************* PLEASE WRITE YOUR NAME CLEARLY USING CAPITAL LETTERS *************		
F U L L N A M E	S T U D E N T I D	6 questions on 4 pages, 120 minutes 100 points in total Justify your answer

1. (7 pts) Prove or disprove the following: For any set x, if x is transitive, then $\bigcup x$ is transitive.

Let x be a set. Suppose that x is transitive. We shall show that $\bigcup x$ is transitive, that is, for every $y \in \bigcup x$, we have $y \subseteq \bigcup x$ Let $y \in \bigcup x$. Then, by definition, $y \in z$ for some $z \in x$. Since x is transitive, $z \subseteq x$ and hence $y \in x$. But then, for any $w \in y$, since $w \in y$ and $y \in x$, we have $w \in \bigcup x$. Hence $y \subseteq \bigcup x$. Thus $\bigcup x$ is transitive.

2. (10 pts) Recall that the recursive definitions of addition and multiplication operations $+$ and $\cdot$ on the set of natural numbers $\mathbb{N}$ are given as follows:

$$\begin{aligned} m+0 &= m \\ m+S(n) &= S(m+n) \end{aligned} \quad \text{and} \quad \begin{aligned} m\cdot 0 &= 0 \\ m\cdot S(n) &= (m\cdot n)+m \end{aligned}$$

for all $m, n \in \mathbb{N}$, where $S(n)$ denotes the successor of the natural number n. You are given that the identity

$$m\cdot(n+p) = m\cdot n + m\cdot p$$

hold for all $m, n, p \in \mathbb{N}$. Prove that $\cdot$ is associative, that is, for all $m, n, p \in \mathbb{N}$, we have

$$(m\cdot n)\cdot p = m\cdot(n\cdot p)$$

[**WARNING:** If you use an identity involving arithmetical operations on $\mathbb{N}$ other than the identities given in the question, you are supposed to prove it.]

Let $m, n \in \mathbb{N}$. We shall prove that $(m\cdot n)\cdot p = m\cdot(n\cdot p)$ for all $p \in \mathbb{N}$ by induction on p.

- **Base case.** By the definition of $\cdot$, we have $(m\cdot n)\cdot 0 = 0 = m\cdot 0 = m\cdot(n\cdot 0)$ and hence the claim holds for $p = 0$.
- **Successor step.** Let $p \in \mathbb{N}$. Suppose that $(m\cdot n)\cdot p = m\cdot(n\cdot p)$. Then, by the definition of $\cdot$, the inductive assumption and the given identity, we have

$$\begin{aligned} (m\cdot n)\cdot S(p) &= ((m\cdot n)\cdot p) + (m\cdot n) \\ &= (m\cdot(n\cdot p)) + (m\cdot n) \\ &= m\cdot((n\cdot p)) + n) \\ &= m\cdot(n\cdot S(p)) \end{aligned}$$

  It follows that the claim holds for $S(p)$. Hence, by the principle of mathematical induction, the claim holds for all $p \in \mathbb{N}$.

3. $(5 \times 7 = 35$ pts) Consider the relation $\preccurlyeq$ defined on $\mathbb{N} \times \mathbb{Z}$ as follows.

$$(m,n) \preccurlyeq (k,\ell) \longleftrightarrow m+n < k+\ell \ \vee \ (m+n = k+\ell \ \wedge \ m \leq k)$$

for all $(m,n),(k,\ell) \in \mathbb{N} \times \mathbb{Z}$.
(a) Show that the relation $\preccurlyeq$ is transitive.

Let $(m,n),(k,\ell),(p,r) \in \mathbb{N} \times \mathbb{Z}$. Suppose that $(m,n) \preccurlyeq (k,\ell)$ and $(k,\ell) \preccurlyeq (p,r)$. By the definition of $\preccurlyeq$, we must have $m+n < k+\ell \ \vee \ (m+n = k+\ell \wedge m \leq k)$ and $k+\ell < p+r \ \vee \ (k+\ell = p+r \wedge k \leq p)$. We now split into four cases.

