************* PLEASE WRITE YOUR NAME CLEARLY USING CAPITAL LETTERS *************		
F U L L N A M E	S T U D E N T I D	Submission deadline: June 21, 2021 4 questions on 6 pages 80 points in total

You can use all the identities and facts regarding the ordinal arithmetic listed in lecture notes and the book, unless the question itself is one of those identities, in which case you should simply prove it via the appropriate technique.

1. (4×6=24 pts) Some parts of this question are independent.

a) Using transfinite induction on γ, prove that for all ordinals α, β, γ if $\alpha \leq \beta$, then $\alpha + \gamma \leq \beta + \gamma$.

b) Using transfinite induction on β, prove that for all ordinals α, β, γ if $\gamma > 0$ and $\alpha < \beta$, then $\gamma \cdot \alpha < \gamma \cdot \beta$.

c) Let α and β be ordinals. Let $\alpha \sqcup \beta$ denote the set $(\alpha \times \{0\}) \cup (\beta \times \{1\})$ which is sometimes called the **disjoint union** of α and β. Consider the relation $\prec$ on the set $\alpha \sqcup \beta$ given by

$$(\gamma, i) \prec (\delta, j) \quad \text{if and only if} \quad i < j \ \vee \ (i = j \ \wedge \ \gamma < \delta)$$

You are **given** that $\prec$ is a strict well-order relation. Show that the order type of the strictly well-ordered set $(\alpha \sqcup \beta, \prec)$ is $\alpha + \beta$ by explicitly finding an order-isomorphism.

d) Let α and β be ordinals. Consider the relation $\prec$ on the set $\alpha \times \beta$ given by

$$(\gamma, \delta) \prec (\gamma', \delta') \quad \text{if and only if} \quad \delta < \delta' \ \vee \ (\delta = \delta' \ \wedge \ \gamma < \gamma')$$

You are **given** that $\prec$ is a strict well-order relation. Show that the order type of the strictly well-ordered set $(\alpha \times \beta, \prec)$ is $\alpha \cdot \beta$ by explicitly finding an order-isomorphism.

What should we learn from this question? Transfinite induction is a fundamental technique that is not only used in set theory but also in other areas of mathematics that often employ transfinite constructions. You should know how to prove certain statements via transfinite induction. Although we defined ordinal addition and multiplication via transfinite recursion in class, these operations also admit "geometric" definitions in the following sense: $\alpha+\beta$ is the order type of the well-ordered set obtained from "attaching β to the end of α" and $\alpha \cdot \beta$ is the order type of the well-ordered set obtained by "attaching β many ordered copies of α back to back". In this question, you prove these facts. Ordinal exponentiation also admits such an equivalent non-recursive definition. You can check out the relevant section in your book if you are curious where some of these theorems are proven.

2. (6×6=36 pts) An ordinal α is said to be a **left divisor** of an ordinal γ if there exists an ordinal β such that $\gamma = \alpha \cdot \beta$. We write $\alpha \mid_\ell \gamma$ to denote that α is a left divisor of γ. Given ordinals α, β and δ, we say that δ is the **greatest common left divisor** of α and β if

- $\delta \mid_\ell \alpha$ and $\delta \mid_\ell \beta$, and
- For every ordinal η with $\eta \mid_\ell \alpha$ and $\eta \mid_\ell \beta$, we have $\eta \mid_\ell \delta$.

a) Let α, β, γ and δ be ordinals such that $\gamma = \alpha + \beta$. Prove that if $\delta \mid_\ell \gamma$ and $\delta \mid_\ell \alpha$, then $\delta \mid_\ell \beta$.

b) Recall that the division theorem for ordinals states that, for every ordinal α, β with $\beta > 0$, there exist unique ordinals η, ρ with $\rho < \beta$ such that

$$\alpha = \beta \cdot \eta + \rho$$

Here β, η and ρ are called the divisor, the quotient and the remainder respectively. Given non-zero ordinals α and β, let us apply the division theorem in an iterative manner as follows.

$$\begin{aligned} \alpha &= \beta \cdot \eta_0 + \rho_0 \\ \beta &= \rho_0 \cdot \eta_1 + \rho_1 \\ \rho_0 &= \rho_1 \cdot \eta_2 + \rho_2 \\ &\ldots \end{aligned}$$

Briefly explain why this process always terminates with 0 remainder after finitely many steps, i.e. there exists $k \in \mathbb{N}$ with $\rho_k = 0$.

c) Let α and β be non-zero ordinals. Prove that the divisor of the last step where we obtain a zero remainder in the process in Part (b) is the greatest common left divisor of α and β. (In other words, the greatest common left divisor is equal to β if $\rho_0 = 0$ and is equal to ρ_k if $\rho_{k+1} = 0$.)

d) Find the greatest common left divisor of $\omega^{\omega^3} + \omega^\omega$ and $\omega^{\omega^2+\omega} + \omega^{\omega+1}$ using Part (b) and Part (c).

An ordinal $\gamma > 1$ is said to be **prime** if there are no ordinals $\alpha, \beta < \gamma$ such that $\gamma = \alpha \cdot \beta$. For example ω is prime because product of any two ordinals less than ω is less than ω.

e) Let $0 < \alpha$ and $1 < k < \omega$ be ordinals. Prove that $\omega \cdot \alpha + k$ is not prime.

f) Prove that $\omega^2 + 1$ is prime.

