Gödel's Incompleteness Theorem for Mathematicians

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- In 1931, Kurt Gödel proved his famous incompleteness theorems and showed that Hilbert's program cannot be achieved.
- In 1936, Alan Turing proved that Hilbert's Entscheindungsproblem cannot be solved, i.e. there is no general algorithm which will decide whether a given mathematical statement is true or not.

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  - For each formula $\varphi(x, y_1, \ldots, y_k)$ in the language of arithmetic,

$$\forall y_1 \ldots \forall y_k \; ((\varphi(0, y_1, \ldots, y_k) \wedge$$

$$\forall x \; \varphi(x, y_1, \ldots, y_k) \rightarrow \varphi(S(x), y_1, \ldots, y_k)) \rightarrow \forall x \; \varphi(x, y_1, \ldots, y_k))$$

Representing recursive sets and functions in PA

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Fact

If $f : \mathbb{N} \rightarrow \mathbb{N}$ is a computable function, then there exists a formula $\varphi(x, y)$ in the language of PA such that for each $n \in \mathbb{N}$ PA proves that

$$\varphi(\overline{n}, y) \leftrightarrow \overline{f(n)} = y$$

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$\exists$ 3	$\wedge$ 9	$=$ 15	S 21	$\cdot$ 27	z 33
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- Then each finite sequence $s_1 s_2 s_3 \dots s_k$ consisting of these symbols can be assigned to the natural number $2^{\ulcorner s_1 \urcorner} \cdot 3^{\ulcorner s_2 \urcorner} \dots p_k^{\ulcorner s_k \urcorner}$ where $\ulcorner s \urcorner$ denotes the natural number assigned to s.

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- Given a formula φ in the language of arithmetic, the corresponding natural number under this assignment will be called the Gödel number of φ and is denoted by $\ulcorner \varphi \urcorner$.
- Using a similar trick, we can also assign natural numbers to finite sequences of formulas.

Constructing a provability predicate

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$$Prov_{PA}(x) := \exists y \ Pr(y,x)$$
- Notice that $Prov_{PA}(\overline{\ulcorner \varphi \urcorner})$ is a sentence written in the language of PA and is a number-theoretic statement.

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- We can also construct the number theoretic statement

$$Con(PA) := \neg Prov_{PA}(\overline{\ulcorner 0 = S(0) \urcorner})$$

which asserts that $0 = S(0)$ is not provable from PA, i.e. PA is consistent.

The heart of the matter

Lemma (The Diagonal Lemma)

Let $\psi(x)$ be a formula in the language of PA with one-free variable. Then there exists a sentence φ such that

$$PA \vdash \varphi \leftrightarrow \psi(\overline{\ulcorner \varphi \urcorner})$$

Proof.

Observe that the function $f : \mathbb{N} \to \mathbb{N}$ which maps $\ulcorner \theta(x) \urcorner$ with one-free variable to $\ulcorner \theta(\overline{\ulcorner \theta(x) \urcorner}) \urcorner$ and which maps other natural numbers to 0 is computable.

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Let $\chi(x)$ be the formula $\exists y(\alpha(x, y) \wedge \psi(y))$ and let φ be $\chi(\overline{\ulcorner \chi(x) \urcorner})$. □

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- Assume that $PA \vdash \neg\varphi$. Then $PA \vdash Prov_{PA}(\overline{\ulcorner \varphi \urcorner})$ by the construction of φ. If PA is ω-consistent, then $PA \vdash Pr(\overline{n}, \overline{\ulcorner \varphi \urcorner})$ for some natural number n and hence $PA \vdash \varphi$, in which case PA cannot be consistent. Thus, if PA is ω-consistent, then PA cannot prove $\neg\varphi$.

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Theorem (Gödel)

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- However, it is not true that φ is true in every model of PA.Indeed, it follows from Gödel's completeness theorem that there exists models of PA in which φ is false. Such models of PA contain non-standard natural numbers for which the provability predicate $Pr(x, y)$ does not capture its intended meaning.

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Let $\mathcal{T} = \{\lceil \varphi \rceil : \mathbb{N} \models \varphi\}$*. Then there is no formula* $\psi(x)$ *in the language of arithmetic such that* $n \in \mathcal{T} \Leftrightarrow \mathbb{N} \models \psi(\overline{n})$*. In other words, arithmetical truth cannot be defined arithmetically.*

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Proof.

Assume to the contrary that there exists such a formula $\psi(x)$. It follows from the Diagonal Lemma that there exists a sentence φ such that PA proves $\varphi \leftrightarrow \neg\psi(\overline{\ulcorner \varphi \urcorner})$. Then $\mathbb{N} \models \varphi$ iff $\mathbb{N} \models \neg\psi(\overline{\ulcorner \varphi \urcorner})$ iff $\mathbb{N} \models \neg\varphi$, which is a contradiction. □

Gödel's theorem revisited

- Since recursive sets are arithmetically definable, it follows from Tarski's theorem that

Corollary

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- We have seen that Gödel's incompleteness theorems apply to ZFC. Since Gödel's original result and the invention of forcing, a technique for which Paul Cohen received the Fields medal, many natural statements have been proven to be independent of ZFC, i.e. if ZFC is consistent, then these statements neither provable nor disprovable from ZFC.
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Every strong measure zero set is countable, where a set $A \subseteq \mathbb{R}$ is said to be strong measure zero if for every sequence (ϵ_n) of positive reals there exist a sequence (I_n) of intervals such that $|I_n| < \epsilon_n$ and $A \subseteq \bigcup_{n=0}^{\infty} I_n$.

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Let $A = \mathbb{C}[x, y, z]$ and $M = \mathbb{C}(x, y, z)$ be its field of fractions. The projective dimension of M as an A-module is 2.

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- **Reality**: This is complete non-sense. Gödel's theorem does not say anything about mathematics being formalizable or not. Indeed, virtually all known mathematics can be formalized in ZFC, which some consider as *the* foundation of mathematics. That some statements are independent of ZFC is a whole nother issue.

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- If one does not insist that the formal system interpret Robinson arithmetic, then one can easily find r.e. theories which are both consistent and complete. For example, Tarski proved that the theory of real closed fields is complete (and indeed decidable).

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- Gödel's theorem prevents any "sufficiently strong and nice" theory from proving its own consistency statement. Nevertheless, it does not preclude the existence of some consistency proof which cannot be formalized within the theory.
- One should also note that given finitely many axioms of ZFC (or of PA), one can prove in ZFC (or in PA) that these axioms are consistent. (However, ZFC does not prove that "every finite subset of ZFC is consistent.")

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- **Reality**: This may be true or false up to interpretation. Gödel's theorem does not directly imply that strong AI is impossible.
- Lucas and Penrose tried to argue using Gödel's theorem that the human mind cannot be simulated by a Turing machine. However, there have been many counter arguments by logicians and philosophers against the Lucas-Penrose argument. For example, you can read Solomon Feferman's criticism on Penrose's argument.

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- Indeed, proofs of variants of Gödel's theorems have been formalized and checked by proof-assistants such as Isabelle. This precludes the possibility that there is a flaw or missing step in the proof.

Thank you!