- Case I: $(m+n < k+\ell$ and $k+\ell < p+r)$ Then $m+n < p+r$ and hence $(m,n) \preccurlyeq (p,r)$.
- Case II: $(m+n < k+\ell$ and $k+\ell = p+r \ \wedge \ k \leq p)$ Then $m+n < k+\ell = p+r$ and hence $(m,n) \preccurlyeq (p,r)$.
- Case III: $(m+n = k+\ell \ \wedge \ m \leq k$ and $k+\ell < p+r)$ Then $m+n = k+\ell < p+r$ and hence $(m,n) \preccurlyeq (p,r)$.
- Case IV: $(m+n = k+\ell \ \wedge \ m \leq k$ and $k+\ell = p+r \ \wedge \ k \leq p)$ Then $m+n = k+\ell = p+r$ and $m \leq p$, and hence $(m,n) \preccurlyeq (p,r)$.

It follows that $\preccurlyeq$ is a transitive relation.

You are now given that $\preccurlyeq$ is a partial order relation on $\mathbb{N} \times \mathbb{Z}$.
(b) Show that $\preccurlyeq$ is a linear order relation on $\mathbb{N} \times \mathbb{Z}$.

Let $(m,n),(k,\ell \in \mathbb{N} \times \mathbb{Z}$. Since $\leq$ is a linear order relation on $\mathbb{Z}$, we may split into three cases:

- Case I: $(m+n < k+\ell)$ Then, by the definition of $\preccurlyeq$, we have $(m,n) \preccurlyeq (k,\ell)$.
- Case II: $(m+n > k+\ell)$ Then, by the definition of $\preccurlyeq$, we have $(k,\ell) \preccurlyeq (m,n)$.
- Case III: $(m+n = k+\ell)$ By the exact same reasoning as above, we may split into two further cases
  - Case III.a: $(m \leq k)$ Then, by the definition of $\preccurlyeq$, we have $(m,n) \preccurlyeq (k,\ell)$.
  - Case III.b: $(k < m)$ Then, by the definition of $\preccurlyeq$, we have $(k,\ell) \preccurlyeq (m,n)$.

In all cases, we obtained $(m,n) \preccurlyeq (k,\ell)$ or $(k,\ell) \preccurlyeq (m,n)$, and hence $\preccurlyeq$ is a linear order relation.

(c) Show that $\preccurlyeq$ is not a well order relation on $\mathbb{N} \times \mathbb{Z}$.

Observe that the sequence $((0,-n))_{n\in\mathbb{N}}$ form an infinite strictly decreasing sequence, namely,

$$(0,0) \succ (0,-1) \succ (0,-2) \succ \dots$$

in the linearly ordered set $(\mathbb{N} \times \mathbb{Z}, \preccurlyeq)$. Therefore $\preccurlyeq$ is not a well-order relation.

(d) Let $\prec$ denote the induced strict partial order relation given by $(m,n) \prec (k,\ell) \leftrightarrow (m,n) \preccurlyeq (k,\ell) \ \wedge \ (m,n) \neq (k,\ell)$ for all $(m,n),(k,\ell) \in \mathbb{N} \times \mathbb{Z}$. Find an element $(a,b) \in \mathbb{N} \times \mathbb{Z}$ such that

$$\text{For all } (k,l) \in \mathbb{N} \times \mathbb{Z}, \text{ if } (a,b) \prec (k,\ell), \text{ then there exists } (m,n) \in \mathbb{N} \times \mathbb{Z} \text{ such that } (a,b) \prec (m,n) \prec (k,l)$$

In other words, (a,b) does not have a successor element with respect to this partial order relation.

THIS PART OF THE QUESTION IS CANCELED DUE TO A TYPO. IN THIS PART OF THE QUESTION, ALL $\preccurlyeq$ SHOULD HAVE BEEN $\succcurlyeq$ AND ALL $\prec$ SHOULD HAVE BEEN $\succ$. SO IT SHOULD HAVE ASKED FOR AN ELEMENT THAT DOES NOT HAVE A PREDECESSOR. CONSEQUENTLY, ALL STUDENTS WILL RECEIVE FULL POINTS FROM THIS QUESTION.

(e) Prove that there exists an order preserving function from $(\mathbb{N}\times\mathbb{Z}, \preccurlyeq)$ to $(\mathbb{R}, \leq)$, that is, there exists a function $f : \mathbb{N}\times\mathbb{Z} \to \mathbb{R}$ such that $(m, n) \preccurlyeq (k, \ell)$ iff $f(m, n) \leq f(k, \ell)$, for all $(m, n), (k, \ell) \in \mathbb{N}\times\mathbb{Z}$.

