METU Department of Mathematics, Math 406, Final Take-Home, June 13, 2020

************* PLEASE WRITE YOUR NAME CLEARLY USING CAPITAL LETTERS *************		
F U L L N A M E	S T U D E N T I D	Submission deadline: July 2, 2020 5 questions on 5 pages 100 points in total

1. (8 × 5 pts) Determine whether the following statements are true or false. Explain your reasoning **very briefly**.

a) **True or False:** There exist first-order theories whose models are all finite.

b) **True or False:** There exist first-order theories whose models are all countably infinite.

For the next question only, consider the language of group theory $\mathcal{L} = \{\cdot, e, ^{-1}\}$ and let Σ be the group axioms.
c) **True or False:** $\Sigma \vdash \forall x((\forall y\ x \cdot y = y \cdot x = y) \rightarrow x = e)$. In other words, that the identity element is unique can be proven from Σ.

d) **True or False:** Any two algebraically closed fields of characteristic 0 are elementarily equivalent.

e) **True or False:** There are structures which have proper substructures but which have no proper elementary substructures.

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2. (10 pts) Recall that we have shown that the theory DLO has quantifier elimination. Let $\varphi(x, y, z)$ be the following formula in the language of linear orders.

$$\varphi(x, y, z): \quad \forall w(w < x \rightarrow w < z) \ \wedge \ \exists u(x < u \wedge u < z) \ \wedge \ \forall t(t < x \leftrightarrow t < y)$$

Find a quantifier free formula $\psi(x, y, z)$ in the language of linear orders such that

$$DLO \vdash \forall x \forall y \forall z(\varphi(x, y, z) \leftrightarrow \psi(x, y, z))$$

Briefly explain your reasoning. (But you do not need to give a formal proof of this sentence from DLO.)

3. (10 pts) Let $\mathcal{L} = \{\cdot\}$ be the language consisting of a single binary function symbol. Consider the theory of semigroups given by

$$T = \{\forall x \forall y \forall z \ (x \cdot y) \cdot z = x \cdot (y \cdot z)\}$$

Show that T is not model-complete and hence, does not admit quantifier elimination.
(**Hint.** Recall what you proved in Q.1.a of Take-Home Exam II.)

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4. (10+10 pts) Let $\mathcal{U}$ be a non-principal ultrafilter on $\mathbb{N}^+$ and consider the ultraproduct

$$\mathcal{Z} = \prod_{n \in \mathbb{N}^+} (\mathbb{Z}_n, +, 0) \;/\; \mathcal{U}$$

which is a group by Łoś's theorem.

a) Show that the order of the element $\overline{(1,1,1,\dots)}$ in the group $\mathcal{Z}$ is infinite.

b) Show that there does not exist a sentence φ in the language $\mathcal{L}$ of group theory such that for every $\mathcal{L}$-structure $\mathcal{A}$, we have that $\mathcal{A} \models \varphi$ if and only if $\mathcal{A}$ is a cyclic group. In other words, show that there is no sentence in the language of group theory which holds exactly for cyclic groups. (**Hint.** You can freely use the fact that the group $\mathcal{Z}$ is uncountable.)

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5. (10+10 pts) Let $\mathcal{U}$ be a non-principal ultrafilter on $\mathbb{N}$ and consider the ultrapower

$$^*\mathcal{R} = \prod_{n\in\mathbb{N}} (\mathbb{R}, +, \cdot, 0, 1, <) \ / \ \mathcal{U}$$

Let $^*\mathbb{R}$, $\mathbf{+}$, $\bullet$, $\mathbf{0}$, $\mathbf{1}$ and $\prec$ denote the universe and the interpretations of the symbols in the structure $^*\mathcal{R}$ respectively.

a) Consider the following imaginary dialogue between Socrates and Glaucon.

...
Socrates: Is $(\mathbb{R}, +, \cdot, 0, 1, <)$ not an ordered field?
Glaucon: Yes, Socrates, it is.
Socrates: Does it then not follow that its ultrapower $^*\mathcal{R}$ is an ordered field as well?
Glaucon: It certainly does. We have established this earlier by Łoś's theorem.
Socrates: Indeed. Now that this is settled, Glaucon, do you agree that, by the ultraproduct construction, we have

$$\overline{(1, 0, 3, 0, 5, \dots)} \bullet \overline{(0, 2, 0, 4, 0, \dots)} = \overline{(0, 0, 0, 0, 0, \dots)} = \mathbf{0}?$$

Glaucon: Clearly.
Socrates: The factors in this product seem very different than $\mathbf{0}$, do they not?
Glaucon: That is not to be denied.
Socrates: But then, Glaucon, does $^*\mathcal{R}$ have a zero divisor contrary to what you just admitted, or, is it that things are not always what they seem to be?

Help Glaucon answer Socrates. More precisely, explain why the factors in the product above are not zero divisors.

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b) Find a sequence $(\mathbf{r_n})_{n\in\mathbb{N}}$ of elements of ${}^*\mathbb{R}$ such that $\mathbf{r_i} + \underbrace{\mathbf{1} + \ldots + \mathbf{1}}_{n\ times} \prec \mathbf{r_{i+1}}$ for all $i \in \mathbb{N}$ and $n \in \mathbb{N}$. In other words,

$$\mathbf{r_0} \prec \mathbf{r_0} + \mathbf{1} \prec \mathbf{r_0} + \mathbf{1} + \mathbf{1} \prec \cdots \prec \mathbf{r_1} \prec \mathbf{r_1} + \mathbf{1} \prec \mathbf{r_1} + \mathbf{1} + \mathbf{1} \prec \cdots \prec \mathbf{r_2} \prec \mathbf{r_2} + \mathbf{1} \prec \mathbf{r_2} + \mathbf{1} + \mathbf{1} \prec \ldots$$

It suffices for you to list the elements $\mathbf{r_n}$ and briefly explain why the inequalities hold without providing a complete proof in detail.