************* PLEASE WRITE YOUR NAME CLEARLY USING CAPITAL LETTERS *************		
F U L L N A M E	S T U D E N T I D	DURATION: 70 MINUTES 5 QUESTIONS ON 2 PAGES TOTAL: 40 POINTS

***(8+2+2=12 pts)* 1.** Consider the relation $\sim$ on $\mathbb{R}^2$ given by

$$(x_1, y_1) \sim (x_2, y_2) \text{ if and only if } x_1 \cdot y_1 = x_2 \cdot y_2$$

for all $(x_1, y_1), (x_2, y_2) \in \mathbb{R}^2$.

(a) Show that $\sim$ is an equivalence relation.

Let $(x, y) \in \mathbb{R}^2$. Then $x \cdot y = x \cdot y$ and hence, $(x, y) \sim (x, y)$. Therefore $\sim$ is reflexive.

Let $(x, y), (\hat{x}, \hat{y}) \in \mathbb{R}^2$. Assume that $(x, y) \sim (\hat{x}, \hat{y})$. Then, by definition, $x \cdot y = \hat{x} \cdot y$ and hence, $\hat{x} \cdot \hat{y} = x \cdot y$ which shows that $(\hat{x}, \hat{y}) \sim (x, y)$. Therefore $\sim$ is symmetric.

Let $(x, y), (\hat{x}, \hat{y}), (\overline{x}, \overline{y}) \in \mathbb{R}^2$. Assume that $(x, y) \sim (\hat{x}, \hat{y})$ and $(\hat{x}, \hat{y}) \sim (\overline{x}, \overline{y})$. Then, by definition, $x \cdot y = \hat{x} \cdot \hat{y}$ and $\hat{x} \cdot \hat{y} = \overline{x} \cdot \overline{y}$. It follows that $x \cdot t = \overline{x} \cdot \overline{y}$ and hence, $(x, y) \sim (\overline{x}, \overline{y})$. Therefore $\sim$ is transitive, which completes the proof that $\sim$ is an equivalence relation.

(b) Describe the set $[(0, 0)]$ in the plane $\mathbb{R}^2$ geometrically.

$[(0, 0)] = \{(x, y) \in \mathbb{R}^2 : (0, 0) \sim (x, y)\} = \{(x, y) \in \mathbb{R}^2 : x \cdot y = 0\} = \{(x, y) \in \mathbb{R}^2 : x = 0 \text{ or } y = 0\}$. Therefore $[(0, 0)]$ is the union of the coordinate axes in $\mathbb{R}^2$.

(c) Find $(a, b) \in \mathbb{R}^2$ such that $[(a, b)] = \left\{(x, y) \in \mathbb{R}^2 \;\middle|\; y = \frac{2}{x}\right\}$.

Choose $(a, b) = (2, 1)$. Then we have that

$$[(2, 1)] = \{(x, y) \in \mathbb{R}^2 : (2, 1) \sim (x, y)\} = \{(x, y) \in \mathbb{R}^2 : x \cdot y = 2\} = \left\{(x, y) \in \mathbb{R}^2 \;\middle|\; y = \frac{2}{x}\right\}$$

***(6 pts)* 2.** Let A be a set and $f : A \to A$ be a function. Show that if f has a left inverse, then $f \circ f$ also has a left inverse.

Solution I: Assume that f has a left inverse, say, $g : A \to A$ is a left inverse for f, that is, $g \circ f = 1_A$. Consider the map $g \circ g : A \to A$. Then, for any $x \in A$, we have that

$$\begin{aligned}((g \circ g) \circ (f \circ f))(x) &= (g \circ g)((f \circ f)(x)) \\ &= (g \circ g)(f(f(x))) \\ &= g(g(f(f(x)))) = g((g \circ f)(f(x))) = g(1_A(f(x))) = g(f(x)) = (g \circ f)(x) = 1_A(x)\end{aligned}$$

Therefore $g \circ g$ is a left inverse for $f \circ f$.

Solution II: Assume that f has a left inverse. Then f is one-to-one. Consequently, $f \circ f$ is one-to-one and hence, has a left inverse.

(3+3+5=11 pts) **3.** Let $f : \mathbb{N} \times \mathbb{N} \to \mathbb{Z}$ be the function defined by $f(m, n) = m - n$ for all $m, n \in \mathbb{N}$.
(a) Is f injective? Explain.

$f(1, 1) = f(2, 2) = 0$ but $(1, 1) \neq (2, 2)$. Therefore f is not one-to-one.

(b) Is f surjective? Explain.

Let $k \in \mathbb{Z}$. Choose $(m, n) = (k+1, 1) \in \mathbb{N} \times \mathbb{N}$ if $k \geq 0$; and choose $(m, n) = (1, 1-k) \in \mathbb{N} \times \mathbb{N}$ if $k < 0$. Then, if $k \geq 0$, then $f(m, n) = f(k+1, 1) = k$; and if $k < 0$, then $f(m, n) = f(1, 1-k) = 1-(1-k) = k$. Hence $f(m, n) = k$. It follows that f is surjective.

