## Gödel's Incompleteness Theorem for Mathematicians

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METU Math Club Student Seminars

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- In 1931, Kurt Gödel proved his famous incompleteness theorems and showed that Hilbert's program cannot be achieved.
- In 1936, Alan Turing proved that Hilbert's Entscheindungsproblem cannot be solved, i.e. there is no general algorithm which will decide whether a given mathematical statement is provable (from a given set of axioms.)

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- For each formula  $\varphi(x, y_1, \dots, y_k)$  in the language of arithmetic,

$$\forall y_1 \ldots \forall y_k ((\varphi(0, y_1, \ldots, y_k) \land$$

$$\forall x \ \varphi(x, y_1, \ldots, y_k) \rightarrow \varphi(S(x), y_1, \ldots, y_k)) \rightarrow \forall x \ \varphi(x, y_1, \ldots, y_k))$$

### Representing recursive sets and functions in PA

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#### Fact

If  $f : \mathbb{N} \to \mathbb{N}$  is a computable function, then there exists a formula  $\varphi(x, y)$  in the language of PA such that for each  $n \in \mathbb{N}$  PA proves that

$$\varphi(\overline{n}, y) \leftrightarrow \overline{f(n)} = y$$

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• Then each finite sequence  $s_1 s_2 s_3 \dots s_k$  consisting of these symbols can be assigned to the natural number  $2^{\lceil s_1 \rceil} \cdot 3^{\lceil s_2 \rceil} \dots p_k^{\lceil s_k \rceil}$  where  $\lceil s \rceil$  denotes the natural number assigned to s.

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- Given a formula φ in the language of arithmetic, the corresponding natural number under this assignment will be called the Gödel number of φ and is denoted by [φ].
- Using a similar trick, we can also assign natural numbers to finite sequences of formulas.

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$$Prov_{PA}(x) := \exists y \ Pr(y, x)$$

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- We can also construct the number theoretic statement

$$Con(PA) := \neg Prov_{PA}(\overline{[0 = S(0)]})$$

which asserts that 0 = S(0) is not provable from PA, i.e. PA is consistent.

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#### Lemma (The Diagonal Lemma)

Let  $\psi(x)$  be a formula in the language of PA with one-free variable. Then there exists a sentence  $\varphi$  such that

$$\mathsf{PA} \vdash \varphi \leftrightarrow \psi(\overline{\lceil \varphi \rceil})$$

#### Proof.

Observe that the function  $f : \mathbb{N} \to \mathbb{N}$  which maps  $\lceil \theta(x) \rceil$  with one-free variable to  $\lceil \theta(\lceil \overline{\theta(x)} \rceil) \rceil$  and which maps other natural numbers to 0 is computable.

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Let  $\chi(x)$  be the formula  $\exists y(\alpha(x,y) \land \psi(y))$  and let  $\varphi$  be  $\chi(\overline{\lceil \chi(x) \rceil})$ .

Image: A math a math

#### Constructing the Gödel sentence

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Assume that PA ⊢ φ. Then PA ⊢ Prov<sub>PA</sub>([φ]) and hence PA ⊢ ¬φ by the construction of φ. It follows that PA is inconsistent. Thus, if PA is consistent, then PA cannot prove φ.
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- Assume that PA ⊢ ¬φ. Then PA ⊢ Prov<sub>PA</sub>([φ]) by the construction of φ. If PA is ω-consistent, then PA ⊢ Pr(n, [φ]) for some natural number n and hence PA ⊢ φ, in which case PA cannot be consistent. Thus, if PA is ω-consistent, then PA cannot prove ¬φ.

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## Theorem (Gödel)

If PA is  $\omega$ -consistent, then PA cannot prove  $\varphi$  or  $\neg \varphi$ .

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## Gödel's incompleteness theorems

 Barkley Rosser found a trick to improve this result by weakening the ω-consistency assumption. Using this trick, one can prove the following form of Gödel's theorem.

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### Theorem

Let  $T \supseteq Q$  be a recursively enumerable theory containing the Robinson Arithmetic Q. If T is consistent, then there exists a sentence  $\varphi$  such that T does not prove  $\varphi$  or  $\neg \varphi$ ; and T does not prove Con(T).

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# Tarski's undefinability theorem

 Another striking application of the Diagonal Lemma is Tarski's theorem of undefinability of truth, from which we can deduce Gödel's first theorem as a corollary.  Another striking application of the Diagonal Lemma is Tarski's theorem of undefinability of truth, from which we can deduce Gödel's first theorem as a corollary.

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Let  $\mathcal{T} = \{ \lceil \varphi \rceil : \mathbb{N} \models \varphi \}$ . Then there is no formula  $\psi(x)$  in the language of arithmetic such that  $n \in \mathcal{T} \Leftrightarrow \mathbb{N} \models \psi(\overline{n})$ . In other words, arithmetical truth cannot be defined arithmetically.

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## Proof.

