M E T U Department of Mathematics

Math 320 Set Theory Final Exam 27 May 2019 9:30		
Last Name : Name : Student No:	Signature : Duration : 120 *minutes*	
6(+1) QUESTIONS ON 4(+1) PAGES	TOTAL 100(+10) POINTS	

1	2	3	4	5	6	**SHOW YOUR WORK**		

1. (8+8+6=22 pts)

a) Recall that the $\beth$ numbers are defined by transfinite recursion as follows.

- $\beth_0 = \aleph_0$
- $\beth_{\alpha+1} = 2^{\beth_\alpha}$ for all ordinals α, and
- $\beth_\gamma = sup\{\beth_\theta : \theta < \gamma\}$ for all limit ordinals γ.

Using transfinite induction, prove that $\aleph_\alpha \leq \beth_\alpha$ for every ordinal α.

b) Prove that, for every ordinal α, if $\beth_\alpha = \alpha$, then $\aleph_\alpha = \alpha$.
(**Hint.** You can assume the statement in part b even if you could not prove it.)

c) Prove that there exists an uncountable cardinal κ such that $2^\lambda < \kappa$ for all cardinals $\lambda < \kappa$.
(**Hint.** Try to find such κ among the $\beth$ numbers.)

2. (6+6+6+6+6=30 pts) Assuming the Generalized Continuum Hypothesis (GCH), that is, the statement $2^{\aleph_\alpha} = \aleph_{\alpha+1}$ for every ordinal α, find the corresponding $\aleph$ numbers of the following computations in cardinal arithmetic. (You can use **all** identities and theorems we learned in class regarding cardinal arithmetic.)

a) $\aleph_3^{\left(\aleph_1^{\aleph_2}\right)} =$

b) $(\aleph_{\aleph_0} \cdot \aleph_{2^{\aleph_0}} + \aleph_7) + ((\aleph_{\aleph_1})^{\aleph_1} \cdot 42) =$

c) $(\aleph_{\aleph_3})^{\aleph_2} =$

d) $|\mathcal{P}(\mathbb{R})|^{|\mathbb{N}|} =$

e) $\sum_{n\in\omega} \aleph_n =$

3. (8 pts) Recall that ω_1 denotes the least uncountable ordinal, that is, ω_1 is the set of all countable ordinals. Prove that the set $\mathcal{F} = \{C \subseteq \omega_1 : C \text{ is countable}\}$ has cardinality $2^{\aleph_0}$.
(**Hint.** You can freely use the fact that $\mathcal{F} = \{C \subseteq \omega_1 : \exists \alpha \in \omega_1 \; C \subseteq \alpha\}$.)

4. (8+8=16 pts) Let $\mathcal{W}$ denote the set of strict well-order relations on $\mathbb{N}$, that is,

$$\mathcal{W} = \{E \in \mathcal{P}(\mathbb{N} \times \mathbb{N}) : E \text{ is a strict well-order relation on } \mathbb{N}\}$$

and let $\sim$ be the relation on $\mathcal{W}$ given by

$$E \sim F \longleftrightarrow \text{There exists an order-isomorphism } f : \mathbb{N} \to \mathbb{N} \text{ from } (\mathbb{N}, E) \text{ to } (\mathbb{N}, F)$$

for every $E, F \in \mathcal{W}$.

a) Prove that $\sim$ is an equivalence relation.

b) Find the cardinality of the quotient set $\mathcal{W}/\sim$.
(**Hint.** Think about the possible order-types of the strictly well-ordered sets of the form $(\mathbb{N}, E)$.)

5. (4+10=14 pts)

a) State the definition of a regular cardinal. An uncountable cardinal κ is regular if ...

b) Let κ be an infinite regular cardinal and $\gamma < \kappa$ be an ordinal such that $\kappa = \bigcup_{\alpha \in \gamma} A_\alpha$ for some indexed system of sets $\{A_\alpha\}_{\alpha \in \gamma}$. Prove that $sup(A_\alpha) = \kappa$ for some $\alpha < \gamma$.

6. (10 pts) List the axioms of ZFC. Each correctly stated axiom will earn you one point.

NAME AND STUDENT ID:

(**BONUS QUESTION**. You can hand-in your solution for this question until 15.30 on 27 May 2019. If you cannot find me in my office, slip your solution under the door. You can discuss ideas with your friends who are taking this course, but cannot write the solutions together. You should avoid any other external help. If you try to use textbook solutions or internet resources such as StackExchange or MathOverflow, be aware that I will know.)

7. (10 pts) Consider the ordinals ω^n where the exponentiation is with respect to ordinal arithmetic. Show that, for every positive $n \in \mathbb{N}$, there exists an order-preserving map $f : \omega^n \to \mathbb{R}$.