MATH 535, Topology, Homework 4

1. (2 pts) Let (X, d) be a sequentially compact metric space and $A, B \subseteq X$ be disjoint closed subsets. Show that $\inf\{d(a, b) : a \in A, b \in B\} > 0$.

2. (2+3+3 pts) Let (X, d) be a metric space.

(a) Let $K \subseteq X$ be a compact subset and $x \in X$. Set $d(x, K) = \inf_{k \in K} d(x, k)$. Show that $d(x, K) = d(x, y)$ for some $y \in K$.

It follows from (a) that we can actually replace inf by min in the definition of $d(x, K)$ and indeed define $d(x, K) = \min_{k \in K} d(x, k)$. Using a similar argument that you used for (a), one can also show that, if $K, L \subseteq X$ are compact, then $\sup_{x \in K} d(x, L) = d(k, L)$ for some $k \in K$. Therefore, we can again replace sup by max and define $d(K, L) = \max_{x \in K} d(x, L)$.

Let $\mathcal{K}(X)$ denote the set of non-empty compact subsets of X and consider the function $\delta_H : \mathcal{K}(X) \times \mathcal{K}(X) \to [0, \infty)$ given by

$$\delta_H(K, L) = \max\{d(K, L), d(L, K)\}$$

You can easily check (not as a part of this homework) that δ_H is a metric on $\mathcal{K}(X)$. This metric is called the **Hausdorff metric** on $\mathcal{K}(X)$ and the topology τ generated by this metric is called the **Vietoris topology** on $\mathcal{K}(X)$.

For the rest of this question, we endow $\mathcal{K}(X)$ with this topology. Before working on the rest of the question, you should get an intuitive feeling of how δ_H works. In order to do that, as an example, choose $X = \mathbb{R}^2$ and play with this. For example, find some compact sets in the ball $B_{\delta_H}([0, 1]^2, 1)$ and try to compute $\delta_H(S^1, [0, 1]^2)$. After you get a feeling of how δ_H works, proceed.

(b) Prove that if (X, d) is totally bounded, then so is $(\mathcal{K}(X), \delta_H)$.

It turns out that if (X, d) is complete, then so is $(\mathcal{K}(X), \delta_H)$. From this fact together with (b), it follows that if (X, d) is compact, then so is $(\mathcal{K}(X), \delta_H)$. If you wish to see the proofs of these, you can find them in textbooks; just search the phrase "space of compact sets".

(c) Prove that $\{K \in \mathcal{K}(X) : |K| \leq 535\}$ is a closed subset of $\mathcal{K}(X)$.

3. (2 pts) Let X be a topological space and $D \subseteq X$ be a dense subset. Consider the equivalence relation $\sim$ on X given by

$$x \sim y \text{ if and only if } x = y \text{ or } x, y \in D$$

In other words, $\sim$ is the equivalence relation whose quotient set is

$$X/\sim \; = \{[x] : x \in X\} = \{\{x\} : x \in D^c\} \cup \{D\}$$

Endow $X/\sim$ with the quotient topology. Show that no distinct two points in $X/\sim$ can be separated by disjoint open sets.