M E T U Department of Mathematics

Math 320 Set Theory Midterm 1 13 April 2019 11:40		
Last Name : Name : Student No:	Signature : Duration : 120 *minutes*	
4 QUESTIONS ON 4 PAGES	TOTAL 100 POINTS	

1	2	3	4			SHOW YOUR WORK		

1. (5+10=15 pts)

Let $\mathcal{D}$ be a non-empty set whose elements are Dedekind cuts of $\mathbb{Q}$. Let $X = \bigcap \mathcal{D} = \bigcap_{S \in \mathcal{D}} S$.

a) Prove or disprove the following statement: For every $q \in X$ and $p \in \mathbb{Q}$, if $p < q$, then $p \in X$.

b) Recall that the recursive definitions of addition and multiplication operations $+$ and $\cdot$ on the set of natural numbers $\mathbb{N}$ are given as follows:

$$\begin{aligned} m + 0 &= m \\ m + S(n) &= S(m+n) \end{aligned} \quad \text{and} \quad \begin{aligned} m \cdot 0 &= 0 \\ m \cdot S(n) &= (m \cdot n) + m \end{aligned}$$

for all $m, n \in \mathbb{N}$, where $S(n)$ denotes the successor of the natural number n. You are given that the identities $0 \cdot m = 0$, $1 \cdot m = m$ and $(m+n) \cdot p = m \cdot p + n \cdot p$ hold for all $m, n, p \in \mathbb{N}$. Prove that $\cdot$ is commutative, that is, for all $m, n \in \mathbb{N}$, we have

$$m \cdot n = n \cdot m$$

[**WARNING:** If you use an identity involving arithmetical operations on $\mathbb{N}$ other than the identities given in the question, you are supposed to prove it.]

2. (8+8+6+8=30 pts) Consider the relation $\preccurlyeq$ on the set $\mathbb{N} \times \mathbb{R}$ given by

$$(n, q) \preccurlyeq (m, r) \longleftrightarrow (n < m \quad \vee \quad (n = m \quad \wedge \quad q \leq r))$$

for all $(n, q), (m, r) \in \mathbb{N} \times \mathbb{R}$. In other words, the relation $(n, q) \preccurlyeq (m, r)$ holds if and only if we have that $n < m$, or, that $n = m$ and $q \leq r$.

(a) Prove that the relation $\preccurlyeq$ is reflexive and transitive.

You are given that $\preccurlyeq$ is a linear order relation.
(b) Prove that $\preccurlyeq$ is not a well-order relation.

(c) Let $\prec$ denote the associated strict linear order relation. Let $(n, q), (n', q')$ be elements of $\mathbb{N} \times \mathbb{R}$ such that $(n, q) \prec (n', q')$. Prove that there exists $(m, r) \in \mathbb{N} \times \mathbb{R}$ such that

$$(n, q) \prec (m, r) \prec (n', q')$$

(d) Prove that there exists an order preserving function from $(\mathbb{N} \times \mathbb{R}, \preccurlyeq)$ to $(\mathbb{R}, \leq)$, that is, there exists a function $f : \mathbb{N} \times \mathbb{R} \to \mathbb{R}$ such that $(n, q) \preccurlyeq (m, r) \leftrightarrow f(n, q) \leq f(m, r)$ for all $(n, q), (m, r) \in \mathbb{N} \times \mathbb{R}$.

3. (10+10+10=30 pts) Consider the relation $\sim$ on $\mathbb{R}$ given by

$$x \sim y \longleftrightarrow x - y \in \mathbb{Q}$$

for all $x, y \in \mathbb{R}$. In other words, two real numbers related under the relation $\sim$ if and only if their difference is a rational number.

(a) Prove that $\sim$ is an equivalence relation.

(b) Show that the equivalence class $[x]$ is countable for every $x \in \mathbb{R}$.

(c) Show that $\mathbb{R}/\sim$ is uncountable.
(**Hint.** Recall the fact that the quotient set $\mathbb{R}/\sim$ is a partition of $\mathbb{R}$.)

4. (5+10+10=25 pts)

(a) State the Axiom of Power Set.

(b) State and prove Cantor's theorem.

(c) Construct a non-empty set U such that for every $x \in U$ there exists an infinite set $y \in U$ such that $|x| < |y|$.