M E T U Department of Mathematics

Math 320 Set Theory Final Exam 29 May 2018 9:30		
Last Name : Name : Student No:	Signature : Duration : 130 *minutes*	
6(+1) QUESTIONS ON 4(+1) PAGES		TOTAL 100(+10) POINTS

1	2	3	4	5	6	SHOW YOUR WORK		

1. (6+6+6=18 pts)

a) Prove or disprove: There exists an infinite cardinal κ such that $2^{cf(\kappa)} = \kappa$ where $cf(\kappa)$ denotes the cofinality of κ.

b) Prove that for all infinite cardinals κ, we have that $(\kappa^+)^\kappa = 2^\kappa$ where κ^+ denotes the least cardinal greater than κ. (**Hint.** Recall that $\kappa < 2^\kappa$ for every cardinal κ.)

c) Recall that the beth numbers are defined by transfinite recursion as follows.

- $\beth_0 = \aleph_0$
- $\beth_{\alpha+1} = 2^{\beth_\alpha}$ for all ordinals α, and
- $\beth_\gamma = sup\{\beth_\theta : \theta < \gamma\}$ for all limit ordinals γ.

Prove that if $2^{\aleph_\alpha} = \aleph_{\alpha+1}$ for every ordinal α, then $\aleph_\alpha = \beth_\alpha$ for every ordinal α.
(**Hint.** Use transfinite induction!)

2. (6+6+6+6+6=30 pts) Assuming the Generalized Continuum Hypothesis (GCH), that is, the statement $2^{\aleph_\alpha} = \aleph_{\alpha+1}$ for every ordinal α, find the corresponding $\aleph$ numbers of the following computations in cardinal arithmetic. (You can use **all** identities and theorems we learned in class regarding cardinal arithmetic.)

a) $\aleph_0^{\aleph_1^{\aleph_2}} =$

b) $(\aleph_{123} + \aleph_{2^{\aleph_0}}) + ((\aleph_{\aleph_0})^5 \cdot 2) =$

c) $(\aleph_{\aleph_\omega})^{2^{\aleph_7}} =$

d) $(\aleph_{\aleph_0})^{\aleph_{\aleph_1}} =$

e) $|\mathbb{R}|^{|\mathbb{Q}|} =$

3. (12 pts) Let α be a limit ordinal. Prove that $cf(\alpha) \leq cf(\aleph_\alpha)$.

4. (8+8=16 pts) Let $\mathcal{C}$ denote the set of functions functions from ω_1 to 2 which eventually take the value 0. In other words, $\mathcal{C}$ is the set

$$\{f \in {}^{\omega_1}2 : \exists \alpha \in \omega_1 \, \forall \beta \in \omega_1 \, (\beta \geq \alpha \rightarrow f(\beta) = 0)\}$$

Consider the relation $\sim$ on $\mathcal{C}$ defined by

$$f \sim g \longleftrightarrow |\{\alpha \in \omega_1 : f(\alpha) \neq g(\alpha)\}| \leq \aleph_0$$

for all $f, g \in \mathcal{C}$.

a) Prove that $\sim$ is an equivalence relation.

b) Find the cardinality of the quotient set $\mathcal{C}/\sim$.
(**Hint.** Try to first find the equivalence class of the zero function.)

5. (6+8=14 pts)

a) State the definition of a cardinal number.

b) Let V_α denote the α-th level of the von Neumann hierarchy of sets for each ordinal α. Using transfinite induction, prove that $|\mathrm{V}_\alpha| \leq \beth_\alpha$ for all ordinals α.

6. (10 pts) Recall that Cantor's theorem states that $|X| < |\mathcal{P}(X)|$ for any set X. Prove Cantor's theorem.

Bonus question (10 pts). Prove that the cardinality of the set of continuous functions from $\mathbb{R}$ to $\mathbb{R}$ is the same as the cardinality of $\mathbb{R}$.
(**Hint.** Recall the basic analysis fact that two continuous functions from $\mathbb{R}$ to $\mathbb{R}$ are the same if and only if they take the same values at rational numbers.)

OR

Bonus question (10 pts). Prove that the dimension of $\mathbb{R}$ as a vector space over $\mathbb{Q}$ is the same as the cardinality of $\mathbb{R}$.
(**Hint.** Recall the basic linear algebra facts that the dimension of a vector space is the cardinality of any of its bases and that if $\mathcal{B} \subseteq \mathbb{R}$ is a basis for $\mathbb{R}$ as a vector space over $\mathbb{Q}$, then every element of $\mathbb{R}$ can be written as a unique linear combination $a_1v_1 + a_2v_2 + \cdots + a_nv_n$ where a_i's are non-zero rational numbers and v_i's are in $\mathcal{B}$.)