(c) Does f have a right inverse? If so, find a right inverse for f. If not, explain why f does not have a right inverse.

Since f is surjective, then f has a right inverse. We shall now find this right inverse. Consider the map $g : \mathbb{Z} \to \mathbb{N} \times \mathbb{N}$ given by

$$g(k) = \begin{cases} (k+1, 1) & \text{if } k \geq 0 \\ (1, 1-k) & \text{if } k < 0 \end{cases}$$

Let $k \in \mathbb{Z}$. If $k \geq 0$, then we have $(f \circ g)(k) = f(g(k)) = f(k+1, 1) = k + 1 - 1 = k$. If $k < 0$, then we have $(f \circ g)(k) = f(g(k)) = f(1, 1-k) = 1 - (1-k) = k$. Therefore, $(f \circ g)(k) = k = 1_{\mathbb{Z}}(k)$, which shows that g is a right inverse for f.

(6 pts) **4.** Let A be a set and let R be a relation on A with the following properties:

- R is reflexive
- For any $x, y, z \in A$, if xRy and yRz, then zRx.

Prove that R is an equivalence relation.

We need to show that R is reflexive, symmetric and transitive. That R is reflexive is given as the first assumption. Let $x, y \in A$. Assume that xRy. As R is reflexive, xRx. Then, since xRx and xRy, by the second assumption, we have that yRx. Thus R is symmetric.

Now let $x, y, z \in A$. Assume that xRy and yRz. Then, by the second assumption, we have zRx. As R is symmetric, this gives xRz. Thus R is transitive, which completes the proof that R is an equivalence relation.

(5 pts) **5.** Let $A = \{1, 2, 3, 4, 5\}$ and let $f : A \to A$ be a function with the following properties:

1. $f^{-1}(\ \{1, 2, 3\}\ ) = \{4, 5\}$
2. $f^{-1}(\{3\}) = \emptyset$
3. $f(4) \neq 1$
4. $f^{-1}(\ \{1\}\ ) \neq \emptyset$
5. $f(\ \{1, 2\}\ ) = \{4\}$
6. $f^{-1}(\ \{5\}\ ) = \emptyset$

Find the values:

$f(1) = 4$ $\quad$ $f(2) = 4$ $\quad$ $f(3) = 4$ $\quad$ $f(4) = 2$ $\quad$ $f(5) = 1$

(For this part only, you DO NOT need to justify your answer.)

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By signing below, I pledge that I will write this examination as my own work and without the assistance of others or the usage of unauthorized material or information. I understand that possession of any kind of electronic device during the exam is prohibited. I also understand that not obeying the rules of the examination will result in immediate cancellation and disciplinary procedures.

Signature ....

(8+2+2=12 pts) **1.** Consider the relation $\sim$ on $\mathbb{R}^2$ given by

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Let $(x, y), (\hat{x}, \hat{y}) \in \mathbb{R}^2$. Assume that $(x, y) \sim (\hat{x}, \hat{y})$. Then, by definition, $x \cdot y = \hat{x} \cdot y$ and hence, $\hat{x} \cdot \hat{y} = x \cdot y$ which shows that $(\hat{x}, \hat{y}) \sim (x, y)$. Therefore $\sim$ is symmetric.

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(c) Find $(a, b) \in \mathbb{R}^2$ such that $[(a, b)] = \left\{(x, y) \in \mathbb{R}^2 \;\middle|\; y = \frac{2}{x}\right\}$.

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(c) Does f have a right inverse? If so, find a right inverse for f. If not, explain why f does not have a right inverse.

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Let $k \in \mathbb{Z}$. If $k \geq 0$, then we have $(f \circ g)(k) = f(g(k)) = f(k+1,1) = k+1-1 = k$. If $k < 0$, then we have $(f \circ g)(k) = f(g(k)) = f(1,1-k) = 1-(1-k) = k$. Therefore, $(f \circ g)(k) = k = 1_{\mathbb{Z}}(k)$, which shows that g is a right inverse for f.

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Now let $x,y,z \in A$. Assume that xRy and yRz. Then, by the second assumption, we have zRx. As R is symmetric, this gives xRz. Thus R is transitive, which completes the proof that R is an equivalence relation.

***(5 pts)* 5.** Let $A = \{1,2,3,4,5\}$ and let $g : A \to A$ be a function with the following properties:

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3. $g(2) \neq 5$
4. $g^{-1}(\ \{5\}\ ) \neq \emptyset$
5. $g(\ \{4,5\}\ ) = \{2\}$
6. $g^{-1}(\ \{1\}\ ) = \emptyset$

Find the values:

$g(1) = 5$ $\quad$ $g(2) = 4$ $\quad$ $g(3) = 2$ $\quad$ $g(4) = 2$ $\quad$ $g(5) = 2$

(For this part only, you DO NOT need to justify your answer.)