M E T U Department of Mathematics

Math 319 Lebesgue Integral Midterm 2 6 December 2017 17:40		
Last Name : Name : Student No:	Signature : Duration : 85 *minutes*	
4 QUESTIONS ON 2 PAGES	TOTAL 60 POINTS	

1	2	3	4			**SHOW YOUR WORK**		

1. (10+10=20 pts) For any measure space (X, Ω, μ), let $L^+(X, \Omega, \mu)$ denote the set of non-negative extended real-valued measurable functions from X to $\bar{\mathbb{R}}$, that is,

$$L^+(X, \Omega, \mu) = \{f : X \to \bar{\mathbb{R}} : 0 \leq f \text{ and } f \text{ is } (\Omega, \mathcal{B}(\bar{\mathbb{R}}))\text{-measurable}\}$$

(a) State the Monotone Convergence Theorem.

(b) Prove or disprove: For any sequence $(f_n)_{n\in\mathbb{N}}$ of functions in $L^+(\mathbb{R}, \mathcal{B}(\mathbb{R}), m)$ with $f_n \geq f_{n+1}$ for all $n \in \mathbb{N}$, we have that

$$\int_{\mathbb{R}} \lim_{n\to\infty} f_n \, dm = \lim_{n\to\infty} \int_{\mathbb{R}} f_n \, dm$$

where m denotes the Lebesgue measure.

2. (12 pts) Prove that the function $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = \max\{n \in \mathbb{Z} : n \leq x\}$ is a Borel measurable function, that is, a $(\mathcal{B}(\mathbb{R}), \mathcal{B}(\mathbb{R}))$-measurable function.

3. (16 pts) Find the limit

$$\lim_{n\to\infty}\int_0^1 f_n(x)dx$$

where $f_n(x) = \begin{cases} \frac{\sqrt[n]{1-x}\ \sin(x^n)}{x^n} & \text{if } 0 < x \le 1 \\ 1 & \text{if } x = 0 \\ 0 & \text{if } x < 0 \text{ or } x > 1 \end{cases}$. Explain and justify each step of your calculation.

4. (12 pts) Let K be a closed subset of $\mathbb{R}$ such that $m(K) = 0$, where m denotes the Lebesgue measure. Prove that the function $h(x)$ given by

$$h(x) = \begin{cases} 154 & \text{if } x \in K \\ 319 & \text{if } x \notin K \end{cases}$$

is Riemann integrable over $[0,1]$ and find its Riemann integral $\int_0^1 h(x)dx$.