************* **PLEASE WRITE YOUR NAME CLEARLY USING CAPITAL LETTERS** *************		
F U L L N A M E	S T U D E N T I D	Submission deadline: May 24, 2021 4 questions on 6 pages 80 points in total

1. (8×5=40 pts) Let $\mathcal{I} = \{f \in {}^{\mathbb{N}}\mathbb{R} : \quad \forall n \in \mathbb{N} \;\; f(n) \leq f(n+1) \quad \wedge \quad \exists M \in \mathbb{R} \; \forall n \in \mathbb{N} \;\; |f(n)| \leq M\}$ be the set of non-decreasing bounded functions with domain $\mathbb{N}$ and codomain $\mathbb{R}$. Recall that each function in $\mathcal{I}$ has a finite limit at infinity by the Monotone Convergence Theorem that you learned in calculus. Now consider the relation $\preccurlyeq$ on $\mathcal{I}$ given by

$$f \preccurlyeq g \quad \text{if and only if} \quad \lim_{n\to\infty} f(n) \leq \lim_{n\to\infty} g(n)$$

for all $f, g \in \mathcal{I}$.

a) Show that $\preccurlyeq$ is reflexive, transitive but **not** antisymmetric.

b) Let E be the relation on $\mathcal{I}$ given by

$$f \; E \; g \quad \text{if and only if} \quad f \preccurlyeq g \; \wedge \; g \preccurlyeq f$$

for all $f, g \in \mathcal{I}$. Show that E is an equivalence relation.

c) Let $h : \mathbb{N} \to \mathbb{R}$ be the constant function given by $h(n) = 1$ for all $n \in \mathbb{N}$. Show that $[h]_E$ is uncountable.

d) Now consider the relation $\preccurlyeq_E$ on the quotient set $\mathcal{I}/E$ given by

$$A \preccurlyeq_E B \quad \text{if and only if} \;\; \exists f \in A \; \exists g \in B \; f \preccurlyeq g$$

for all $A, B \in \mathcal{I}/E$. In other words, an equivalence class A is $\preccurlyeq_E$-related to another equivalence class B iff some representative of A is $\preccurlyeq$-related to some representative of B. Prove that $\preccurlyeq_E$ is a partial order relation on $\mathcal{I}/E$.

e) Suppose that F is another equivalence relation on $\mathcal{I}$ such that the relation on the quotient set $\mathcal{I}/F$ given by

$$A \preccurlyeq_F B \quad \text{if and only if} \;\; \exists f \in A \; \exists g \in B \; f \preccurlyeq g$$

for all $A, B \in \mathcal{I}/F$, is a partial order relation on $\mathcal{I}/F$. Show that $E \subseteq F$.

f) Using the definition, show that $\preccurlyeq_E$ is a linear order relation.

g) Using the definition, show that $\preccurlyeq_E$ is **not** a well-order relation.

h) Show that the linearly ordered set $(\mathcal{I}/E, \preccurlyeq_E)$ is order-isomorphic to $(\mathbb{R}, \leq)$ by explicitly constructing an order-isomorphism between these two linearly ordered sets.

What did we learn in this question? We started with a relation which is reflexive and transitive. Such relations are known as **quasi-order** relations. Starting with an arbitrary quasi-order relation $\preccurlyeq$ on X, you can get a partial order relation $\preccurlyeq_E$ on a suitable quotient set X/E in general, by mimicking our construction here. We indeed determined the finest such equivalence relation E in this question. You should Google the phrase "quasi-order" if you want to learn more about these.

2. (5+5+5 pts) Consider the following passage that appears in an imaginary mathematical textbook.

> ...therefore, the set K of constructible numbers is the smallest subset of $\mathbb{R}$ that contains $\mathbb{Q}$ and is closed under the square roots of its positive elements. But why does there exist such a subset in the first place? Let us now show that there indeed exists a smallest subset $K \subseteq \mathbb{R}$ such that $\mathbb{Q} \subseteq K$ and $\sqrt{x} \in K$ for all positive $x \in K$.
>
> Set $K_0 = \mathbb{Q}$ and, for every $n \in \mathbb{N}$, define $K_{n+1} = \{t \in \mathbb{R} \mid t \in K_n \ \vee \ \exists k \in K_n \ \ (k > 0 \ \wedge \ t = \sqrt{k}) \ \}$. We claim that the set $K = \bigcup_{n\in\mathbb{N}} K_n$ is as desired. Clearly $\mathbb{Q} = K_0 \subseteq K$. Let $x \in K$ be positive. Then, since $x \in K$, we have that $x \in K_m$ for some $m \in \mathbb{N}$. It now follows from the construction that $\sqrt{x} \in K_{m+1} \subseteq K$. Therefore, K contains $\mathbb{Q}$ and closed under square roots of its positive elements.
>
> Next will be shown that the set K we constructed above is the smallest such subset with respect to subset inclusion. Let $L \subseteq \mathbb{R}$ be such that $\mathbb{Q} \subseteq L$ and $\sqrt{x} \in L$ for all positive $x \in L$. We wish to prove that $K \subseteq L$. Since $K = \bigcup_{n\in\mathbb{N}} K_n$, in order to conclude $K \subseteq L$, it suffices to prove that, for every $n \in \mathbb{N}$, we have...

