M E T U Department of Mathematics

Math 319 Lebesgue Integral Midterm 1 27 October 2017 17:40		
Last Name : Name : Student No:	Signature : Duration : 85 *minutes*	
4 QUESTIONS ON 2 PAGES		TOTAL 60 POINTS

1	2	3	4			SHOW YOUR WORK		

1. (8+12=20 pts)

(a) State the definition of a σ-algebra on a set X. A collection $\Omega \subseteq \mathcal{P}(X)$ is called a σ-algebra if ...

(b) Let Ω be the σ-algebra on $\mathbb{R}$ generated by the collection

$$\{\ [a,b) \cup \{\sqrt{2}\}\ \ : a,b \in \mathbb{R}\ \wedge\ a < b\}$$

Prove that $\mathcal{B}(\mathbb{R}) \subseteq \Omega$, where $\mathcal{B}(\mathbb{R})$ is the Borel σ-algebra of $\mathbb{R}$. (**Hint.** You can use the fact that $\mathcal{B}(\mathbb{R})$ is generated by the collection of sets of the form (a,b) where $a,b \in \mathbb{R}$ and $a < b$.)

2. (12 pts) Let (X, Ω, μ_a) be a measure space such that $\{x\} \in \Omega$ for every $x \in X$, where $a \in X$ is a fixed element and μ_a is the Dirac measure concentrated at a, that is,

$$\mu_a(S) = \begin{cases} 1 & \text{if } a \in S \\ 0 & \text{if } a \notin S \end{cases}$$

Let $(X, \overline{\Omega}, \overline{\mu_a})$ denote the completion of (X, Ω, μ_a). Prove that $\overline{\Omega} = \mathcal{P}(X)$.

3. (8+8 pts) Let X be an uncountable set. Consider the function $\nu^* : \mathcal{P}(X) \to [0, +\infty]$ given by

$$\nu^*(S) = \begin{cases} 1 & \text{if } S \text{ is uncountable} \\ 0 & \text{if } S \text{ is countable} \end{cases}$$

(a) Prove that ν^* is an outer measure.

(b) Prove that if $A \subseteq X$ is ν^*-measurable, then A is either countable or cocountable.

4. (12 pts) Let m^* denote the usual outer measure on $\mathcal{P}(\mathbb{R})$ defined by

$$m^*(A) = \inf\{ \sum_{k=0}^{\infty} |I_k| : A \subseteq \bigcup_{k=0}^{\infty} I_k \text{ and for each k, } I_k = (a_k, b_k) \text{ for some real numbers } a_k < b_k\}$$

where $|I|$ denotes the length of the interval I. Prove that for every positive real number $r \in \mathbb{R}$ and $A \subseteq \mathbb{R}$, we have that $m^*(rA) = r \cdot m^*(A)$, where $rA = \{rx : x \in A\}$.