Consider the function $f : \mathbb{N}\times\mathbb{Z} \to \mathbb{R}$ given by

$$f(m, n) = m + n - \frac{1}{m + 1}$$

for all $m \in \mathbb{N}$ and $n \in \mathbb{Z}$. We shall show that f is order-preserving. Recall that, since $\preccurlyeq$ is a linear order relation, it suffices to show that $(m, n) \preccurlyeq (k, \ell)$ implies $f(m, n) \leq f(k, \ell)$ for all $(m, n), (k, \ell) \in \mathbb{N}\times\mathbb{Z}$.

Let $(m, n), (k, \ell) \in \mathbb{N}\times\mathbb{Z}$. Suppose $(m, n) \preccurlyeq (k, \ell)$. Then, by definition, $m + n < k + \ell$ or $m + n = k + \ell \ \wedge \ m \leq k$. We split into cases.

- Case I: $(m + n < k + \ell)$. Then we have

$$f(m, n) = m + n - \frac{1}{m + 1} \leq m + n \leq k + \ell - 1 \leq k + \ell - \frac{1}{k + 1} = f(k, \ell)$$

- Case II: $(m + n = k + \ell \ \wedge \ m \leq k)$ Then we have

$$f(m, n) = m + n - \frac{1}{m + 1} = k + \ell - \frac{1}{m + 1} \leq k + \ell - \frac{1}{k + 1} = f(k, \ell)$$

In both cases, we obtain $f(m, n) \leq f(k, \ell)$. Therefore f is order-preserving.

4. (4 × 7 = 28 pts) Recall that ${}^{\mathbb{Z}}2$ is the set of functions from $\mathbb{Z}$ to 2. In other words, ${}^{\mathbb{Z}}2$ is the set of 0-1 sequences indexed by $\mathbb{Z}$. Consider the relation E on ${}^{\mathbb{Z}}2$ given by

$$f \ E \ g \longleftrightarrow \exists k \in \mathbb{Z} \ \forall n \in \mathbb{Z} \ f(n + k) = g(n)$$

for all $f, g \in {}^{\mathbb{Z}}2$.

(a) Show that the relation E is an equivalence relation on ${}^{\mathbb{Z}}2$.

Let $f \in {}^{\mathbb{Z}}2$. Then, choosing $k = 0$, we have $f(n + k) = f(n)$ for all $n \in \mathbb{Z}$, and hence $f \ E \ f$. Thus E is reflexive.

Let $f, g \in {}^{\mathbb{Z}}2$. Assume that $f \ E \ g$. Then there exists $k \in \mathbb{Z}$ such that $f(n + k) = g(n)$ for all $n \in \mathbb{Z}$. But this implies that $g(n + K) = f(n)$ for all $n \in \mathbb{Z}$ where $K = -k \in \mathbb{Z}$. Hence $g \ E \ f$. Thus E is symmetric.

Let $f, g, h \in {}^{\mathbb{Z}}2$. Assume that $f \ E \ g$ and $g \ E \ h$. Then, by definition, there exist $k, \ell \in \mathbb{Z}$ such that $f(n + k) = g(n)$ and $g(n + \ell) = h(n)$ for all $n \in \mathbb{Z}$. It follows that $f(n + K) = h(n)$ for all $n \in \mathbb{Z}$ where $K = k + \ell \in \mathbb{Z}$. Hence $f \ E \ h$. Thus E is transitive.

(b) Show that the equivalence class $[f]_E$ is countable for all $f \in {}^{\mathbb{Z}}2$.

Observe that, for all $g \in [f]_E$, since $f \ E \ g$, we can choose some $k_g \in \mathbb{Z}$ such that $f(n + k_g) = g(n)$ for all $n \in \mathbb{Z}$. Consider the function $\varphi : [f]_E \to \mathbb{Z}$ given by $\varphi(g) = k_g$ for all $g \in [f]_E$.

