METU Department of Mathematics - Math 111 - Midterm II - November 15, 2019

PLEASE WRITE YOUR NAME CLEARLY USING CAPITAL LETTERS		
F U L L N A M E	S T U D E N T I D	DURATION 70 MINUTES
5 QUESTIONS ON 2 PAGES		TOTAL 40 POINTS

By signing below, I pledge that I will write this examination as my own work and without the assistance of others or the usage of unauthorized material or information. I understand that possession of any kind of electronic device during the exam is prohibited. I also understand that not obeying the rules of the examination will result in immediate cancellation and disciplinary procedures.

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***(3 pts)* 1.** Prove or disprove: If a, b, c are integers such that $a|c$ and $b|c$, then $ab|c$.

We shall disprove this statement by providing a counterexample. Let $a = b = c = 2$. Then $2|2$ and $2|2$ but $2 \cdot 2 = 4 \nmid 2$. Therefore, this statement is false.

***(9 pts)* 2.** Using a proof by **contrapositive**, prove that if $A \subseteq B$, then $\mathcal{P}(A) \subseteq \mathcal{P}(B)$.

Assume that $\mathcal{P}(A) \nsubseteq \mathcal{P}(B)$. Then, by definition, there exists a set $C \in \mathcal{P}(A)$ such that $C \notin \mathcal{P}(B)$. Since $C \notin \mathcal{P}(B)$, we have $C \nsubseteq B$ and hence there exists $c \in C$ such that $c \notin B$. On the other hand, since $C \in \mathcal{P}(A)$, we have $C \subseteq A$ and consequently $c \in A$. Since $c \in A$ and $c \notin B$, we have that $A \nsubseteq B$.

***(8 pts)* 3.**

Prove that there exists a unique set W such that for every set A we have $W \cup A = A$.

We first show the existence of such a set. Choose $W = \emptyset$. Then, by properties of the empty set, for any set A, we have that $\emptyset \cup A = A$.

We will now show that such a set is unique and any such set necessarily equals the empty set. Let $\overline{W}$ be a set with the property that for every set A, we have $\overline{W} \cup A = A$. Since this statement is true for any A, it is true for $A = \emptyset$ and hence $\overline{W} \cup \emptyset = \emptyset$. On the other hand, it follows from the properties of the empty set that $\overline{W} \cup \emptyset = \overline{W}$. Therefore, $\overline{W} = \emptyset$.

***(8pts)* 4.** Let A and B be sets such that $A \subsetneq B$ (that is, A is a proper subset of B) and let C be a set such that $A \times C = B \times C$. Prove that $C = \emptyset$.

We shall show this via a proof by contradiction. Assume towards a contradiction that $C \neq \emptyset$. Then, C being non-empty, there exists $c \in C$. Since $A \subsetneq B$, there exists $b \in B$ such that $b \notin A$. Then, we have that $(b, c) \in B \times C$ and $(b, c) \notin A \times C$. Therefore $A \times C \neq B \times C$, which contradicts the given assumption.

***(12 pts)* 5.** Let A and B be two subsets of a set C. Show that the following statements are equivalent by proving the implications $a \Rightarrow b \Rightarrow c \Rightarrow a$.

(a) $A \cup B = C$

(b) $(C - A) \cap (C - B) = \emptyset$

(c) $(C - A) \subseteq B$

$(a \Rightarrow b)$:
We shall prove this via a proof by contradiction. Assume that $A \cup B = C$ and $(C - A) \cap (C - B) \neq \emptyset$. Since $(C - A) \cap (C - B) \neq \emptyset$, there exists $c \in (C - A) \cap (C - B)$. This means that $c \in C - A$ and $c \in C - B$, which imply that $c \in C$ and $c \notin A$ and $c \notin B$. But then, $c \in C$ and $c \notin A \cup B$, which contradicts the assumption that $A \cup B = C$.

$(b \Rightarrow c)$:
Assume that $(C - A) \cap (C - B) = \emptyset$. Let $x \in C - A$. Since $x \notin \emptyset = (C - A) \cap (C - B)$, we have that $x \notin C - A$ or $x \notin C - B$. On the other hand, we know that $x \in C - A$. Therefore, we have that $x \in C - B$. This shows that $C - A \subseteq C - B$.

$(c \Rightarrow a)$:
Assume that $(C - A) \subseteq B$. We wish to show that $A \cup B = C$. We shall do this by showing the inclusions $A \cup B \subseteq C$ and $C \subseteq A \cup B$.

Since $A \subseteq C$ and $B \subseteq C$, we have that $A \cup B \subseteq C$ by a theorem proven in class. We now show the other inclusion. Let $c \in C$. We split into two cases.

Case 1 $(c \in A)$. Assume that $c \in A$. Then, as $A \subseteq A \cup B$, we have $c \in A \cup B$.
Case 2 $(c \notin A)$. Assume that $c \notin A$. Then, we have $c \in C - A$. It now follows from our initial assumption that $c \in B$. But then, as $B \subseteq A \cup B$, we have $c \in A \cup B$.

In all cases, we reached $c \in A \cup B$. Therefore, $c \in A \cup B$. This completes the proof that $C \subseteq A \cup B$.