A gentle introduction to non-standard analysis

Burak Kaya

METU

burakk@metu.edu.tr

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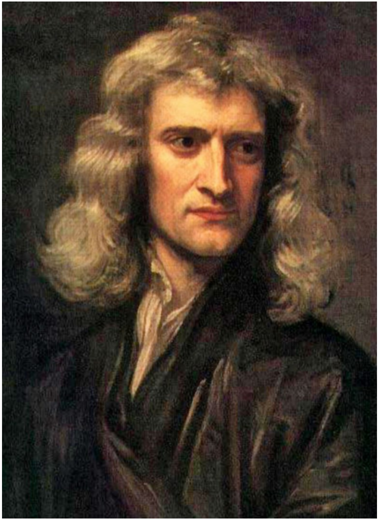
The birth of infinitesimal calculus



Sir Isaac Newton
(My 16th academic great-grandfather)

The birth of infinitesimal calculus

Gottfried Wilhelm von Leibniz

The complete ordered field of real numbers

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- For all x, y, z we have that $(x + y) \cdot z = x \cdot z + y \cdot z$.

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- For all x, y we have that if $0 \leq x, y$, then $0 \leq a \cdot b$.
- (Dedekind-completeness) Any non-empty subset of $\mathbb{R}$ with an upper bound has a least upper bound.

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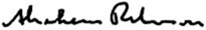
Question

Are there ordered fields "similar" to real numbers which has non-zero infinitesimals?

The rise of infinitesimal calculus

Courtesy of Yale University News Bureau



Abraham Robinson

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Example

$\forall x \in \mathbb{R}\ (0 \leq x \to \exists y \in \mathbb{R}\ x = y \cdot y)$ iff
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- $({}^*\mathbb{R}, {}^*+, {}^*\cdot, {}^*0, {}^*1, {}^*\leq)$ has non-zero infinitesimals.

Standard parts

Theorem

For every limited hyperreal $r \in {}^*\mathbb{R}$ *there exists a unique real number* $s \in \mathbb{R}$ *such that* $r \approx s$.

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Proof.

Let $A = \{x \in \mathbb{R} : x < r\}$. Since r is limited, A is bounded above and hence has a least upper bound $s \in \mathbb{R}$. Let $\delta > 0$ be any real number. Then we have $s + \delta \geq r$ since $s + \delta \notin A$. Similarly, we have $s - \delta < r$ since s is the least upper bound. Thus, for any real number $\delta > 0$, $-\delta < r - s < \delta$ and hence $r \approx s$. $\square$

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Given a limited hyperreal $r \in {}^*\mathbb{R}$, the unique real number to which it is infinitely close is called its standart part and denoted by $st(r)$.

Limits via infinitesimals

Theorem

Let $f : \mathbb{R} \to \mathbb{R}$ *be a function and* $c \in \mathbb{R}$. *Then* $\lim_{x \to c} f(x) = L$ *if and only if* $f(c + \epsilon) \approx L$ *for every non-zero infinitesimal* ϵ.

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Let us find the limit of $f(x) = x^2$ at $c = 4$ using this characterization. Let ϵ be a non-zero infinitesimal.

$$\lim_{x \to 4} x^2 = st((4+\epsilon)^2) = st(4^2 + 8\epsilon + \epsilon^2) = 16$$

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Theorem

Let $f : \mathbb{R} \to \mathbb{R}$ be a function. Then $\lim\limits_{x \to +\infty} f(x) = L$ if and only if $f(w) \approx L$ for every positive unlimited hyperreal w.

Derivatives via infinitesimals

Theorem

Let $x \in \mathbb{R}$ and $f : \mathbb{R} \to \mathbb{R}$ be a function. Then $f'(x) = L$ if and only if for every non-zero infinitesimal ϵ we have

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$$f'(x) = st\left(\frac{(x+\epsilon)^2 - x^2}{\epsilon}\right) = st\left(\frac{x^2 + 2x\epsilon + \epsilon^2 - x^2}{\epsilon}\right) = st(2x+\epsilon) = 2x$$

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Theorem

If f is Riemann integrable on $[a, b]$, then $\int_a^b f(x)dx = st(S_f(\epsilon))$ for any positive infinitesimal ϵ.

Uniform continuity via infinitesimals

Theorem

A function $f : A \to \mathbb{R}$ is uniformly continuous if and only if for all hyperreals $x, y \in {}^*A$ we have that if $x \approx y$, then $f(x) \approx f(y)$.

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Proof.

Let $x, y \in {}^*[a, b]$ such that $x \approx y$. Then $st(x) = c \in [a, b]$. It follows from the continuity of f that $f(x) \approx f(c)$ and $f(y) \approx f(c)$. Therefore, $f(x) \approx f(y)$ and hence f is uniformly continuous. $\square$

Intermediate Value Theorem via infinitesimals

The full form of Intermediate Value Theorem can easily be derived from the following statement.

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If $f : [a, b] \to \mathbb{R}$ is a continuous function and $d \in \mathbb{R}$ such that $f(a) < d < f(b)$, then there exists $c \in (a, b)$ such that $f(c) = d$.

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Proof.

Moreover, for each $n \in \mathbb{N}$, we have that

$$a \leq S(n) < b \text{ and } f(S(n)) < d \leq f(S(n) + \frac{b-a}{n})$$

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we have that $f(c) \approx d$.

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$$f({}^*S(N)) < d \leq f({}^*S(N) + \frac{b-a}{N})$$

we have that $f(c) \approx d$. However, both $f(c)$ and d are in $\mathbb{R}$ and hence $f(c) = d$. $\square$

Compactness via infinitesimals

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Theorem (Robinson's criteria of compactness)

A subset $A \subseteq \mathbb{R}$ is compact if and only if for every $x \in {}^*A$ there exists $y \in A$ such that $x \approx y$.

Construction of a hyperreal field

- Let $\mathcal{U}$ be a non-principal ultrafilter on $\mathbb{N}$.
- Let ${}^*\mathbb{R}$ be the set of equivalence classes of the relation $\sim$ defined on $\mathbb{R}^{\mathbb{N}}$ given by

$$(p_n) \sim (q_n) \longleftrightarrow \{n \in \mathbb{N} : p_n = q_n\} \in \mathcal{U}$$

- $\mathbb{R}$ embeds into ${}^*\mathbb{R}$ diagonally.
- Each function and relation on $\mathbb{R}$ can be canonically extended to ${}^*\mathbb{R}$. For example,

$$[(p_n)] \ {}^* \leq [(q_n)] \longleftrightarrow \{n \in \mathbb{N} : p_n \leq q_n\} \in \mathcal{U}$$

  Let ${}^*+, {}^*\cdot, {}^*0, {}^*1, {}^*\leq$ be the canonical extensions of $+, \cdot, 0, 1, \leq$.
- The structure $({}^*\mathbb{R}, {}^*+, {}^*\cdot, {}^*0, {}^*1, {}^*\leq)$ is a hyperreal field.

Thank you!