MATH 535, Topology, Homework 2

1. (2+1 pts) Let (X_i, τ_i) be separable topological spaces for all $i \in \mathbb{N}$.

a) Show that $\prod_{i\in\mathbb{N}} X_i$ together with the product topology is separable.

b) Show that $\prod_{i\in\mathbb{N}} \{0,1\}$ together with the box topology where each component has the discrete topology is not separable.

2. (2+1+4 pts)

Recall that the Baire space $\mathbb{N}^{\mathbb{N}}$ is the set of functions from $\mathbb{N}$ to $\mathbb{N}$ whose topology is induced by the metric

$$d(f,g) = \begin{cases} \dfrac{1}{\min\{i \in \mathbb{N} : f(i) \neq g(i)\} + 1} & \text{if } f \neq g \\ 0 & \text{if } f = g \end{cases}$$

It follows from a theorem we proved in class that the Baire space is a complete metric space. We can consider the infinite symmetric group $Sym(\mathbb{N}) \subseteq \mathbb{N}^{\mathbb{N}}$ as a subspace of $\mathbb{N}^{\mathbb{N}}$. In this case, the subspace topology of $Sym(\mathbb{N})$ is induced by the restriction of the metric d to $Sym(\mathbb{N}) \times Sym(\mathbb{N})$.

a) Let $f_n = (0, 1, \ldots, n)$ for each $n \in \mathbb{N}$. Here we use the standard cycle notation, that is, $f_n : \mathbb{N} \to \mathbb{N}$ is the element of $Sym(\mathbb{N})$ which sends 0 to 1, 1 to 2,..., n to 0 and fixes everything else.

Show that the sequence $(f_n)_{n\in\mathbb{N}}$ is a Cauchy sequence which is not convergent in the metric space $(Sym(\mathbb{N}), d)$.

b) Is the subspace $Sym(\mathbb{N})$ completely metrizable?

For the remainder of this question, let us first recall the definition of a **topological group**. A topological group is a group G together with a topology, with respect to which the multiplication operation $(g,h) \mapsto g \cdot h$ from the product space $G \times G$ to G and the inversion operation $g \mapsto g^{-1}$ from G to G are continuous maps. For example, the group $(\mathbb{R}, +)$ is a topological group since the addition map $(x,y) \mapsto x + y$ from $\mathbb{R} \times \mathbb{R}$ to $\mathbb{R}$ is a continuous map and the map $x \mapsto -x$ from $\mathbb{R}$ to $\mathbb{R}$ is also a continuous map. Recall that $Sym(\mathbb{N})$ together with composition of functions as the multiplication forms a group.

c) Prove that $Sym(\mathbb{N})$ is a topological group.
(**Hint.** First try to argue that composition of bijections is continuous as a map from $Sym(\mathbb{N}) \times Sym(\mathbb{N})$ to $\mathbb{N}^{\mathbb{N}}$. Similarly, argue that inversion is continuous as a map from $Sym(\mathbb{N})$ to $\mathbb{N}^{\mathbb{N}}$. Then argue that these maps will stay continuous if we shrink the codomains to $Sym(\mathbb{N})$. In order to do the former, ask yourself this: When is a function from a topological space to a product space continuous? Try to remember the theorem we proved in class regarding this. It may give you a faster solution than trying to use the definition of continuity or the equivalent ϵ-δ condition on the whole product space.)