = \frac{V_{out}^{(2)}}{3}$ has the maximum value of 6V

$V_d^{(2)} = 18 - V_-^{(2)} > 6 \longrightarrow V_d^{(2)} > 0$ (Op-Amp always in +sat)

$V_{out}^{(1)} = V_{out}^{(2)} = 18V$

From (I): $\frac{11}{3} V_-^{(1)} = 2V_{in} + 18 + 4 \longrightarrow V_-^{(1)} = \frac{6V_{in}}{11} + 6$

D. cont

$V_d^{(1)} > 0 \longrightarrow V_-^{(1)} < 3V$

$\downarrow$

$\left(\frac{6}{11}V_{in} + 6\right) < 3$

$\boxed{V_{out}^{(1)} = V_{out}^{(2)} = 18V}$ $\boxed{V_{in} < -\frac{33}{6} = -\frac{11}{2}}$

E

Op-Amp (1)	(2)
-sat	→ -sat

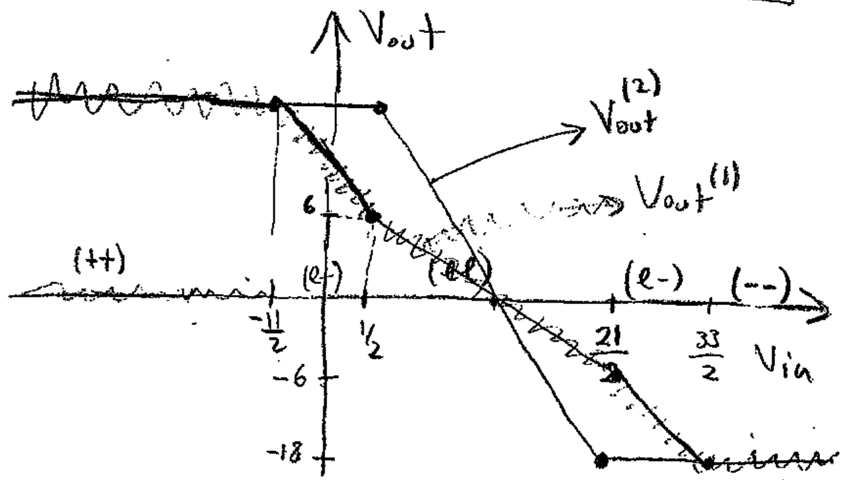
$V_{out}^{(1)} = V_{out}^{(2)} = -18V$

$V_-^{(1)} > 3V$

$V_-^{(1)} = \frac{6}{11}V_{in} - 6$

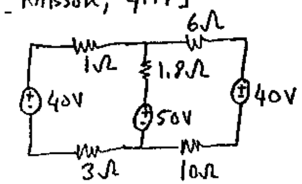
$V^{(1)} > 3 \longrightarrow \frac{6}{11}V_{in} - 6 > 3$

$\boxed{V_{out}^{(1)} = V_{out}^{(2)} = -18V}$ $\boxed{V_{in} > \frac{33}{2}}$



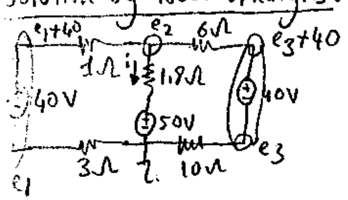
Node - Mesh - Superposition - Source Transformation:

[Nilsson, 4.17]



Find the power delivered by the 50V source.

Solution by Node Analysis:



$$\frac{e_1+40-e_2}{1}+\frac{e_1}{3}=0$$

$$\frac{e_2-(e_1+40)}{1}+\frac{e_2-50}{1.8}+\frac{e_2-(e_3+40)}{6}=0 \Rightarrow \begin{bmatrix} 4 & -3 & 0 \\ -18 & 31 & -3 \\ 0 & -5 & 8 \end{bmatrix}\begin{bmatrix} e_1 \\ e_2 \\ e_3 \end{bmatrix}=\begin{bmatrix} 1340 \\ -120 \\ -200 \end{bmatrix}$$

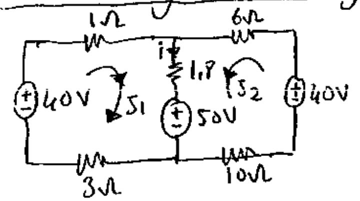
$$\frac{e_3+40-e_2}{6}+\frac{e_3}{10}=0$$

$\boxed{\text{3eqn's , 3 unknowns}}$

$e_1 = 4.8V$; $e_2 = 46.4V$; $e_3 = 4V$.