Your aim in this question is to justify certain steps of the argument in this passage.

a) Why does the sequence $(K_n)_{n\in\mathbb{N}}$ exist? What axioms or theorems can be used to justify the existence of such a sequence? Explain **briefly.**

b) Given that $(K_n)_{n\in\mathbb{N}}$ exists, why does the set K exist? What axioms or theorems can be used to justify the existence of K? Explain **briefly.**

c) Prove that $K \subseteq L$. (**Hint.** Look at the last sentence given in the passage to guess what you are supposed to do.)

Why did we solve this question? One of the many objectives of this course is to teach you how certain basic "every day constructions" are justified via the axioms of set theory. Although one does not write a detailed justification for every step in an informal proof in textbooks or papers, one should be able to do so if pressed. As you can see, the question has nothing special to do with constructible numbers per se. I just wanted to work with a fun passage! That said, all the claims written in the passage above are true. The constructible numbers are the real numbers which you can obtain via the "straightedge-and-ruler constructions" that ancient Greeks worked on. To learn about the connection between straightedge-and-ruler constructions and taking square roots, you can Google this phrase later once you are done with this exam. (Clearly this has nothing to do with set theory.) It is possible that the algebraic properties of this field K is covered in MATH368 or maybe in MATH367.

3. (5+5+5 pts) Let $\mathbb{Z}[x]$ denote the set of polynomials whose coefficients are in $\mathbb{Z}$.

a) Prove that $\mathbb{Z}[x]$ is countable.

b) A real number r is called **algebraic** if it is the root of some non-zero polynomial in $\mathbb{Z}[x]$, that is, $f(r) = 0$ for some non-zero $f \in \mathbb{Z}[x]$. Prove that the set of algebraic real numbers is countable.

(**Hint.** You can freely use the algebra fact that the number of roots of a non-zero polynomial $f \in \mathbb{Z}[x]$ is at most the degree of the polynomial f.)

c) A real number is called **transcendental** if it is not algebraic. Prove that there are uncountably many transcendental real numbers.

What should you learn from this question? Another objective of this course is to teach you how to compute cardinalities of some fundamental objects appearing in various areas of mathematics. You should know some basic coding techniques and indirect arguments which show that certain sets are countable. I chose this specific example due to both its simplicity and its historical relevance to the birth of the notions of countable and uncountable. (Indeed, Cantor's first set theory paper shows that algebraic numbers are countable. Of course, you are **not** expected to do Cantor's original argument and indeed you **should not**; because it is more complicated than it needs to be. With the tools and theorems that you learned in class, you can prove (a) easily.)

4. (10 pts) Let ${}^{\mathbb{N}}\mathbb{N}$ denote the set of the sequences over $\mathbb{N}$ indexed by $\mathbb{N}$. Fix a set $A \subseteq {}^{\mathbb{N}}\mathbb{N}$ which will be referred to as the *payoff set*. Burak and Kaya are playing a two-player game as follows.

- Kaya plays a natural number k_0 as his first move.

$$\begin{array}{lc} \text{Kaya} & k_0 \\ \text{Burak} & \end{array}$$

- Then Burak responds by playing a natural number b_0 as his first move.

$$\begin{array}{lcc} \text{Kaya} & k_0 & \\ \text{Burak} & & b_0 \end{array}$$

- Kaya now plays another natural number k_1 as his second move.

$$\begin{array}{lccc} \text{Kaya} & k_0 & & k_1 \\ \text{Burak} & & b_0 & \end{array}$$

...

Kaya and Burak keep playing natural numbers alternately **forever** as follows.

$$\begin{array}{lccccccc} \text{Kaya} & k_0 & & k_1 & & k_2 & & \dots \\ \text{Burak} & & b_0 & & b_1 & & b_2 & \dots \end{array}$$

After infinitely many moves are made and players are "done" making their moves, Kaya and Burak will have created an element of ${}^{\mathbb{N}}\mathbb{N}$, namely, the sequence $(k_0, b_0, k_1, b_1, k_2, b_2, \dots)$. Kaya wins the game if $(k_0, b_0, k_1, b_1, k_2, b_2, \dots) \in A$ and Burak wins the game if $(k_0, b_0, k_1, b_1, k_2, b_2, \dots) \notin A$.

