M E T U Department of Mathematics

Math 320 Set Theory Midterm 1 04 April 2018 17:40								
Last Name : Name : Student No:						Signature : Duration : 120 *minutes*		
6 QUESTIONS ON 4 PAGES						TOTAL 100 POINTS		
1	2	3	4	5	6	**SHOW YOUR WORK**		

1. (5+10=15 pts)

a) Prove or disprove the following statement: For any non-empty set $\mathcal{C}$, if every element of $\mathcal{C}$ is an inductive set, then $\bigcap \mathcal{C}$ is inductive.

b) Recall that the recursive definitions of addition and multiplication operations $+$ and $\cdot$ on the set of natural numbers $\mathbb{N}$ are given as follows:

$$\begin{aligned} m + 0 &= m \\ m + S(n) &= S(m+n) \end{aligned} \quad \text{and} \quad \begin{aligned} m \cdot 0 &= 0 \\ m \cdot S(n) &= (m \cdot n) + m \end{aligned}$$

for all $m, n \in \mathbb{N}$, where $S(n)$ denotes the successor of the natural number n. You are given that $+$ is commutative and associative, that is, the identities $m + n = n + m$ and $(m + n) + p = m + (n + p)$ hold for all $m, n, p \in \mathbb{N}$. Prove that $\cdot$ distributes over $+$ from right, that is, for all $m, n, p \in \mathbb{N}$, we have

$$(m + n) \cdot p = m \cdot p + n \cdot p$$

[**WARNING:** If you use an identity involving arithmetical operations on $\mathbb{N}$ other than the identities given in the question, you are supposed to prove it.]

2. (8+7+7+8=30 pts) Recall that ${}^{\mathbb{N}}\mathbb{N}$ is the set of functions from $\mathbb{N}$ to $\mathbb{N}$. Consider the relation $\preccurlyeq$ on the set ${}^{\mathbb{N}}\mathbb{N}$ given by

$$f \preccurlyeq g \longleftrightarrow \exists m \in \mathbb{N}\ \forall n \in \mathbb{N}\ (n \geq m \to f(n) \leq g(n))$$

for all $f, g \in {}^{\mathbb{N}}\mathbb{N}$. In other words, the relation $f \preccurlyeq g$ holds if and only if $f(n) \leq g(n)$ for all sufficiently large natural numbers n.

(a) Show that the relation $\preccurlyeq$ is reflexive and transitive.

(b) Show that the relation $\preccurlyeq$ is not antisymmetric.

(c) Determine whether every two elements of ${}^{\mathbb{N}}\mathbb{N}$ are comparable with respect to the relation $\preccurlyeq$. (Recall that two functions f and g are said to be comparable with respect to $\preccurlyeq$ if $f \preccurlyeq g$ or $g \preccurlyeq f$.)

(d) Prove that for every sequence $(f_i)_{i \in \mathbb{N}}$ over ${}^{\mathbb{N}}\mathbb{N}$, there exists $g \in {}^{\mathbb{N}}\mathbb{N}$ such that $f_i \preccurlyeq g$ for all $i \in \mathbb{N}$.

3. (10 pts) Show that the set $\mathcal{F} = \{A \subseteq \mathbb{N} : A \text{ is finite}\}$ consisting of finite subsets of $\mathbb{N}$ is countable. (**Hint.** You can freely use the fact that $\mathcal{F}$ equals $\{A \subseteq \mathbb{N} : \exists n \in \mathbb{N}\ A \subseteq n\}$.)

4. (8+4+8=20 pts) Consider the relation $\sim$ on $\mathcal{P}(\mathbb{N})$ given by

$$A \ \sim \ B \longleftrightarrow \exists n \in \mathbb{N} \ \ |A \Delta B| = n$$

for all $A, B \in \mathcal{P}(\mathbb{N})$. In other words, two subsets of $\mathbb{N}$ are related under the relation $\sim$ if and only if their symmetric difference is finite. You are given the fact that symmetric difference is associative, that is, $(x \Delta y)\Delta z = x\Delta(y \Delta z)$ for all sets x, y, z.

(a) Prove that $\sim$ is an equivalence relation.

(b) Write the definition of the quotient set $\mathcal{P}(\mathbb{N})/{\sim}$.

(c) You are given that there exists an injection from $\mathcal{P}(\mathbb{N})$ to $\mathcal{P}(\mathbb{N})/{\sim}$. Show that $|\mathcal{P}(\mathbb{N})| = |\mathcal{P}(\mathbb{N})/{\sim}|$. (**Hint.** Try first to argue that there is an injection from $\mathcal{P}(\mathbb{N})/{\sim}$ to $\mathcal{P}(\mathbb{N})$.)

5. (5+5=10 pts)

(a) State either the Axiom of Infinity or the Axiom of Choice.

For the next question, recall that the Axiom of Foundation states that for every non-empty set X, there exists $x \in X$ such that $x \cap X = \emptyset$.

(b) Let A be a set. Prove that there does not exist a function $f : \mathbb{N} \to A$ such that $f(i+1) \in f(i)$ for all $i \in \mathbb{N}$.

6. (15 pts) Let X be a set. Prove that there does not exist a surjection $f : X \to \mathcal{P}(X)$.