	Math 3	20 S	Set The	eory	Final	Exam	29 N	May 2	2018	9	:30	
Last Name:				Sigr	nature	:						
Name : Student No:				Dur	ation	: 130 m	ninutes	s				
6(+1) QUESTIONS ON 4(+1) PAGES TOTAL 100(+										-10) P	OINTS	
1 2 3	4	5	6		SHOW	YOUR	R WO	RK				

M E T U Department of Mathematics

1. (6+6+6=18 pts)

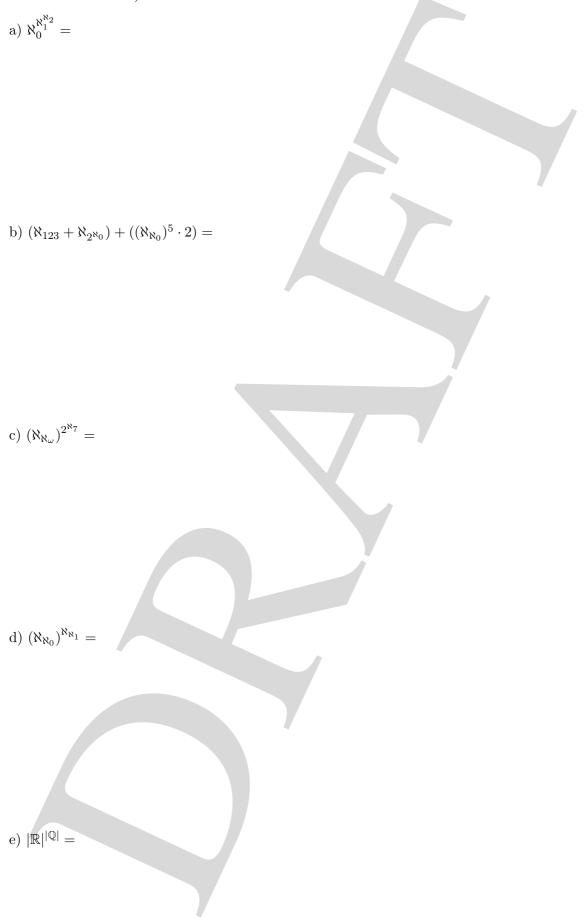
a) Prove or disprove: There exists an infinite cardinal κ such that $2^{cf(\kappa)} = \kappa$ where $cf(\kappa)$ denotes the cofinality of κ .

b) Prove that for all infinite cardinals κ , we have that $(\kappa^+)^{\kappa} = 2^{\kappa}$ where κ^+ denotes the least cardinal greater than κ . (**Hint.** Recall that $\kappa < 2^{\kappa}$ for every cardinal κ .)

c) Recall that the beth numbers are defined by transfinite recursion as follows.

- $\beth_0 = \aleph_0$
- $\beth_{\alpha+1} = 2^{\beth_{\alpha}}$ for all ordinals α , and
- $\beth_{\gamma} = \sup\{\beth_{\theta} : \theta < \gamma\}$ for all limit ordinals γ .

Prove that if $2^{\aleph_{\alpha}} = \aleph_{\alpha+1}$ for every ordinal α , then $\aleph_{\alpha} = \beth_{\alpha}$ for every ordinal α . (**Hint.** Use transfinite induction!) 2. (6+6+6+6=30 pts) Assuming the Generalized Continuum Hypothesis (GCH), that is, the statement $2^{\aleph_{\alpha}} = \aleph_{\alpha+1}$ for every ordinal α , find the corresponding \aleph numbers of the following computations in cardinal arithmetic. (You can use **all** identities and theorems we learned in class regarding cardinal arithmetic.)



3. (12 pts) Let α be a limit ordinal. Prove that $cf(\alpha) \leq cf(\aleph_{\alpha})$.



4. (8+8=16 pts) Let C denote the set of functions from ω_1 to 2 which eventually take the value 0. In other words, C is the set

$$\{f \in {}^{\omega_1}2 : \exists \alpha \in \omega_1 \; \forall \beta \in \omega_1 \; (\beta \ge \alpha \to f(\beta) = 0)\}$$

Consider the relation \sim on $\mathcal C$ defined by

$$f \sim g \longleftrightarrow |\{\alpha \in \omega_1 : f(\alpha) \neq g(\alpha)\}| \le \aleph_0$$

for all $f, g \in \mathcal{C}$.

a) Prove that \sim is an equivalence relation.

b) Find the cardinality of the quotient set C/\sim . (**Hint.** Try to first find the equivalence class of the zero function.)

5. (6+8=14 pts)

a) State the definition of a cardinal number.

b) Let V_{α} denote the α -th level of the von Neumann hierarchy of sets for each ordinal α . Using transfinite induction, prove that $|V_{\alpha}| \leq \beth_{\alpha}$ for all ordinals α .

6. (10 pts) Recall that Cantor's theorem states that $|X| < |\mathcal{P}(X)|$ for any set X. Prove Cantor's theorem.

Bonus question (10 pts). Prove that the cardinality of the set of continuous functions from \mathbb{R} to \mathbb{R} is the same as the cardinality of \mathbb{R} .

(**Hint.** Recall the basic analysis fact that two continuous functions from \mathbb{R} to \mathbb{R} are the same if and only if they take the same values at rational numbers.)



Bonus question (10 pts). Prove that the dimension of \mathbb{R} as a vector space over \mathbb{Q} is the same as the cardinality of \mathbb{R} .

(**Hint.** Recall the basic linear algebra facts that the dimension of a vector space is the cardinality of any of its bases and that if $\mathcal{B} \subseteq \mathbb{R}$ is a basis for \mathbb{R} as a vector space over \mathbb{Q} , then every element of \mathbb{R} can be written as a unique linear combination $a_1v_1 + a_2v_2 + \cdots + a_nv_n$ where a_i 's are non-zero rational numbers and v_i 's are in \mathcal{B} .)

