************* PLEASE WRITE YOUR NAME CLEARLY USING CAPITAL LETTERS *************		
F U L L N A M E	S T U D E N T I D	Submission deadline: June 8, 2020 4 questions on 5 pages 60 points in total

1. (5+5+5+5+5 pts) Let $\mathcal{L} = \{\cdot\}$ be the language consisting of a single binary function symbol and let

$$\mathcal{L}' = \mathcal{L} \cup \{p \in \mathbb{N}^+ : p \textit{ is prime}\}$$

where each p is a constant symbol. For this question, consider the $\mathcal{L}$-structures $\mathcal{A} = (\mathbb{N}^+, \cdot^{\mathcal{A}})$ and $\mathcal{B} = (2\mathbb{N}^+, \cdot^{\mathcal{B}})$ where $\cdot$ is interpreted as the usual multiplication in both structures.

a) Prove that $\mathcal{B}$ is not an elementary substructure of $\mathcal{A}$.

We now expand $\mathcal{A}$ into an $\mathcal{L}'$-structure: Consider the $\mathcal{L}'$-structure $\mathcal{N} = (\mathbb{N}^+, \cdot^{\mathcal{N}}, 2^{\mathcal{N}}, 3^{\mathcal{N}}, 5^{\mathcal{N}}, 7^{\mathcal{N}}, \dots)$ where $\cdot^{\mathcal{N}}$ is the usual multiplication and $p^{\mathcal{N}} = p$ for each prime p.

b) Let $\mathcal{M}$ be a substructure of $\mathcal{N}$ such that $1 \in M$. Show that $\mathcal{M} = \mathcal{N}$.
(**Hint.** Some theorems are "fundamental", some are not.)

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c) Prove the structure $\mathcal{A}$ has uncountably many automorphisms.
(**Hint/Fact.** There are uncountably many permutations on the set of prime numbers. Try to construct an automorphism of $\mathcal{A}$ for each permutation of primes.)

d) Let $X \subseteq \mathbb{N}^+$ be a set definable (without parameters) in the structure $\mathcal{A}$ such that $p \in X$ for some prime p. Show that $p \in X$ for all primes p. (**Hint.** There is a reason that you solved part (c) of this question.)

e) Show that the set of prime numbers is definable (without parameters) in the structure $\mathcal{A}$ by explicitly writing down a formula $\varphi(x)$ with one free-variable defining this set.

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2. (5+5 pts) Let $\mathcal{L} = \{+, \cdot, 0, 1\}$ be the language of fields and let p be a fixed prime number. You are given the algebra fact that, for every $n \in \mathbb{N}^+$, there exists a unique finite field $\mathbb{F}_{p^n}$ whose order is p^n. Set

$$T = \bigcap_{n \in \mathbb{N}^+} \mathrm{Th}(\mathbb{F}_{p^n})$$

In other words, T is the set of all $\mathcal{L}$-sentences which are true in every finite field of characteristic p.

a) Show that there exists an infinite model of T. (**Hint.** “Compact” solutions are better than messy ones.)

b) Show that, for every infinite cardinal κ, there exists a model of T whose cardinality is κ.

3. (5 pts) Let $\mathcal{L}$ be a language and let φ be an $\mathcal{L}$-sentence such that any two models of φ are isomorphic. (Recall that you have seen examples of such sentences in Take-Home Exam I.) Show that the theory $T = \{\varphi\}$ is complete.

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4. (5+5+5+5 pts) Let $\mathcal{L} = \{<\}$ be the language of linear orders and consider the theory T consisting of the following sentences:

$$\forall x\ x \not< x$$
$$\forall x \forall y \forall z\ (x < y \wedge y < z) \rightarrow x < z$$
$$\forall x \forall y\ x < y \vee x = y \vee y < x$$
$$\forall x \forall y\ (x < y \rightarrow (\exists z\ x < z \wedge z < y))$$
$$\exists x \forall y\ (x = y \vee x < y)$$
$$\exists x \forall y\ (x = y \vee y < x)$$
$$\exists x \exists y\ x < y$$

That is, T is the theory of dense linear orders with endpoints (with at least 2 elements.)

a) Show that T is $\aleph_0$-categorical. (You can use all theorems that we proved in lectures.)

b) Argue that T has no finite models and conclude that T is complete.

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c) Set $T' = T - \{\exists x \exists y \; x < y\}$. Show that T' is not complete.

Warning. Be careful here. In general, just because we removed a sentence from a complete theory does not make the resulting theory incomplete; because the removed sentence may already be provable from the remaining theory. **Hint.** Remember how you solved Question 3.b from Take-Home Exam I. If you want to show that a suitable sentence is not proved or refuted by a given theory, simply find two models of that theory where that sentence is true and false respectively.

d) Recall that DLO is the $\mathcal{L}$-theory of dense linear orders without endpoints. Show that DLO is not $|\mathbb{R}|$-categorical.