# MATH 535, Topology, Homework 3

**1. (2 pts)** Let $X$ be a connected metrizable space with at least two elements. Show that $X$ is uncountable.

**Hint.** Let $x_0 \in X$ be fixed and $d$ be a compatible metric on $X$. Consider the function $x \mapsto d(x, x_0)$ from $X$ to $\mathbb{R}$. What properties does this function have? How can its image look like? What can you conclude?

**2. (2 pts)** Consider the Hausdorff topology $\tau$ on the set $\mathbb{N}^+$ of positive natural numbers generated by the basis

$$\mathcal{B} = \{a\mathbb{N} + b : \gcd(a, b) = 1, a, b \in \mathbb{N}^+\}$$

Show that the topological space $(\mathbb{N}^+, \tau)$ is connected.

**Hint.** This space is known as the Golomb space. If you get stuck, then you can Google this term to get help from internet sources e.g. Golomb's own paper. That said, should you decide to use arguments that you read online, you better *completely understand and not just copy.*

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**Moral of the story.** There are countable connected Hausdorff spaces but not metrizable ones with more than one point.

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**3. (2+2+2 pts)**

a) Show that the image of a meager set under a homeomorphism is meager.

For the rest of this question, let $G$ be a topological group.

b) Let $g \in G$ be fixed. Prove that the left multiplication map $\mathrm{m}_g : G \to G$ given by $\mathrm{m}_g(x) = gx$ is a homeomorphism.

c) Suppose that the topology of $G$ is completely metrizable and let $H \leqslant G$ be a meager subgroup. Show that $|G : H|$ is uncountable.