Math 320 Set Theory Midterm 1 04 April 2017 17:40		
Last Name : Name : Student No:	Signature : Duration : 120 *minutes*	
5 QUESTIONS ON 4 PAGES		TOTAL 100 POINTS

1	2	3	4	5		SHOW YOUR WORK		

1. (5+10=15 pts)

a) Prove or disprove the following statement: For any inductive set I we have that $I \subseteq \bigcup I$.

b) Recall that the recursive definition of multiplication operation $\cdot$ on the set of natural numbers $\mathbb{N}$ is given as follows:

$$m \cdot 0 = 0$$

$$m \cdot S(n) = (m \cdot n) + m$$

for all $m, n \in \mathbb{N}$, where $S(n)$ denotes the successor of the natural number n. You are given that the identity $m \cdot (n + p) = (m \cdot n) + (m \cdot p)$ holds for all $m, n, p \in \mathbb{N}$. Prove that $\cdot$ is associative, that is, for all $m, n, p \in \mathbb{N}$, we have

$$m \cdot (n \cdot p) = (m \cdot n) \cdot p$$

[**WARNING:** If you use an identity involving arithmetical operations on $\mathbb{N}$ other than the identity given in the question, you are supposed to prove it.]

2. (9+6+6+6=27 pts) Recall that ${}^{\mathbb{N}}\mathbb{N}$ is the set of functions from $\mathbb{N}$ to $\mathbb{N}$. Let $\mathcal{I}$ be the set of functions in ${}^{\mathbb{N}}\mathbb{N}$ which are strictly increasing, that is,

$$\mathcal{I} = \{f \in {}^{\mathbb{N}}\mathbb{N} \ : \forall i, j \in \mathbb{N} \ \ i < j \leftrightarrow f(i) < f(j) \ \}$$

Consider the relation $\preccurlyeq$ on the set $\mathcal{I}$ given by

$$f \preccurlyeq g \longleftrightarrow \exists h \in \mathcal{I} \ \ g = f \circ h$$

for all $f, g \in \mathcal{I}$. In other words, if you consider functions in $\mathcal{I}$ as strictly increasing sequences over $\mathbb{N}$ indexed by $\mathbb{N}$, the relation $f \preccurlyeq g$ holds if and only if $(g(n))_{n\in\mathbb{N}}$ is a subsequence of $(f(n))_{n\in\mathbb{N}}$.

(a) Show that $\preccurlyeq$ is a partial order relation on $\mathcal{I}$.

(b) Show that $\preccurlyeq$ is not a linear order relation on $\mathcal{I}$.

(c) Determine whether there exists a least element of $\mathcal{I}$ with respect to the partial order relation $\preccurlyeq$.

(d) Let $\prec$ denote the induced strict partial order relation given by $f \prec g \leftrightarrow f \preccurlyeq g \ \wedge \ f \neq g$ for all $f, g \in \mathcal{I}$. Find a sequence $(f_i)_{i\in\mathbb{N}}$ over $\mathcal{I}$ such that for all $i \in \mathbb{N}$, we have that $f_i \prec f_{i+1}$ and there does not exist $g \in \mathcal{I}$ such that $f_i \prec g \prec f_{i+1}$.

3. (8+7+6+7=28 pts) Recall that ${}^{\mathbb{N}}2$ is the set of functions from $\mathbb{N}$ to the set $2 = \{0, 1\}$. Consider the relation E on ${}^{\mathbb{N}}2$ given by

$$f \; E \; g \longleftrightarrow \exists m \in \mathbb{N} \; \forall n \in \mathbb{N} \; (n \geq m \rightarrow f(n) = g(n))$$

for all $f, g \in {}^{\mathbb{N}}2$. In other words, if you consider functions in ${}^{\mathbb{N}}2$ as sequences over 2 indexed by $\mathbb{N}$, the relation $f \; E \; g$ holds if and only if the sequences $(f(n))_{n \in \mathbb{N}}$ and $(g(n))_{n \in \mathbb{N}}$ have the same values at sufficiently large indices.

(a) Prove that E is an equivalence relation.

(b) Prove that the set $[f]_E$ is countable for any $f \in {}^{\mathbb{N}}2$.

(c) Prove that there exists an injective function from the quotient set ${}^{\mathbb{N}}2/E$ to the set ${}^{\mathbb{N}}2$.

(d) Explicitly construct an injective function from the set ${}^{\mathbb{N}}2$ to the quotient set ${}^{\mathbb{N}}2/E$.

4. (5+5=10 pts)

(a) State the Axiom of Foundation.

(b) Prove that there do not exist sets x, y, z such that $x \in y$, $y \in z$ and $z \in x$.

5. (20 pts)

State and prove Cantor's theorem.