P.1: [Griffiths’ p-vector algorithm] (Hayes 9.14)
Griffiths has developed an algorithm for LMS adaptive filters. The method is based on a variation of LMS in which the desired signal is not explicitly present at the filter output but a-priori knowledge of the cross-correlation between the input and the desired output is given. Specifically, recall that the LMS algorithm has the coefficient update equation as follows:

\[ w_{n+1} = w_n + \mu e(n)x(n) = w_n + \mu d(n)x(n) - \mu \left[w_n^T x(n)\right] x(n) \]

If we replace \( d(n)x(n) \) with its expected value \( p = E\{d(n)x(n)\} \), we get the p-vector algorithm of Griffiths’:

\[ w_{n+1} = w_n + \mu p - \mu \left[w_n^T x(n)\right] x(n) \]

Note that this update does not require \( d(n) \) explicitly.

a) Derive an expression on \( \mu \) for convergence in the mean.

b) Develop a leaky p-vector algorithm by writing down the filter coefficient update equation. Determine the range of \( \mu \) values for its convergence in the mean and find \( \lim_{n \to \infty} E\{w_n\} \).

c) [Computer Exercise] Let \( x(n) \) be a process that is generated according to the difference equation

\[ x(n) = 1.2728x(n-1) - 0.81x(n-2) + v(n) \]

where \( v(n) \) is unit variance white Gaussian noise. Suppose we would like to implement a linear predictor for \( x(n) \):

\[ \hat{x}(n) = w_n(1)x(n-1) + w_n(2)x(n-2) \]

i) Generate a sequence \( x(n) \) of length \( N = 500 \) and determine a step size so that \( w_n \) converges in 200 iterations (set the time constant \( \tau = \frac{1}{\mu_{\text{min}}} \) to 200).

ii) Implement the LMS adaptive predictor and plot \( w_n(k) \) versus \( n \) for \( k = 1, 2 \).

iii) Find the p-vector, \( p = E\{d(n)x(n)\} \). And implement the p-vector algorithm (modify llms.m) Plot \( w_n(k) \) and compare with part ii).

iv) Investigate the sensitivity of the p-vector algorithm to the errors in the p-vector. Make some small changes to the p-vector of part iii) and re-run the algorithm.
Consider a system identification problem.

The unknown plant has the transfer function of the form:

$$H_f(z) = \frac{0.05 - 0.4z^{-1}}{1 - 1.314z^{-1} + 0.25z^{-2}}$$

The adaptive that is used to model $H_f(z)$ has two free parameters $a$ and $b$.

$$H_a(z) = \frac{b}{1 - az^{-1}}$$

The input to the both systems is unit variance white noise. The goal is to find $a$ and $b$ that minimizes $\varepsilon = E\{(e(n))^2\}$ where $e(n) = d(n) - y(n)$. The error function for the minimization has a global minimum at $(b, a) = (-0.311, 0.906)$ and a local minimum at $(b, a) = (0.114, -0.519)$.

In order for the filter coefficients $a_n$ and $b_n$ in $H_a(z)$ to converge in the mean using Feintuch’s algorithm, it is necessary that

$$\lim_{n \to \infty} E\{e(n)x(n)\} = 0$$

and

$$\lim_{n \to \infty} E\{e(n)y(n-1)\} = 0$$

a) Find the values of $E\{e(n)x(n)\}$ and $E\{e(n)y(n-1)\}$ at the global minimum of $\varepsilon$. What does this imply about the Feintuch’s Algorithm?

b) Find the stationary point of the Feintuch’s adaptive filter, i.e. $(a, b)$ for which $E\{e(n)x(n)\} = 0$ and $E\{e(n)y(n-1)\} = 0$.

(Hint: $d(n) = \sum_{k=0}^{\infty} h_f(k)x(n-k)$ and $y(n) = \sum_{k=0}^{\infty} h_a(k)x(n-k)$. $x(n)$ is unit variance white noise. $e(n) = d(n) - y(n)$)
P.3: [Equation Error Method] (Hayes)

Consider the system identification set-up of Problem 2. Assume that the plant has the transfer function of
\[ G(z) = \frac{1}{1 - 0.5z^{-1}} \]
and the constructed adaptive filter is of the form
\[ H_a(z) = \frac{b}{1 - az^{-1}} \]

Assume that the plant output \( d(n) \) is corrupted with white noise of variance \( \rho_v^2 \).

The Wiener solution minimizing the error \( \varepsilon = E\{(d(n) - y(n))^2\} \) is expected to be \((b, a) = (1, 0.5)\) and the minimum error corresponding to optimal coefficients is \( \rho_v^2 \).

In this problem, we will see that the equation error method is biased when the plant output is noisy. To find the optimum \((a, b)\) according to the equation error method, we need to solve the equation:
\[ R_u \Theta = p \]  \hspace{1cm} (1)

where \( u(n) = [x(n) \ d(n-1)]^T \) and
\[ R_u = \begin{bmatrix} r_{xx}(0) & r_{xd}(1) \\ r_{xd}(1) & r_{dd}(0) \end{bmatrix}; \quad \Theta = \begin{bmatrix} b \\ a \end{bmatrix}; \quad p = \begin{bmatrix} r_{dx}(0) \\ r_{dd}(1) \end{bmatrix} \]

The autocorrelation values can be calculated as:
\[ r_{dx}(0) : \quad E\{d(n)x(n)\} = \frac{1}{2\pi i} \oint P_x(z)G(z)\frac{dz}{z} = \frac{1}{2\pi i} \oint \frac{1}{1 - 0.5z^{-1}} \frac{dz}{z} = \frac{1}{2\pi i} \oint \frac{1}{z - 0.5} dz = \text{residue of } \frac{1}{z - 0.5} \text{ at } z = 0.5 = 1 \]
\[ r_{xd}(1) : \quad E\{x(n)d(n-1)\} = 0 \]
\[ r_{dd}(0) : \quad E\{d(n)d(n)\} = 4/3 + \rho_v^2 \]
\[ r_{dd}(1) : \quad E\{d(n)d(n-1)\} = 2/3 \]

Then the solution of equation system is:
\[ \Theta = \begin{bmatrix} b \\ a \end{bmatrix} = \begin{bmatrix} 1 \\ \frac{1}{1 + \rho_v^2} \end{bmatrix} \]

The solution of the equation error method is therefore biased if \( \rho_v^2 \neq 0 \). Note that the bias is proportional to the noise variance. For zero-noise conditions the equation error system converges to the known Wiener solution \((b, a) = (1, 0.5)\).
a) Assume that the input to the unknown plant is not white but has the correlation of

\[ P_x(e^{jw}) = \frac{1}{|1 - 0.3e^{-jw}|^2} \]

Find the Wiener filter coefficients \((b, a)\) minimizing the error \(E\{e^2(n)\}\).

b) Find the values for \(a\) and \(b\) that are optimum according to the equation error method.

c) Repeat part a) and b) for

\[ P_x(e^{jw}) = \frac{1}{|1 - 0.8e^{-jw}|^2} \]

d) What do you observe with respect to the biases in the coefficients \(a\) and \(b\)? How are the biases affected by the shape of the power spectrum \(P_x(e^{jw})\)?

e) Write Matlab programs to implement output error method and equation error method and confirm your results derived in parts (a) - (d).