M E T U Department of Mathematics

Math 320 Set Theory Midterm 2 11 May 2019 12:00		
Last Name : Name : Student No:	Signature : Duration : 120 *minutes*	
6 QUESTIONS ON 4 PAGES	TOTAL 100 POINTS	

1	2	3	4	5	6	SHOW YOUR WORK		

1. (10+15=25 pts)

a) For this question only, assume that all axioms of ZFC **except the Axiom of Foundation** and that there exist a set x such that $x = \{x\}$. Prove that there exists a transitive set with exactly two elements which is different than $\{\emptyset, \{\emptyset\}\}$.

a) Recall that the addition $+$, the multiplication $\cdot$ and the exponentiation on ordinal numbers are recursively defined as follows.

$$\begin{aligned} \alpha + 0 &= \alpha & \quad \alpha \cdot 0 &= 0 & \quad \alpha^0 &= 1 \\ \alpha + S(\beta) &= S(\alpha + \beta) \quad \text{and} & \alpha \cdot S(\beta) &= (\alpha \cdot \beta) + \alpha \quad \text{and} & \alpha^{S(\beta)} &= \alpha^\beta \cdot \alpha \\ \alpha + \gamma &= sup\{\alpha + \theta : \theta \in \gamma\} & \alpha \cdot \gamma &= sup\{\alpha \cdot \theta : \theta \in \gamma\} & \alpha^\gamma &= sup\{\alpha^\theta : \theta \in \gamma\} \end{aligned}$$

for all ordinals α, β and limit ordinals γ. You are given that

$$\alpha^{sup(X)} = sup\{\alpha^\theta : \theta \in X\}$$

for all ordinals $\alpha > 1$ and for all any non-empty sets X of ordinals and that $\alpha^\beta \cdot \alpha^\gamma = \alpha^{\beta+\gamma}$ for all ordinals α, β, γ. Prove that, for all ordinals $\alpha > 1$ and β, γ, we have that

$$\left(\alpha^\beta\right)^\gamma = \alpha^{\beta \cdot \gamma}$$

WARNING: If you use an identity involving ordinal arithmetic other than the identities given in the question, you are supposed to prove it.

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2. (6+6+6+6=24 pts) Find the Cantor normal forms of the results of the following computations in ordinal arithmetic. (You can use **all** the identities we learned in class regarding ordinal arithmetic in Cantor normal form.)

a) $(\omega^{\omega_1} \cdot 3 + \omega^{\omega^\omega + \omega} \cdot 8 + \omega^{\omega^{320}} \cdot 1 + 6) + (\omega^{\omega^\omega} \cdot 2 + \omega^\omega \cdot 5 + 1) =$

b) $(\omega^{\omega \cdot 2} + \omega^{\omega+7} \cdot 4 + \omega^3) + (\omega^{\omega+\omega} + \omega^2 \cdot 3) =$

c) $(\omega^{\omega^\omega} \cdot 2 + \omega^2) \cdot (\omega^{\omega^{\omega^2}} + 2) =$

d) $\sup\{\omega + 1,\ \omega^2 + \omega,\ \omega^3 + \omega^2,\ \omega^4 + \omega^3,\ \dots\} =$

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3. (13 pts) Let $\mathbb{N}^+$ denote the set of positive natural numbers. Consider the following subset of real numbers

$$\mathcal{S} = \left\{ 1 - \frac{1}{n} \in \mathbb{R} : n \in \mathbb{N}^+ \right\} \cup \{1, 2, 3, 4, \dots\}$$

You are given that the set S together with the usual order relation $\leq$ on the set of real numbers $\mathbb{R}$ forms a well-ordered set. Explicitly construct an order isomorphism $f : \alpha \to \mathcal{S}$ where $\alpha = ot(S, \leq)$ is the order-type of the well ordered set $(S, \leq)$. (You are **not required** to show that the map you defined is an order isomorphism.)

4. (13 pts) Let ω_1 denote the first uncountable ordinal and ω denote the first infinite ordinal. Consider the set ${}^{\omega_1}\omega$ of all functions from ω_1 to ω. Prove that for every $f \in {}^{\omega_1}\omega$ there exists an uncountable set $C \subseteq \omega_1$ such that $f \restriction C$ is constant, that is, there exists $k \in \omega$ such that $f(\alpha) = k$ for all $\alpha \in C$. (**Hint.** Recall that a countable union of countable sets is countable.)

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5. (6+7=13 pts)

a) State the definition of an ordinal number.

b) Complete the following statement of the principle of transfinite induction: A property $\varphi(x)$ of sets holds for all ordinal numbers if

- $\varphi(0)$ holds.
- $\varphi(S(\alpha))$ holds whenever $\varphi(\alpha)$ holds, for all ordinals α.
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6. (12 pts) Construct a function $f : \omega_1 \to \omega_1$ such that

- For all $0 \neq \alpha \in \omega_1$, we have that $f(\alpha) < \alpha$, and
- For every $\alpha \in \omega_1$ there exists $\beta \in f[\omega_1]$ such that $\alpha < \beta$, that is, $f[\omega_1]$ is unbounded in ω_1.

(**Hint.** Try to define such a function using the coefficients in the Cantor normal forms of α.)

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