M E T U Department of Mathematics

Math 320 Set Theory Midterm 2 09 May 2018 17:40		
Last Name : Name : Student No:	Signature : Duration : 120 *minutes*	

6 QUESTIONS ON 4 PAGES						TOTAL 100 POINTS		
1	2	3	4	5	6	**SHOW YOUR WORK**		

1. (10+15=25 pts)

a) Prove or disprove: If X is a set whose elements are transitive sets, then $\bigcup X$ is transitive.

a) Recall that the addition $+$ and the multiplication $\cdot$ on ordinal numbers are recursively defined as follows.

$$\begin{aligned} \alpha + 0 &= \alpha & \qquad \alpha \cdot 0 &= 0 \\ \alpha + S(\beta) &= S(\alpha + \beta) & \text{and} \qquad \alpha \cdot S(\beta) &= (\alpha \cdot \beta) + \alpha \\ \alpha + \gamma &= sup\{\alpha + \theta : \theta \in \gamma\} & \qquad \alpha \cdot \gamma &= sup\{\alpha \cdot \theta : \theta \in \gamma\} \end{aligned}$$

for all ordinals α, β and limit ordinals γ. You are given that

$$\alpha + sup(X) = sup\{\alpha + \theta : \theta \in X\} \text{ and } \alpha \cdot sup(X) = sup\{\alpha \cdot \theta : \theta \in X\}$$

for all ordinals α and for all any non-empty sets X of ordinals. Moreover, you are given the fact that $+$ and $\cdot$ are associative. Prove that, for all ordinals α, β, γ, we have that

$$\alpha \cdot (\beta + \gamma) = (\alpha \cdot \beta) + (\alpha \cdot \gamma)$$

WARNING: If you use an identity involving ordinal arithmetic other than the identities given in the question, you are supposed to prove it.

2. (6+6+6+7=25 pts) Find the Cantor normal forms of the results of the following computations in ordinal arithmetic. (You can use **all** the identities we learned in class regarding ordinal arithmetic in Cantor normal form.)

a) $(\omega^{\omega+3} \cdot 5 + \omega^{\omega^2+\omega} \cdot 3 + \omega^{320} \cdot 2 + 7) + (\omega^{\omega} \cdot 3 + \omega \cdot 5 + 6) =$

b) $(\omega^{\omega^{\omega^2}} + \omega^{\omega^{\omega}}) + (\omega^{\omega^{\omega^2}} \cdot 119 + \omega^2 \cdot 2 + 1) =$

c) $(\omega^{\omega^5+2} \cdot 2 + 1) \cdot (\omega^{\omega} + 2) =$

d) $\omega^{\omega^{\omega}} \cdot \omega^{\omega_1} =$ (**Hint.** You can use the result which is supposed to be proven in Problem 5.)

3. (12 pts) Let $\mathbb{N}^+$ denote the set of positive natural numbers. Consider the following subset of real numbers

$$\mathcal{S} = \left\{ m - \frac{1}{n} \in \mathbb{R} : m, n \in \mathbb{N}^+ \right\}$$

You are given that the set S together with the usual order relation $\leq$ on the set of real numbers $\mathbb{R}$ forms a well-ordered set. Find the order type of the well-ordered set $(S, \leq)$. (**Hint.** Try to plot the set $\mathcal{S}$ on the real number line in order to understand its order structure.)

4. (13 pts) Let ω_1 denote the first uncountable ordinal and ω denote the first infinite ordinal. You are given the fact that if α is a countable ordinal, then the ordinal $\alpha \cdot n$ is countable for all $n \in \omega$. Using transfinite induction, prove that the ordinal ω^α is countable for all $\alpha < \omega_1$.

5. (6+6=12 pts)

a) State the definition of an ordinal number.

b) State either one of the Axiom of Choice, Zorn's Lemma or the Well-Ordering Theorem.

6. (13 pts) Let $(X, \mathcal{P})$ be a partially ordered set. Consider the set

$$\mathbb{P} = \{R \subseteq X \times X : \ \mathcal{P} \subseteq R \text{ and "} R \text{ is a partial order relation on } X\text{"}\}$$

You are given that the relation $\preccurlyeq$ on $\mathbb{P}$ defined by

$$R \preccurlyeq S \longleftrightarrow R \subseteq S$$

for all $R, S \in \mathbb{P}$, is a partial order relation on $\mathbb{P}$.
a) Show that the partially ordered set $(\mathbb{P}, \preccurlyeq)$ has the property that every chain in $\mathbb{P}$ has an upper bound in $\mathbb{P}$.

For the next part of this question, you are given the following fact:

- If x and y are elements of X which are incomparable with respect to some partial order relation R on X, then there exists a partial order relation $\widehat{R} \supsetneq R$ on X with respect to which x and y are comparable. (In other words, any two incomparable elements in a partially ordered set can be made comparable with respect to an extended partial order relation.)

b) Prove that there exists a linear order relation $\mathcal{L}$ on X such that $\mathcal{P} \subseteq \mathcal{L}$.
(This fact is sometimes called **Szpilrajn extension theorem**.)