Assume to the contrary that there exists such a formula  $\psi(x)$ . It follows from the Diagonal Lemma that there exists a sentence  $\varphi$  such that PA proves  $\varphi \leftrightarrow \neg \psi(\overline{\lceil \varphi \rceil})$ . Then  $\mathbb{N} \models \varphi$  iff  $\mathbb{N} \models \neg \psi(\overline{\lceil \varphi \rceil})$  iff  $\mathbb{N} \models \neg \varphi$ , which is a contradiction.

• Since recursive sets are arithmetically definable, it follows from Tarski's theorem that

Corollary The set  $Th(\mathbb{N}, +, \cdot, S, 0)$  of true sentences in the structure  $(\mathbb{N}, +, \cdot, S, 0)$  is not recursive.

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## Sketch proof.

Assume that PA is both sound and complete. It follows that given any sentence  $\varphi$ , since either  $\varphi$  or  $\neg \varphi$  will be eventually provable, we can decide with a Turing machine whether  $\varphi$  is true or not in the structure  $(\mathbb{N}, +, \cdot, S, 0)$  by enumerating all valid PA-proofs.

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### Corollary

PA cannot be both sound and complete.

## Sketch proof.

Assume that PA is both sound and complete. It follows that given any sentence  $\varphi$ , since either  $\varphi$  or  $\neg \varphi$  will be eventually provable, we can decide with a Turing machine whether  $\varphi$  is true or not in the structure  $(\mathbb{N}, +, \cdot, S, 0)$  by enumerating all valid PA-proofs.

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## Incompleteness phenomenon in mathematics

- We have seen that Gödel's incompleteness theorems apply to ZFC. Since Gödel's original result and the invention of forcing, a technique for which Paul Cohen received the Fields medal, many natural statements have been proven to be independent of ZFC, i.e. if ZFC is consistent, then these statements neither provable nor disprovable from ZFC.
- There are hundreds of independence results. Here we only present some famous statements that are independent of ZFC.

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There does not exist a set A such that  $|\mathbb{N}| < |A| < |\mathbb{R}|$ .

### Statement (Borel's conjecture)

Every strong measure zero set is countable, where a set  $A \subseteq \mathbb{R}$  is said to be strong measure zero if for every sequence  $(\epsilon_n)$  of positive reals there exist a sequence  $(I_n)$  of intervals such that  $|I_n| < \epsilon_n$  and  $A \subseteq \bigcup_{n=0}^{\infty} I_n$ .

## Statement (Kaplansky's conjecture)

Every algebra homomorphism from the Banach algebra C(X) to any other Banach algebra is continuous, where X is a compact Hausdorff space and C(X) is the space of continuous complex valued functions on X.

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### Statement

Let  $A = \mathbb{C}[x, y, z]$  and  $M = \mathbb{C}(x, y, z)$  be its field of fractions. The projective dimension of M as an A-module is 2.

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- **Reality**: This is complete non-sense. Gödel's theorem does not say anything about mathematics being formalizable or not. Indeed, virtually all known mathematics can be formalized in ZFC, which some consider as *the* foundation of mathematics. That some statements are independent of ZFC is a whole nother issue.

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- **Misconception**: "Gödel's theorem shows that no formal system can be both complete and consistent".
- **Reality**: No! No! No! In order to apply Gödel's theorem, one should have a formal system with recursively enumerable axioms which can interpret the Robinson arithmetic. In particular, the formal system should be able to define its own provability predicate and prove (a specific instance of) the Diagonal Lemma.

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- If one does not insist that the axiom set be recursively enumerable, then one can easily find extensions of PA which are both complete and consistent. Indeed, Lindenbaum's lemma states that any consistent first-order theory has a complete extension.
- If one does not insist that the formal system interpret Robinson arithmetic, then one can easily find r.e. theories which are both consistent and complete. For example, Tarski proved that the theory of real closed fields is complete (and indeed, decidable).

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- Gödel's theorem prevents any "sufficiently strong and nice" theory from proving its own consistency statement. Nevertheless, it does not preclude the existence of some consistency proof which cannot be formalized within the theory.
- One should also note that given finitely many axioms of ZFC (or of PA), one can prove in ZFC (or in PA) that these axioms are consistent. (However, ZFC does not prove that "every finite subset of ZFC is consistent.")

Burak Kaya (METU)

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- **Reality**: This may be true or false up to interpretation. Gödel's theorem does not directly imply that strong AI is impossible.
- Lucas and Penrose tried to argue using Gödel's theorem that the human mind cannot be simulated by a Turing machine. However, there have been many counter arguments by logicians and philosophers against the Lucas-Penrose argument. For example, you can read Solomon Feferman's criticism on Penrose's argument.

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- **Reality**: Since Gödel's original theorem, the incompleteness phenomenon has been studied extensively and is very well-understood. Mathematical community has accepted and appreciated Gödel's theorems.
- Indeed, proofs of variants of Gödel's theorems have been formalized and checked by proof-assistants such as Isabelle. This precludes the possibility that there is a flaw or missing step in the proof.

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# Thank you!

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