Observe that $ran(\varphi) \subseteq \mathbb{Z}$ is countable since it is a subset of a countable set. We claim that φ is injective. Let $g, h \in [f]_E$. Suppose that $\varphi(g) = \varphi(h)$. Then, for all $n \in \mathbb{Z}$, we must have

$$g(n) = f(n + k_g) = f(n + k_h) = h(n)$$

and hence $g = h$. This shows that φ is injective from which it follows that $|[f]_E| = |ran(\varphi)|$. But since the latter set is countable, we have that $[f]_E$ is countable as well.

(c) Show that there exists an injection from ${}^{\mathbb{Z}}2/E$ to ${}^{\mathbb{Z}}2$.

Since each equivalence class is non-empty, by the axiom of choice, for each $S \in {}^{\mathbb{Z}}2/E$, we can choose some $f_S \in S$. Consider the function $\psi : {}^{\mathbb{Z}}2/E \to {}^{\mathbb{Z}}2$ given by $\psi(S) = f_S$ for all $S \in {}^{\mathbb{Z}}2/E$. Then ψ is an injection because if $f_S = \psi(S) = \psi(S') = f_{S'}$, then $f_S \in S \cap S'$ and hence $S = S'$ since two equivalence classes are either identical or disjoint.

(d) Show that there exists an injection from ${}^{\mathbb{Z}}2$ to ${}^{\mathbb{Z}}2/E$.

Consider the map $\gamma : {}^{\mathbb{Z}}2 \to {}^{\mathbb{Z}}2$ given by $\gamma(f)(n) = \begin{cases} 0 & \text{if } n < 0 \\ 1 & \text{if } n = 0 \\ f\left(-\frac{n}{2}\right) & \text{if } n > 0 \text{ is even} \\ f\left(\frac{n-1}{2}\right) & \text{if } n > 0 \text{ is odd} \end{cases}$. In other words, considered as a sequence, γ is the sequence

$$\gamma(f) = (\dots, 0, 0, 0, 1, f(0), f(-1), f(1), f(-2), f(2), \dots)$$

indexed by $\mathbb{Z}$, where the leftmost entry 1 is at index 0.

We claim that, for all $f, g \in {}^{\mathbb{Z}}2$, if $\gamma(f)\ E\ \gamma(g)$, then $f = g$. Let $f, g \in {}^{\mathbb{Z}}2$. Suppose $\gamma(f)\ E\ \gamma(g)$. Then there exists $k \in \mathbb{Z}$ such that $\gamma(f)(n + k) = \gamma(g)(n)$ for all $n \in \mathbb{Z}$. Observe that the leftmost entry 1 appears at index -1 in both sequences $\gamma(f)$ and $\gamma(g)$. Consequently, we must have $k = 0$. Otherwise, if $k > 0$, then $\gamma(g)(-k) = \gamma(f)(0) = 1$ and so $\gamma(g)$ would have entry 1 at index $-k$; and if $k < 0$, then $\gamma(f)(k) = \gamma(g)(0) = 1$ and $\gamma(f)$ would have entry 1 at index k. But then, since $k = 0$, we have $\gamma(f) = \gamma(g)$ from which it follows that $f(n) = g(n)$ for all $n \in \mathbb{Z}$. Hence $f = g$.

Now, consider $\eta : {}^{\mathbb{Z}}2 \to {}^{\mathbb{Z}}2/E$ given by $\eta(f) = [\gamma(f)]_E$. By the claim above, we have that $f \neq g$ implies $\neg\ \gamma(f)\ E\ \gamma(g)$ which subsequently implies $\eta(f) \neq \eta(g)$ and hence, η is an injection.

5. (5+5=10 pts) a) State the Axiom of Pairing or the Axiom of Union.

Read the lecture notes for statements of both axioms.

b) Let x be a set. By referring to the relevant axioms ZFC, explain why its successor $S(x) = x \cup \{x\}$ exists, i.e. explain how you construct $S(x)$ using the relevant axioms of ZFC.

Given a set x, by the Axiom of Pairing, we can pair x with itself and obtain the singleton $\{x\}$. Now, applying the Axiom of Pairing with sets x and $\{x\}$, we obtain the set $\{x, \{x\}\}$. Finally, applying the Axiom of Union to the set $\{x, \{x\}\}$, we obtain a set that consists of precisely the elements of elements of $\{x, \{x\}\}$, that is, $S(x) = x \cup \{x\}$.

6. (10 pts) State and prove Cantor's theorem.

Read the lecture notes for a statement and proof of Cantor's theorem.