************* **PLEASE WRITE YOUR NAME CLEARLY USING CAPITAL LETTERS** *************		
F U L L N A M E	S T U D E N T I D	Submission deadline: April 20, 2020 4 questions on 5 pages 60 points in total

1. (7+7 pts) Let $\mathcal{L} = \{\cdot, e\}$ be the language (of group theory) where $\cdot$ is a binary function symbol and e is a constant symbol.

a) Prove that for every integer $n \geq 2$, there exists an $\mathcal{L}$-sentence φ_n such that, for every $\mathcal{L}$-structure $\mathfrak{A} = (A, \cdot^{\mathfrak{A}}, e^{\mathfrak{A}})$ we have that $\mathfrak{A} \models \varphi_n$ if and only if $|A| = n$.

b) Prove that there exists an $\mathcal{L}$-sentence φ such that every $\mathcal{L}$-structure $\mathfrak{A}$ with $\mathfrak{A} \models \varphi$ is isomorphic to $(\mathbb{Z}_4, +, 0)$.
Hint. You can freely use the basic algebra fact that every group of order 4 is isomorphic to either $\mathbb{Z}_4$ or to $\mathbb{Z}_2 \times \mathbb{Z}_2$.

************* PLEASE WRITE YOUR NAME CLEARLY USING CAPITAL LETTERS *************		
F U L L N A M E	S T U D E N T I D	Submission deadline: April 20, 2020 4 questions on 5 pages 60 points in total

2. (7+7 pts) In this question, you must use the proof system established in Section 2 of our textbook "A Friendly Introduction to Mathematical Logic." Let $\mathcal{L}$ be a language.

a) Show that for every formula $\varphi(x)$ with one free variable x, we have

$$\vdash (\exists x\ \neg\varphi) \longrightarrow \neg(\forall x\ \varphi) \quad \text{and} \quad \vdash \neg(\forall x\ \varphi) \longrightarrow (\exists x\ \neg\varphi)$$

by explicitly constructing two proofs. Justify each step of each proof by referring to the relevant inference rules and axioms.

b) Let $\Sigma \subseteq Form_{\mathcal{L}}$ and $\varphi \in Sent_{\mathcal{L}}$. Prove that if $\Sigma \nvdash \neg\varphi$, then $\Sigma \cup \{\varphi\}$ is consistent.
Hint. The deduction theorem is your friend!

************* PLEASE WRITE YOUR NAME CLEARLY USING CAPITAL LETTERS *************		
F U L L N A M E	S T U D E N T I D	Submission deadline: April 20, 2020 4 questions on 5 pages 60 points in total

3. (7+7 pts) In this question, you must use the proof system established in Section 2 of our textbook "A Friendly Introduction to Mathematical Logic.". Let $\mathcal{L}$ be a language.

(In part a, you are going to show an implication between two versions of Gödel's completeness theorem, so you should only assume what is given to you, not the form of Gödel's completeness theorem that we proved in class.)

a) Let $\Sigma \subseteq Form_{\mathcal{L}}$. Assume that, for every formula $\varphi \in Form_{\mathcal{L}}$, we have $\Sigma \models \varphi$ implies $\Sigma \vdash \varphi$. Prove that if Σ is consistent, then Σ has a model.

b) $\mathcal{L} = \{\cdot, e\}$ be the language (of group theory) where $\cdot$ is a binary function symbol and e is a constant symbol. Let $\Sigma = \left\{ \forall x \forall y \forall z \ (x \cdot y) \cdot z = x \cdot (y \cdot z) \ \ , \ \ \forall x \ x \cdot e = x \ \ , \ \ \forall x \forall y \ x \cdot y = y \cdot x \right\}$. Show that

$$\Sigma \nvdash \forall x \exists y \ x \cdot y = e$$

Hint. Recall that $\Sigma \cup \{\neg\varphi\}$ being consistent implies that $\Sigma \nvdash \varphi$. Now, what can you tell about Σ and $\forall x \exists y \ x \cdot y = e$ if there exists a model $\mathfrak{A}$ of Σ such that $\mathfrak{A} \models \neg(\forall x \exists y \ x \cdot y = e)$?

************* PLEASE WRITE YOUR NAME CLEARLY USING CAPITAL LETTERS *************		
F U L L N A M E	S T U D E N T I D	Submission deadline: April 20, 2020 4 questions on 5 pages 60 points in total

4. (7+4+7 pts) Let $\mathcal{L} = \{+, \cdot, 0, 1, <\}$ be the language arithmetic where $+, \cdot$ are binary function symbols, $0, 1$ are constant symbols and $<$ is a binary relation symbol. Consider the structure $\mathfrak{N} = (\mathbb{N}, +, \cdot, 0, 1, <)$ where the symbols in the language are interpreted in the usual way. Set $T = Th(\mathfrak{N})$.

a) Let $\mathcal{L}' = \mathcal{L} \cup \{c\}$ where c is a constant symbol. Show that there exists an $\mathcal{L}'$-structure $\mathfrak{A} = (A, +^{\mathfrak{A}}, \cdot^{\mathfrak{A}}, 0^{\mathfrak{A}}, 1^{\mathfrak{A}}, <^{\mathfrak{A}}, c^{\mathfrak{A}})$ such that $\mathfrak{A} \models T$ and there exists an element $c \in A$ such that for every positive integer n,

$$\mathfrak{A} \models \underbrace{1 + 1 + \cdots + 1}_{n \ times} < c$$

b) Let $\mathfrak{A} = (A, +^{\mathfrak{A}}, \cdot^{\mathfrak{A}}, 0^{\mathfrak{A}}, 1^{\mathfrak{A}}, <^{\mathfrak{A}}, c^{\mathfrak{A}})$ be as in part a. Is it true that $\mathfrak{A} \models \forall x (x \neq 0 \rightarrow \exists y \ \ x = y + 1)$?

************* PLEASE WRITE YOUR NAME CLEARLY USING CAPITAL LETTERS *************		
FULL NAME	STUDENT ID	Submission deadline: April 20, 2020 4 questions on 5 pages 60 points in total

c) Let $\mathfrak{A} = (A, +^{\mathfrak{A}}, \cdot^{\mathfrak{A}}, 0^{\mathfrak{A}}, 1^{\mathfrak{A}}, <^{\mathfrak{A}}, c^{\mathfrak{A}})$ as in part a. Prove that there exists a map $f : \mathbb{Z} \to A$ such that

$$x < y \text{ if and only if } f(x) <^{\mathfrak{A}} f(y)$$

for all $x, y \in \mathbb{Z}$.