PLEASE WRITE YOUR NAME CLEARLY USING CAPITAL LETTERS		
FULL NAME	STUDENT ID	DURATION 120 MINUTES
3 QUESTIONS ON 4 PAGES		TOTAL 120 POINTS

By signing below, I pledge that I will write this examination as my own work and without the assistance of others or the usage of unauthorized material or information. I understand that possession of any kind of electronic device during the exam is prohibited. I also understand that not obeying the rules of the examination will result in immediate cancellation and disciplinary procedures.

Signature ....

(15+15+15+15=60 pts) **1.** Throughout this question, we shall work in the language $\mathcal{L} = \{\cdot, e\}$ of groups where $\cdot$ is a binary function symbol and e is a constant symbol.

a) Let $n \geq 2$ be an integer. Write down a sentence $\varphi_n \in Sent_{\mathcal{L}}$ such that for every $\mathcal{L}$-structure $\mathcal{G} = (G, \cdot^{\mathcal{G}}, e^{\mathcal{G}})$, we have that $\mathcal{G} \models \varphi_n$ if and only if $\mathcal{G}$ is a group and every non-identity element of $\mathcal{G}$ has order n.

b) Write down a theory $\Sigma \subseteq Sent_{\mathcal{L}}$ such that for every $\mathcal{L}$-structure $\mathcal{G} = (G, \cdot^{\mathcal{G}}, e^{\mathcal{G}})$, we have that $\mathcal{G} \models \Sigma$ if and only if $\mathcal{G}$ is a group and any two non-identity element of $\mathcal{G}$ have the same order.

c) Let $T \subseteq Sent_{\mathcal{L}}$ be the theory given by

$$T = \bigcap_{\substack{p\in\mathbb{N} \\ p \text{ is prime}}} Th_{\mathcal{L}}(\mathbb{Z}_p, +, 0) = \bigcap_{\substack{p\in\mathbb{N} \\ p \text{ is prime}}} \{\varphi \in Sent_{\mathcal{L}} : \ (\mathbb{Z}_p, +, 0) \models \varphi\}$$

where the structure $(\mathbb{Z}_p, +, 0)$ is the usual cyclic group of order p. In other words, T is the set of all $\mathcal{L}$-sentences that are true in every finite cyclic group of prime order.

Let $\mathcal{L}' = \mathcal{L} \cup \{c\}$ where c is a new constant symbol. Show that there exists a group $\mathcal{H} = (H, \cdot^{\mathcal{H}}, e^{\mathcal{H}}, c^{\mathcal{H}})$ such that $\mathcal{H} \models T$ and $c^{\mathcal{H}}$ has infinite order in H.

d) Fix some group $\mathcal{H} = (H, \cdot^{\mathcal{H}}, e^{\mathcal{H}}, c^{\mathcal{H}})$ as in part (c) of this question. Show that *every* non-identity element of H is of infinite order.

(15+15+15=45 pts) **2.** Let $\mathcal{L}$ be a language and suppose that we have a proof system for the first-order logic over the language $\mathcal{L}$ which satisfies the **Soundness Theorem**, that is, for every $\varphi \in Form_{\mathcal{L}}$ and $\Sigma \subseteq Form_{\mathcal{L}}$, we have $\Sigma \vdash \varphi$ implies $\Sigma \models \varphi$.

a) Let $\Sigma \subseteq Sent_{\mathcal{L}}$ be a theory. Show that if Σ has a model, then Σ is consistent.

For the rest of this question, consider the language of rings $\mathcal{L} = \{+, \cdot, 0, 1\}$ where $+$ and $\cdot$ are binary function symbols and 0 and 1 are constant symbols. Let $\Sigma_{ring} \subseteq Sent_{\mathcal{L}}$ be the set of ring axioms which the instructor will write down on the blackboard during the exam.

b) Show that $\Sigma_{ring} \nvdash 0 \neq 1$.

c) Show that $\Sigma_{ring} \nvdash \forall x \forall y (x \cdot y = 0 \to (x = 0 \vee y = 0))$.

(8+7=15 pts) **3.** Fix a language $\mathcal{L}$. In this question, you must use the proof system established in Section 2 of our textbook "A Friendly Introduction to Mathematical Logic." The instructor will distribute sheets that contain the axioms and inference rules of this proof system during the exam.

a) Let $\varphi(x) \in Form_{\mathcal{L}}$ be an $\mathcal{L}$-formula with one free variable x. Show that

$$\vdash \ (\forall x\ \varphi(x)) \to (\exists x\ \varphi(x))$$

b) Recall that the deduction theorem states that, for all $\varphi \in Sent_{\mathcal{L}}$, $\psi \in Form_{\mathcal{L}}$ and $\Sigma \subseteq Form_{\mathcal{L}}$ we have that $\Sigma \vdash \varphi \to \psi$ if and only if $\Sigma \cup \{\varphi\} \vdash \psi$. Explain briefly in your own words how you think this theorem (in particular, its right-to-left direction) arises in mathematical practice.