************* PLEASE WRITE YOUR NAME CLEARLY USING CAPITAL LETTERS *************		
FULL NAME	STUDENT ID	DURATION: 120 MINUTES 8 QUESTIONS ON 4 PAGES TOTAL: 80 POINTS

By signing below, I pledge that I will write this examination as my own work and without the assistance of others or the usage of unauthorized material or information. I understand that possession of any kind of electronic device during the exam is prohibited. I also understand that not obeying the rules of the examination will result in immediate cancellation and disciplinary procedures.

Signature ....

***(5 pts)* 1.** Let $\mathbb{N} = \{1, 2, 3, \dots\}$ and let $\preceq$ be the relation on $\mathbb{N} \times \mathbb{N}$ given by

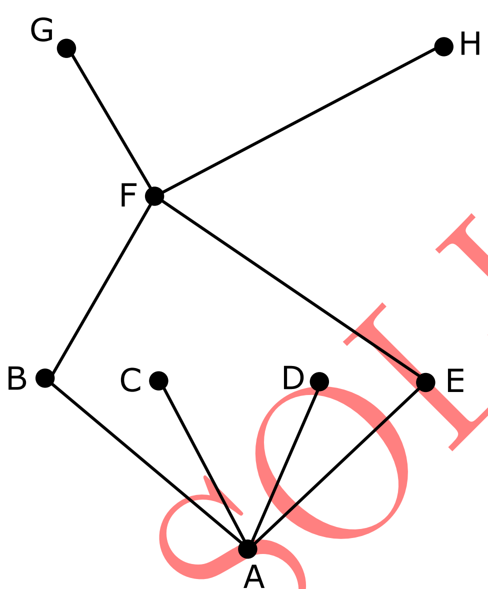
$$(m_1, n_1) \preceq (m_2, n_2) \text{ if and only if } m_1 \leq m_2 \text{ and } n_1 | n_2$$

for all $(m_1, n_1), (m_2, n_2) \in \mathbb{N} \times \mathbb{N}$. Show that $\preceq$ is anti-symmetric.

Let $(m, n), (m', n') \in \mathbb{N} \times \mathbb{N}$. Assume that $(m, n) \preceq (m', n')$ and $(m', n') \preceq (m', n')$. By the definition of $\preceq$, we have that $m \leq m'$ and $n|n'$, and, $m' \leq m$ and $n'|n$.

Since $m \leq m'$ and $m' \leq m$, we have $m = m'$. Since $n|n'$ and $n'|n$, we have that $n = n'k$ and $n' = n\ell$ for some integers $k, l \in \mathbb{Z}$. But then, $n = nk\ell$ and so $1 = k\ell$. This implies that $k = 1$ or $k = -1$. On the other hand, as $n, n' > 0$, we can't have $k = -1$. Hence $k = 1$ and consequently, $n = n'$. Therefore $(m, n) = (m', n')$. So $\preceq$ is anti-symmetric.

***(4×2+4=12 pts)* 2.** Consider the partial order relation $\preceq$ on the set $X = \{A, B, C, D, E, F, G, H\}$ whose Hasse diagram is given below.



a) If they exist, find the following elements of X. (For this part only, you do **not** need to justify your answer. Also **no** partial credits will be given.)

- Maximal element(s) G,H,C,D
- Minimal element(s) A
- Least element A
- Greatest lower bound of the subset $\{G, C\}$ A

b) Is $(X, \preceq)$ a linearly (totally) ordered set? Explain.

If $\preceq$ were a linear order relation, then for any $x, y \in X$ we would have $y \preceq x$ or $x \preceq y$. On the other hand, as it can be seen from the Hasse diagram, we have $C \not\preceq D$ and $D \not\preceq C$. Hence $\preceq$ is not a linear order relation.

(6+6=12 pts) **3.** The parts of this question are **unrelated**.

a) Let A, B, C be sets. Prove that if $A \preccurlyeq B$ and $B \sim C$, then $A \preccurlyeq C$.

Assume that $A \preccurlyeq B$ and $B \sim C$. Then, by definition, there exist an injection $f : A \to B$ and a bijection $g : B \to C$. Consider the function $g \circ f : A \to C$. Since the composition of two injections is an injection, the map $g \circ f$ is an injection. Hence $A \preccurlyeq C$.

b) Is $(\mathbb{R} \times \mathbb{R}) - (\mathbb{Q} \times \mathbb{Q})$ countable? Explain.

No, it is not. Assume towards a contradiction that $(\mathbb{R} \times \mathbb{R}) - (\mathbb{Q} \times \mathbb{Q})$ is countable. Since the cartesian product of two countable sets is countable and $\mathbb{Q}$ is countable, $\mathbb{Q} \times \mathbb{Q}$ is countable. But then, as the union of two countable sets is countable, $(\mathbb{Q}\times\mathbb{Q}) \cup ((\mathbb{R}\times\mathbb{R}) - (\mathbb{Q}\times\mathbb{Q})) = \mathbb{R}\times\mathbb{R}$ is countable.

