M E T U Department of Mathematics

Math 319 Lebesgue Integral Midterm 3 21 December 2017 17:40		
Last Name : Name : Student No:	Signature : Duration : 85 *minutes*	
4 QUESTIONS ON 2 PAGES	TOTAL 60 POINTS	

1	2	3	4			**SHOW YOUR WORK**		

1. (10+10=20 pts) For any measure space (X, Ω, μ), let $M(X, \Omega, \mu)$ denote the set of real-valued measurable functions from X to $\mathbb{R}$, that is,

$$M(X, \Omega, \mu) = \{f : X \to \mathbb{R} : f \text{ is } (\Omega, \mathcal{B}(\mathbb{R}))\text{-measurable}\}$$

(a) State the definition of almost uniform convergence: Let $(f_n)_{n\in\mathbb{N}}$ be a sequence of functions in $M(X, \Omega, \mu)$ and f be a function in $M(X, \Omega, \mu)$. We say that $f_n \longrightarrow f$ almost uniformly if and only if

(b) Prove or disprove: For every sequence of integrable functions $(f_n)_{n\in\mathbb{N}}$ in $M(X, \Omega, \mu)$ and an integrable function f in $M(X, \Omega, \mu)$, if $f_n \to f$ almost everywhere, then $f_n \longrightarrow f$ in L^1.

2. (14 pts) Let $(f_n)_{n\in\mathbb{N}}$ and $(g_n)_{n\in\mathbb{N}}$ be a sequence of functions in $M(X, \Omega, \mu)$. Prove that if $f_n \longrightarrow 0$ in measure and $g_n \longrightarrow 0$ in measure, then $f_n + g_n \longrightarrow 0$ in measure.

3. (14 pts) Let $\mathcal{B}(\mathbb{R}) \otimes \mathcal{B}(\mathbb{R})$ denote the product σ-algebra on $\mathbb{R} \times \mathbb{R}$. Throughout this question, you are **not** allowed to use the fact that $\mathcal{B}(\mathbb{R}) \otimes \mathcal{B}(\mathbb{R}) = \mathcal{B}(\mathbb{R} \times \mathbb{R})$.

(a) Prove that, for every $r \in \mathbb{R}$ and $\epsilon > 0$, the set

$$\{(x, y) \in \mathbb{R} \times \mathbb{R} : |x - r| < \epsilon \text{ and } |y - r| < \epsilon\}$$

is in the σ-algebra $\mathcal{B}(\mathbb{R}) \otimes \mathcal{B}(\mathbb{R})$.

(b) Prove that the set $\Delta = \{(r, r) \in \mathbb{R} \times \mathbb{R} : r \in \mathbb{R}\}$ is in the σ-algebra $\mathcal{B}(\mathbb{R}) \otimes \mathcal{B}(\mathbb{R})$.
(**Hint**. Try to write Δ as a countable intersection of sets which are countable unions of rectangles. Try to use the sets in part a.)

4. (12 pts) Let X be an uncountable set and consider the measure space (X, Ω, ν) where Ω is the σ-algebra of countable and cocountable sets and the measure $\nu : \Omega \to [0, +\infty]$ is given by

$$\nu(S) = \begin{cases} 1 & \text{if } S \text{ is uncountable} \\ 0 & \text{if } S \text{ is countable} \end{cases}$$

Let $(f_n)_{n \in \mathbb{N}}$ and f be functions from X to $\mathbb{R}$ which are $(\Omega, \mathcal{B}(\mathbb{R}))$-measurable. Prove that if $f_n \to f$ almost everywhere, then $f_n \longrightarrow f$ uniformly on $X - C$ for some countable set C.
(**Hint.** Realize the obvious fact that $\nu(X) < +\infty$.)