Why did we solve this question? Ordinal numbers generalize natural numbers and the concept of "counting objects in order." Consequently, we were able to generalize the basic arithmetic operations on natural numbers, which may be essentially seen as counting procedures applied back to back, to the class of ordinals. While some properties of arithmetic on natural numbers, such as commutativity, do not generalize to ordinal arithmetic, some properties do generalize and this allows one to develop some basic "number theory" of ordinal arithmetic. One of the motivations of this question is to let you know that there is a whole world of transfinite numbers which is interesting on its own.

3. (6+6 pts) Let $B \subseteq {}^{\mathbb{R}}\mathbb{R}$ be the smallest set of functions that contain all continuous functions from $\mathbb{R}$ to $\mathbb{R}$ and is closed under taking pointwise limits of sequences of functions. More precisely, let $B \subseteq {}^{\mathbb{R}}\mathbb{R}$ be such that

a. $C(\mathbb{R}, \mathbb{R}) = \{f \in {}^{\mathbb{R}}\mathbb{R} : f \text{ is continuous}\} \subseteq B$,

b. If $\{f_n : n \in \mathbb{N}\} \subseteq B$, $f \in {}^{\mathbb{R}}\mathbb{R}$ and $f(x) = \lim_{n\to\infty} f_n(x)$ for all $x \in \mathbb{R}$, then $f \in B$,

c. If $\widehat{B} \subseteq {}^{\mathbb{R}}\mathbb{R}$ satisfies a. and b. when B is replaced by $\widehat{B}$, then $B \subseteq \widehat{B}$.

Our aim in this question is to give a stratification of the set B with respect to "when a function in B is born". Let us introduce some notation first. Given $K \subseteq {}^{\mathbb{R}}\mathbb{R}$, set

$$\mathbf{PL}(K) = \{f \in {}^{\mathbb{R}}\mathbb{R} : \text{ There exists a sequence } (g_n)_{n\in\mathbb{N}} \text{ of functions in } K \text{ such that } f(x) = \lim_{n\to\infty} g_n(x) \text{ for all } x \in \mathbb{R}\}$$

In other words, $\mathbf{PL}(K)$ is the set of functions that are **pointwise limits** of sequences of functions in K.

Define B_α for all ordinals $\alpha < \omega_1$ by transfinite recursion as follows.

- $B_0 = C(\mathbb{R}, \mathbb{R})$ and
- $B_\alpha = \mathbf{PL}\left(\bigcup_{\xi<\alpha} B_\xi\right)$ for all ordinals $0 < \alpha < \omega_1$.

a) Prove that $\bigcup_{\alpha<\omega_1} B_\alpha \subseteq B$. In order to do that, it suffices to show $B_\alpha \subseteq B$ for all $\alpha < \omega_1$.

b) Prove that $B \subseteq \bigcup_{\alpha<\omega_1} B_\alpha$.

Why did we solve this question? The use of ordinal numbers is not limited to set theory. Ordinals provide a canonical notion of "rank" for various classes of mathematical objects that can be obtained by transfinite means. For example, we have the Cantor-Bendixson rank of a topological space, the rank of a Borel set, the Furstenberg rank of a minimal distal dynamical system, the height of the automorphism tower of a group etc. These are all ordinal numbers attached to various mathematical objects that encode some information about these objects. The functions in the set B in this question are called **Baire functions**. The functions in B_α are called **Baire functions of class** α. In a sense, the least ordinal α for which $f \in B_\alpha$ tells us "how many" iterations of taking pointwise limits we have to do in order to obtain f starting from continuous functions.

4. (8 pts) Burak and Kaya are playing a two-player game of perfect information of length ω on ω_1 as follows.

- Burak plays a countable ordinal α_0 as his first move.

Burak	α_0	
Kaya		

- Then Kaya responds by playing a countable ordinal β_0 as his first move.

Burak	α_0	
Kaya		β_0

- Burak now plays a countable ordinal α_1 as his second move.

Burak	α_0		α_1
Kaya		β_0	

...

Burak and Kaya keep playing countable ordinals alternately **forever** as follows.

Burak	α_0		α_1		α_2		$\dots$
Kaya		β_0		β_1		β_2	$\dots$

Both players see each other's prior moves at every stage. After infinitely many moves are made and players are "done" making their moves,

Kaya wins the game if and only if the set of chosen ordinals $\{\alpha_n : n \in \mathbb{N}\} \cup \{\beta_n : n \in \mathbb{N}\}$ is itself an ordinal.

Show that Kaya has a winning strategy.

Warning. Since we have not formally defined what a winning strategy is, you can informally describe Kaya's strategy as a sequence of moves, possibly based on Burak's earlier moves.

Hint. A set X of ordinals is an ordinal if and only if it is closed downwards, i.e. for all $\beta \in X$, $\alpha \in X$ whenever $\alpha < \beta$. Suppose for the moment that Burak is so confident of himself that he tells Kaya his moves ahead of the game, say, he tells Kaya that he is going to play the ordinals $\omega,\ \omega \cdot 2,\ \omega \cdot 3,\ \dots$ in this order. Now devise a strategy for Kaya to win the game in this special case where he knows his opponent's moves ahead of time. After thinking about this very concrete case, try to generalize your idea without assuming that you know your opponent's all moves prior to the game.