MATH 319 2017-1: Take-Home Assignment

- In this question, we shall prove that
$$\int_{[0,1]} e^{x^2} dm = \sum_{k=0}^{\infty} \frac{1}{k!(2k+1)}$$
Consider the following argument:
$$\begin{aligned}
\int_{[0,1]} e^{x^2} dm &= \int_{[0,1]} \sum_{k=0}^{\infty} \frac{x^{2k}}{k!}\ dm && (1)\\
&= \int_{[0,1]} \lim_{n\to\infty} \sum_{k=0}^{n} \frac{x^{2k}}{k!} dm && (2)\\
&= \lim_{n\to\infty} \int_{[0,1]} \sum_{k=0}^{n} \frac{x^{2k}}{k!} dm && (3)\\
&= \lim_{n\to\infty} \int_0^1 \sum_{k=0}^{n} \frac{x^{2k}}{k!} dx && (4)\\
&= \lim_{n\to\infty} \left(\sum_{k=0}^{n} \frac{x^{2k+1}}{k!(2k+1)} \right) \Bigg|_0^1 && (5)\\
&= \lim_{n\to\infty} \sum_{k=0}^{n} \frac{1}{k!(2k+1)} && (6)\\
&= \sum_{k=0}^{\infty} \frac{1}{k!(2k+1)} && (7)
\end{aligned}$$
Explain briefly why (1) holds and **explain in detail** why we are able to pass from (2) to (3) and from (3) to (4) by referring to related theorems and checking that their conditions hold.

- Consider the measure space $(\mathbb{N}, \mathcal{P}(\mathbb{N}), \nu)$ where ν is the counting measure defined on $\mathcal{P}(\mathbb{N})$ given by

$$\nu(S) = \begin{cases} +\infty & \text{if } S \text{ is infinite} \\ |S| & \text{if } S \text{ is finite} \end{cases}$$

Let $f : \mathbb{N} \to \mathbb{R}$ be **any** positive function.

a. Explain why the function f is $(\mathcal{P}(\mathbb{N}), \mathcal{B}(\mathbb{R}))$-measurable.

b. Find a sequence of simple functions $f_n : \mathbb{N} \to \mathbb{R}$ such that $f_n \leq f_{n+1}$ for all $n \in \mathbb{N}$ and $f_n \to f$ pointwise as $n \to \infty$.

(**Hint.** Consider the functions $f(x)\chi_{\{0,1,2,\ldots,n\}}(x)$. Try to show that these functions are simple and find their standard representations.)

c. Using Part (b) and the Monotone Convergence Theorem, conclude that

$$\int_{\mathbb{N}} f(x) d\nu = \sum_{k=0}^{\infty} f(k)$$

For the rest of this question, we shall work in the product space

$$(\mathbb{R} \times \mathbb{N}, \mathcal{B}(\mathbb{R}) \otimes \mathcal{P}(\mathbb{N}), m \times \nu)$$

where m is the usual Lebesgue measure on $\mathbb{R}$ and ν is the counting measure.

d. Assuming that the function $f(x, y) = |x| \cdot y^2$ from $\mathbb{R} \times \mathbb{N}$ to $\mathbb{R}$ is measurable, explain why we have

$$\begin{aligned} \int_{\mathbb{R}\times\mathbb{N}} f(x,y) d(m \times \nu) &= \int_{\mathbb{N}} \left(\int_{\mathbb{R}} f(x,y) dm \right) d\nu \\ &= \int_{\mathbb{R}} \left(\int_{\mathbb{N}} f(x,y) d\nu \right) dm \end{aligned}$$

by referring to the relevant theorem and checking that its conditions hold.

e. Prove that

$$\int_{[0,1]\times\{1,2,3\}} f(x,y) \; d(m \times \nu) = 7$$

by iterating this integral with respect to dm first and then $d\nu$.

f. Prove that

$$\int_{[0,1]\times\{1,2,3\}} f(x,y) \; d(m \times \nu) = 7$$

by iterating this integral with respect to $d\nu$ first and then dm.

- Throughout this question, we shall work in the product space

$$(\mathbb{R} \times \mathbb{R}, \mathcal{B}(\mathbb{R}) \otimes \mathcal{B}(\mathbb{R}), m \times m)$$

where m denotes the usual Lebesgue measure on $\mathbb{R}$. Let T be the triangle

$$\{(x, y) \in \mathbb{R} \times \mathbb{R} : x > 0 \text{ and } y > 0 \text{ and } x + y < 1\}$$

a. Find rectangles $R_1, R_2, \ldots, R_n \subseteq \mathbb{R} \times \mathbb{R}$ such that

$$\bigcup_{k=1}^{n} R_k \supseteq T \text{ and } (m \times m)\left(\bigcup_{k=1}^{n} R_k\right) = \frac{n+1}{2n}$$

(**Hint.** Think of the (actual) rectangles you would have drawn if one asked you to form a Riemann sum consisting n rectangles with equal widths over the interval $[0, 1]$ using left end points for the function $f(x) = 1 - x$.)

b. Using Part (a), prove that $(m \times m)(T) \leq \frac{1}{2}$