$i = \frac{(46.4-50)}{1.8} = -2A$; $P_{50V} = 50(-2) = -100$ Watts

Solution by Mesh Analysis:



$-40 + (3+1+1.8)J_1 + 1.8J_2 + 50 = 0$ (1)

$-40 + 1.8J_1 + (6+10+1.8)J_2 + 50 = 0$ (2)

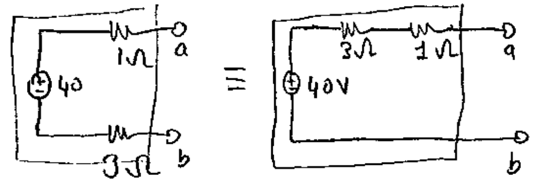
(1) - (2) : $4J_1 = 16J_2 \rightarrow J_1 = 4J_2$.

From (1) : $(5.8)(4J_2) + 1.8J_2 = -10 \rightarrow \underline{J_2 = -0.4A}$, $\underline{J_1 = -1.6A}$

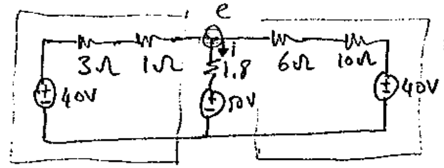
$\underline{i = J_1 + J_2 = -2A} \longrightarrow P_{50V} = \underline{-100 \text{ Watts}}$

$\boxed{\text{2eqn's 2 unknowns}}$

Solution by circuit simplification: (Equivalent circuits)



Both circuits have the same (i,v) characteristics. (Thevenin Equivalent or exchanging series branches)



Since we are only interested in "i", we can substitute the rest of branches and their nodes with simpler but equivalent circuits.

Node Equation : $\frac{e-50}{1.8}+\frac{e-40}{4}+\frac{e-40}{16}=0 \rightarrow \frac{\cancel{5}(e-50)}{9}+\frac{\cancel{5}(e-40)}{16}=0$

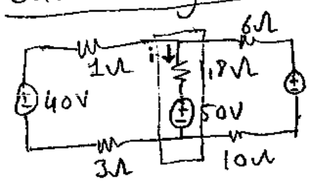
$25e = 360+800$

$e = (14+0.4)+32$

$\underline{e = 46.4V} \rightarrow$

$\boxed{\text{1 equation, 1 unknown}}$ $\rightarrow \underline{i = \frac{46.4-50}{1.8}} = \underline{-2A}$; $\underline{P_{50V} = -100 \text{ Watts}}$

Solution by Source Transformation and Superposition:



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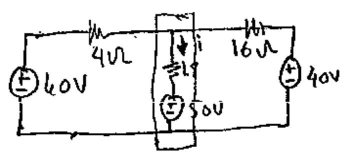

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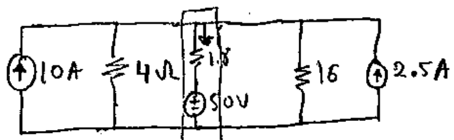

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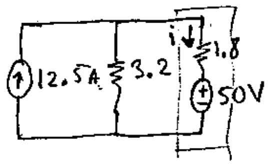

$\boxed{\text{Direct solution No unknowns, no equations}}$

By superposition

$$i = \underbrace{(12.5)\frac{3.2}{5}}_{\text{current division}} - \underbrace{\frac{50}{5}}_{\frac{50}{\dots}} = 8 - 10 = \underline{-2A} \quad ; \quad \underline{P_{50V} = -100 \text{ Watts}}$$