Clearly the game cannot end in a draw. Prior to the game, both players know what the payoff set A is and they see each other's prior moves at every stage. Depending on what A is, one of the players may or may not have a winning strategy. Before we state the question, let us see some examples so that you understand what is going on better.

- Suppose for the moment that $A = \{(a_n)_{n\in\mathbb{N}} \in {}^{\mathbb{N}}\mathbb{N} : \forall n \in \mathbb{N} \quad a_{n+1} = 2a_n\}$. In this case, Burak has a winning strategy: Burak's strategy is simply to play 1 for all his moves. Then Burak guarantees that $(k_0, 1, k_1, 1, k_2, 1, \dots) \notin A$ no matter what Kaya plays.

- Now suppose that $A = \{(a_n)_{n\in\mathbb{N}} \in {}^{\mathbb{N}}\mathbb{N} : (a_n)_{n\in\mathbb{N}} \text{ is not bounded}\}$. In this case, Kaya has a winning strategy: Kaya will play n for his move k_n, which would guarantee that the resulting sequence $(0, b_0, 1, b_1, 2, b_2, \dots)$ is not bounded and so $(0, b_0, 1, b_1, 2, b_2, \dots) \in A$ no matter what Burak plays.

- Now suppose that $A = \left\{(a_n)_{n\in\mathbb{N}} \in {}^{\mathbb{N}}\mathbb{N} : \sum_{n=0}^{\infty} \frac{a_{2n}}{a_{2n+1}} \text{ diverges}\right\}$. In this case, Burak has a winning strategy: In order to make his move b_n, Burak will first look at the natural number k_n which Kaya played last. He will then play $b_n = k_n^3 + 1$ as his move. This makes sure that $\sum_{n=0}^{\infty} \frac{k_n}{b_n} = \sum_{n=0}^{\infty} \frac{k_n}{k_n^3+1} \leq \sum_{n=1}^{\infty} \frac{1}{n^2} < \infty$ and so $(k_0, b_0, k_1, b_1, \dots) \notin A$. Therefore, no matter what Kaya plays, Burak wins the game.

In this question, you are expected to prove the existence of a winning strategy for countable payoff sets. More specifically, prove that if A is at most countable, then Burak has a winning strategy.

Warning. Since we have not formally defined what a winning strategy is, you can informally describe Burak's strategy as a sequence of moves, possibly based on Kaya's earlier moves as well as the elements of A.

Hint. Burak's aim is to make sure that whatever sequence Kaya and he create at the end is not in the payoff set. Recall that both players know what the payoff set A is prior to the game. If it were that $A = \{(0, 0, 0, \dots)\}$, what would Burak do to win? If it were that $A = \{(0, 0, 0, \dots), (1, 2, 3, \dots)\}$, what would Burak do to win? Now try to generalize your answer to these questions to the countable case. When A is at most countable, what can Burak do to make sure that the sequence they are constructing "differs from each element of A" at every stage?

What is all this about? This question is related to one of the active research areas of modern set theory, namely, determinacy. A set $A \subseteq {}^{\mathbb{N}}\mathbb{N}$ is called *determined* if one of the two players has a winning strategy in the two-player game G_A of perfect information of length ω. In this question, you are proving that countable subsets of ${}^{\mathbb{N}}\mathbb{N}$ are determined. The axiom of choice implies that some subsets of ${}^{\mathbb{N}}\mathbb{N}$ are not determined, that is, neither player has a winning strategy for certain games G_A. (This is in contrast with *finite* perfect information games where one of the players must have a strategy if the game cannot end in tie. This is known as Zermelo's theorem.) The next obvious question would be the following: Which subsets of ${}^{\mathbb{N}}\mathbb{N}$ are determined? One of the most celebrated achievements of modern set theory is the result that Borel subsets of ${}^{\mathbb{N}}\mathbb{N}$ are determined, where the topology of ${}^{\mathbb{N}}\mathbb{N}$, known as *the Baire space*, is the product topology arising from the discrete topology in each component. (If you do not know what these words mean, do not worry; they have nothing to do with this specific question.) Determinacy beyond Borel sets requires axioms beyond ZFC. There is a huge community working on this topic. If you are curious, you should Google the term "determinacy".