M E T U Department of Mathematics

Math 320 Set Theory Midterm 2 05 May 2017 17:40		
Last Name : Name : Student No:	Signature : Duration : 100 *minutes*	

6 QUESTIONS ON 4 PAGES						TOTAL 100 POINTS		
1	2	3	4	5	6	**SHOW YOUR WORK**		

1. (5+15=20 pts)

a) Prove or disprove: There exists a non-zero ordinal number γ such that $\omega \cdot \gamma = \gamma$

a) You are given that $\alpha \cdot (\beta + \gamma) = \alpha \cdot \beta + \alpha \cdot \gamma$ for all ordinals α, β, γ and that

$$\alpha \cdot sup(X) = sup\{\alpha \cdot \theta : \theta \in X\}$$

for any non-empty set X of ordinals. Prove that ordinal multiplication is associative, that is, for all ordinals α, β, γ, we have that

$$\alpha \cdot (\beta \cdot \gamma) = (\alpha \cdot \beta) \cdot \gamma$$

WARNING: If you use an identity involving ordinal arithmetic other than the identities given in the question, you are supposed to prove it.

2. (6+6+6+7=25 pts) Find the Cantor normal forms of the results of the following computations in ordinal arithmetic. (You can use **all** the identities we learned in class regarding ordinal arithmetic in Cantor normal form.)

a) $(\omega^{\omega^{\omega}+2}\cdot 2+\omega^{5}\cdot 3+\omega\cdot 2+7)+(\omega^{\omega}\cdot 3+\omega\cdot 5+6)=$

b) $(\omega^{\omega^{3}+\omega^{2}+\omega+1}\cdot 7+\omega^{\omega}\cdot 3+\omega^{2}\cdot 5)+(\omega^{\omega^{3}+\omega^{2}+\omega+1}\cdot 3+\omega^{2}\cdot 2+1)=$

c) $(\omega^{\omega^{\omega}+\omega^{2}+2}\cdot 3+\omega^{2})\cdot(\omega^{\omega^{3}+5}+5)=$

d) $(\omega^{\omega^{\omega}}+\omega^{\omega}+\omega+1)+\epsilon=$ where ϵ is an ordinal satisfying $\omega^{\epsilon}=\epsilon$.

3. (10 pts) Let $\mathcal{C}$ denote the set of subsets of $\mathbb{N}$ that have two elements, that is,

$$\mathcal{C} = \{S : S \subseteq \mathbb{N} \ \wedge \ |S| = 2\}$$

Consider the relation $\prec$ on the set $\mathcal{C}$ defined by

$$S \prec T \longleftrightarrow \min(S) < \min(T) \ \vee \ (\min(S) = \min(T) \ \wedge \ \max(S) < \max(T))$$

for all $S, T \in \mathcal{C}$ where $<$ denotes the usual linear order relation on $\mathbb{N}$. For example, we have that $\{0, 10\} \prec \{2, 5\}$ and $\{3, 7\} \prec \{3, 8\}$. You are given the fact that $(\mathcal{C}, \prec)$ is a strictly well-ordered set.

Prove that there exists a function $f : \mathcal{C} \longrightarrow \omega \cdot \omega$ such that $S \prec T \rightarrow f(S) \in f(T)$.

Hint. In order to understand its order structure, try to list the elements of $\mathcal{C}$ in increasing order.

4. (15 pts) Recall that ${}^{x}y$ denotes the set of functions from the set x to the set y, and that ω and ω_1 denote the first infinite ordinal and the first uncountable ordinal respectively. Prove that

$${}^{\omega}\omega_1 = \bigcup_{\alpha \in \omega_1} {}^{\omega}\alpha$$

5. (5+6+4=15 pts)

a) State the definition of an ordinal number.

b) Using only the definition of an ordinal number, prove that if α is an ordinal number, then every element of α is transitive. (**Hint.** Recall that there cannot be sets x, y, z with $x \in y$, $y \in z$ and $z \in x$. Similarly, there are no sets x, y such that $x \in y$ and $y \in x$.)

c) Give an example of a transitive set that is not an ordinal number.

6. (15 pts) State and prove Cantor's theorem.