M E T U Department of Mathematics

Math 319 Lebesgue Integral Final Exam 17 January 2018 13:30								
Last Name : Name : Student No:						Signature : Duration : 140 *minutes*		
6 QUESTIONS ON 4 PAGES						TOTAL 90 POINTS		
1	2	3	4			**SHOW YOUR WORK**		

1. (15 pts) Find the limit

$$\lim_{n\to\infty} \int_{[0,1]} \frac{\sin\left(\frac{x+1}{n}\right)}{\frac{x+1}{n}} \left(1-\frac{x}{n}\right)^n dm$$

where m denotes the Lebesgue measure on $\mathbb{R}$. Carefully explain each step of your calculation and state which theorems you use.

2. (12 pts) For every positive integer n, define f_n to be the Borel measurable function

$$f_n(x) = n^2 \cdot e^{-n} \cdot \chi_{[n,n^2]}(x)$$

Prove that $f_n \to 0$ in L^1 as $n \to \infty$.

3. (7+8=15 pts) (a) State the definition of a σ-finite measure space. A measure space (X, Ω, μ) is said to be σ-finite if

For Part (b) of this question, consider the measure space $(\mathbb{R}, \Omega, \mu)$ where Ω and μ are given as follows:

$$\Omega = \{S \subseteq \mathbb{R} : S \text{ is countable or co-countable}\} \text{ and } \mu(S) = \begin{cases} +\infty & \text{if } S \text{ is uncountable} \\ 0 & \text{if } S \text{ is countable} \end{cases}$$

(b) Prove or disprove: The measure space $(\mathbb{R}, \Omega, \mu)$ is σ-finite.

4. (5+8 pts) Throughout this question, we shall work in the product space

$$(\mathbb{R} \times \mathbb{R}, \mathcal{B}(\mathbb{R}) \otimes \mathcal{B}(\mathbb{R}), m \times m)$$

where m denotes the usual Lebesgue measure on $\mathbb{R}$. Consider the plane region

$$T = \{(x, y) \in \mathbb{R} \times \mathbb{R} : x^2 + 4y^2 < 4 \text{ and } y > 0\}$$

(a) Explain why the set T is an element of the product σ-algebra $\mathcal{B}(\mathbb{R}) \otimes \mathcal{B}(\mathbb{R})$.

(b) Using the definition of the product measure $m \times m$, prove that

$$(m \times m)(T) \leq 2 + \sqrt{3}$$

5. (8+12 pts) (a) Find a sequence of $(\mathcal{P}(\mathbb{N}), \mathcal{B}(\mathbb{R}))$-measurable functions $f_n : \mathbb{N} \to \mathbb{R}$ such that each f_n is simple, $f_n \leq f_{n+1}$ for all $n \in \mathbb{N}$ and $\lim_{n\to\infty} f_n(y) = \pi 2^y$ for all $y \in \mathbb{N}$.

For the rest of this question, we shall work in the product space

$$(\mathbb{R} \times \mathbb{N}, \mathcal{B}(\mathbb{R}) \otimes \mathcal{P}(\mathbb{N}), m \times \mu)$$

where m is the usual Lebesgue measure on $\mathbb{R}$ and μ is the measure defined on $\mathcal{P}(\mathbb{N})$ given by

$$\mu(S) = \sum_{n \in S} \frac{1}{3^n}$$

(b) You are given that the function $f(x,y) = \dfrac{2^y}{1+x^2}$ from the product space $\mathbb{R} \times \mathbb{N}$ to $\mathbb{R}$ is measurable. Evaluate the integral

$$\int_{\mathbb{R}\times\mathbb{N}} \frac{2^y}{1+x^2} \, d(m \times \mu)$$

Justify each step of your calculation.

6. (3+6+6 (+6) pts) Consider the Borel measurable function

$$f(x) = \begin{cases} \dfrac{(-1)^{\lfloor x \rfloor}}{\lfloor x \rfloor + 1} & \text{if } x \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

where $\lfloor x \rfloor$ is the floor function (the greatest integer function), that is, $\lfloor x \rfloor = \max\{n \in \mathbb{Z} : n \leq x\}$.

(a) Roughly sketch the graph of $f(x)$.

(b) Find a sequence $(f_n)_{n\in\mathbb{N}}$ of simple functions such that $f_n \leq f_{n+1}$ for all $n \in \mathbb{N}$ and $f_n \to |f|$ pointwise as $n \to \infty$.

(c) Determine whether or not f is a (Lebesgue) integrable function.
(**Hint.** Consider the following question first: Is $|f|$ integrable?)

(d) (**Bonus.**) Determine whether or not the improper Riemann integral $\displaystyle\int_0^\infty f(x)dx$ exists.