M E T U Department of Mathematics

Math 320 Set Theory Final Exam 4 June 2017 9:30		
Last Name : Name : Student No:	Signature : Duration : 120 *minutes*	

6 QUESTIONS ON 4 PAGES						TOTAL 100 POINTS		
1	2	3	4	5	6	**SHOW YOUR WORK**		

1. (6+7+7=20 pts)

a) Prove or disprove: If κ is an infinite cardinal number, then we have

$$\kappa = sup\{\lambda : \lambda < \kappa \ \wedge \ \text{“}\lambda \text{ is a cardinal number”}\}$$

For Part (b) and Part (c) of this question, you are given that

- $\kappa + \kappa = \kappa \cdot \kappa = \kappa$ for all infinite cardinal κ,
- $\lambda \leq \kappa \longrightarrow \lambda + \theta \leq \kappa + \theta$ for all cardinals λ, κ and θ,
- Cardinal addition is commutative.
- $2 \leq \lambda \leq \kappa \longrightarrow 2^\theta \leq \lambda^\theta \leq \kappa^\theta$ for all cardinals λ, κ and θ,
- $\kappa < 2^\kappa$ for all cardinals κ,
- $(\kappa^\lambda)^\theta = \kappa^{\lambda \cdot \theta}$ for all cardinals κ, λ and θ.

b) Prove that for all infinite cardinals κ and λ we have

$$\kappa + \lambda = \max\{\kappa, \lambda\}$$

c) Prove that for all infinite cardinals κ and λ such that $2 \leq \lambda \leq \kappa$ we have

$$2^\kappa = \lambda^\kappa$$

WARNING: If you use identities or properties involving cardinal arithmetic other than the ones given in the question, you are supposed to prove it.

2. (7+6+6+6=25 pts) Assuming the Generalized Continuum Hypothesis (GCH), find the corresponding $\aleph$ numbers of the following computations in cardinal arithmetic. (You can use **all** identities and theorems we learned in class regarding cardinal arithmetic.)

a) $((\aleph_0 + \aleph_1) \cdot \aleph_2) + 2^{\aleph_0} =$

b) $(\aleph_3 \cdot \aleph_7) + ((\aleph_{\aleph_0})^2 \cdot \aleph_{641}) =$

c) $(\aleph_{\aleph_1})^{\aleph_2} =$

d) $(\aleph_0 + \aleph_{\aleph_0} + \aleph_{\aleph_{\aleph_0}}) \cdot \kappa =$ where κ is a cardinal satisfying $\aleph_\kappa = \kappa$.

3. (12 pts) Let α be a limit ordinal. Prove that $cf(\aleph_\alpha) \leq cf(\alpha)$.

4. (6+12=18 pts) Let $\preceq$ be the relation on $\mathcal{P}(\mathbb{Q})$ defined by $A \preceq B \longleftrightarrow A \subseteq B$ for all $A, B \in \mathcal{P}(\mathbb{Q})$.

a) Prove that $\preceq$ is a partial order relation.

b) Find $\mathcal{C} \subseteq \mathcal{P}(\mathbb{Q})$ such that $|\mathcal{C}| = 2^{\aleph_0}$ and $\mathcal{C}$ is a chain, that is, any two elements of $\mathcal{C}$ are comparable.
Hint. Recall that $|\mathbb{R}| = 2^{\aleph_0}$ and for each $r \in \mathbb{R}$ consider the set of rational numbers less than r.

5. (5+6+4=15 pts)

a) State the Continuum Hypothesis (CH).

b) Recall that the von Neumann hierarchy of sets is defined via transfinite recursion as follows.

- $\mathrm{V}_0 = \emptyset$
- $\mathrm{V}_{\alpha+1} = \mathcal{P}(\mathrm{V}_\alpha)$ for all ordinals α, and
- $\mathrm{V}_\gamma = \bigcup_{\beta<\gamma} \mathrm{V}_\beta$ for all limit ordinals γ.

Using transfinite induction, prove that V_α is transitive for all ordinals α.

c) Give an example of an infinite cardinal number κ such that $cf(\kappa) < \kappa$.

6. (10 pts) Recall that Cantor's theorem states that $|X| < |\mathcal{P}(X)|$ for any set X. Prove Cantor's theorem.