PLEASE WRITE YOUR NAME CLEARLY USING CAPITAL LETTERS		
F U L L N A M E	S T U D E N T I D	DURATION 150 MINUTES
6 QUESTIONS ON 4 PAGES		TOTAL 100 POINTS

By signing below, I pledge that I will write this examination as my own work and without the assistance of others or the usage of unauthorized material or information. I understand that possession of any kind of electronic device during the exam is prohibited. I also understand that not obeying the rules of the examination will result in immediate cancellation and disciplinary procedures.

Signature ....

(5+5+5 pts) **1.** Let $(\mathbf{X}, \mathcal{M}, \mu)$ be a measure space. Let $(f_n)_{n\in\mathbb{N}}$ be a sequence of functions in $L(\mathbf{X}, \mathcal{M}, \mu)$ and let $f \in L(\mathbf{X}, \mathcal{M}, \mu)$. State the following definitions.

a) $(\mathbf{X}, \mathcal{M}, \mu)$ is a complete measure space iff ...

b) f is a simple function iff ...

c) $f_n \to f$ in measure iff ...

(10 pts) **2.** Consider the completion $(\mathbb{R}\times\mathbb{R}, \overline{\mathcal{B}(\mathbb{R})\otimes\mathcal{B}(\mathbb{R})}, \overline{\mathbf{m}\times\mathbf{m}})$ of the product space $(\mathbb{R}\times\mathbb{R}, \mathcal{B}(\mathbb{R})\otimes\mathcal{B}(\mathbb{R}), \mathbf{m}\times\mathbf{m})$. Show that there exists a set $K \in \overline{\mathcal{B}(\mathbb{R})\otimes\mathcal{B}(\mathbb{R})}$ such that $K_x \in \mathcal{B}(\mathbb{R})$ for every $x \in \mathbb{R}$ and $K_y \notin \mathcal{B}(\mathbb{R})$ for some $y \in \mathbb{R}$.

***(4+6+5+12+12 pts)* 3.** Consider the measure spaces $(\mathbb{N}, \mathcal{P}(\mathbb{N}), \mu)$ and $(\mathbb{N}, \mathcal{P}(\mathbb{N}), \nu)$ where μ is the counting measure and $\nu : \mathcal{P}(\mathbb{N}) \to [0, \infty]$ is the measure given by

$$\nu(S) = \sum_{n \in S} \frac{1}{3^n}$$

a) Show that $(\mathbb{N}, \mathcal{P}(\mathbb{N}), \mu)$ is a σ-finite space.

b) Show that $\mathcal{P}(\mathbb{N}) \otimes \mathcal{P}(\mathbb{N}) = \mathcal{P}(\mathbb{N} \times \mathbb{N})$.

c) Show that every function $f : \mathbb{N} \times \mathbb{N} \to \mathbb{R}$ is measurable, where $\mathbb{N} \times \mathbb{N}$ is endowed with the product σ-algebra $\mathcal{P}(\mathbb{N}) \otimes \mathcal{P}(\mathbb{N})$ and $\mathbb{R}$ is endowed with the σ-algebra $\mathcal{B}(\mathbb{R})$.

d) Compute the integral

$$\int_{\{1,2\} \times \mathbb{N}} f(i, j) \; d(\mu \times \nu)$$

where $f : \mathbb{N} \times \mathbb{N} \to \mathbb{R}$ is given by $f(i, j) = i \cdot 2^j$ by applying Tonelli's theorem. **Make sure to check the conditions of Tonelli's theorem explicitly**.

e) Compute the integral

$$\int_{\{1,2\}\times\mathbb{N}} g(i,j)\ d(\mu\times\nu)$$

where $g:\mathbb{N}\times\mathbb{N}\to\mathbb{R}$ is given by $g(i,j)=(-1)^i\cdot i\cdot(-2)^j$ by applying Fubini's theorem. **Make sure to check the conditions of Fubini's theorem explicitly**.

***(12 pts)* 4.** Let $\mathcal{E}=\{(a,b)-\{\frac{a+b}{2}\}:\ a,b\in\mathbb{R},\ a<b\}$ and let $\mathcal{M}(\mathcal{E})$ denote the σ-algebra on $\mathbb{R}$ generated by $\mathcal{E}$. Show that $\mathcal{B}(\mathbb{R})\subseteq\mathcal{E}$.

***(12 pts)* 5.** Compute $\lim_{n\to\infty} \int_{(\pi,4\pi)} \sin(x) \cdot \arctan\left(\frac{x^n}{n^x}\right) \, d\mathbf{m}$. Briefly justify each step.

***(12 pts)* 6.** Let $\mu : \mathcal{P}(\mathbb{N}) \to [0,\infty]$ and $\nu : \mathcal{P}(\mathbb{N}) \to [0,\infty]$ be the measures on the measurable space $(\mathbb{N}, \mathcal{P}(\mathbb{N}))$ given by

$$\mu(A) = \sum_{n\in A} n^3 \quad \text{and} \quad \nu(A) = \sum_{n\in A} n$$

You are **given** that $\mu \ll \nu$. Find the Radon-Nikodym derivative $\frac{d\mu}{d\nu}$.