Note that $\mathbb{R} \preccurlyeq \mathbb{R} \times \mathbb{R}$ because the map $f : \mathbb{R} \to \mathbb{R} \times \mathbb{R}$ given by $f(x) = (x, 0)$ is an injection. Then $f(\mathbb{R})$ is countable as it is a subset of a countable set. But now, because $\mathbb{R} \sim f(\mathbb{R})$, we have that $\mathbb{R}$ is countable, which is a contradiction. Thus $(\mathbb{R} \times \mathbb{R}) - (\mathbb{Q} \times \mathbb{Q})$ is uncountable.

In the next question, you are supposed to express the sets given in the question using the **set notation**. If you do not remember what this notation is, consider the following example.

Example. The set of all real numbers whose square is greater than 3 or whose square is equal to 2 is

$$\{x \in \mathbb{R} \mid x^2 > 3 \ \vee \ x^2 = 2\}$$

(3+3+3=9 pts) **4.** Express the following sets in the set notation.

a) The set of points in $\mathbb{R}^2$ which lie below the x-axis is

$$\left\{(x, y) \in \mathbb{R}^2 \middle| y < 0\right\}$$

b) The set of integers which are divisible by 2 **or** which divide every integer is

$$\left\{k \in \mathbb{Z} \middle| \ 2|k \quad \vee \quad \forall \ell \in \mathbb{Z} \ \ k|\ell\right\}$$

c) The set of subsets of $\mathbb{R}$ which contain an irrational number

$$\left\{S \in \mathcal{P}(\mathbb{R}) \middle| \ \exists r \in \mathbb{R} - \mathbb{Q} \quad r \in S\right\}$$

<u>***(10 pts)* 5.**</u> Prove that

$$2 \cdot 2^1 + 3 \cdot 2^2 + 4 \cdot 2^3 + \cdots + (n+1) \cdot 2^n = n \cdot 2^{n+1}$$

for all natural numbers $n \geq 1$.

We shall prove this by induction on n.

Base step. We have $2 \cdot 2^1 = 4 = 1 \cdot 2^{1+1}$ and so the claim holds for $n = 1$.

Inductive step. Let $n \geq 1$ be a natural number. Suppose that the claim holds for n, that is,

$$2 \cdot 2^1 + 3 \cdot 2^2 + 4 \cdot 2^3 + \cdots + (n+1) \cdot 2^n = n \cdot 2^{n+1}$$

By adding $(n+2) \cdot 2^{n+1}$ to both sides, we get that

$$\begin{aligned} 2 \cdot 2^1 + 3 \cdot 2^2 + 4 \cdot 2^3 + \cdots + (n+1) \cdot 2^n + (n+2) \cdot 2^{n+1} &= n \cdot 2^{n+1} + (n+2) \cdot 2^{n+1} \\ &= n \cdot 2^{n+1} + n \cdot 2^{n+1} + 2 \cdot 2^{n+1} \\ &= n \cdot 2 \cdot 2^{n+1} + 2^{n+2} \\ &= n \cdot 2^{n+2} + 2^{n+2} \\ &= (n+1) \cdot 2^{(n+1)+1} \end{aligned}$$

Hence the claim holds for $n + 1$.

It now follows from the principle of mathematical induction that the claim holds for all natural numbers $n \geq 1$.

<u>***(6+6=12 pts)* 6.**</u> The parts of this question are **unrelated**.

a) Let A, B be sets and let $f : A \to B$ be a function. Complete the following definition:

f is said to be surjective if For all $b \in B$ there exists $a \in A$ such that $f(a) = b$

b) Let $\mathbb{N} = \{1, 2, 3, \dots\}$ and let $g : \mathbb{N} \times \mathbb{Z} \to \mathbb{Z}$ be the function given by

$$g(n, k) = 2^n + n + k$$

for all $n \in \mathbb{N}$ and for all $k \in \mathbb{Z}$. Prove that g is surjective.

Let $\ell \in \mathbb{Z}$. Choose $(n, k) = (1, \ell - 3)$. Then $(n, k) \in \mathbb{N} \times \mathbb{Z}$ and moreover,

$$g(n, k) = g(1, \ell - 3) = 2^1 + 1 + \ell - 3 = \ell$$

Therefore g is surjective.

(5+5+5=15 pts) **7.** You are given some proofs of theorems below. These given proofs **MAY BE CORRECT OR INCORRECT**. If a proof is correct, then write only "the proof is correct". If a proof is incorrect, then write "the proof is incorrect" and briefly **explain the mistake** in the proof.

Theorem. Let A, B, C be sets. If $A \nsubseteq C$ and $B \subseteq C$, then $A \nsubseteq B$.

Proof. Assume that $A \nsubseteq C$ and $B \subseteq C$. Let $x \in A$ be arbitrary. Since $A \nsubseteq C$, we have that $x \notin C$. On the other hand, as $B \subseteq C$, every element of B is in C and hence, $x \notin C$ implies that $x \notin B$. We have found x such that $x \in A$ and $x \notin B$. Thus $A \nsubseteq B$.

Answer: The proof is incorrect. The mistake in the proof is that $x \notin C$ does not follow. The reason is that, $A \nsubseteq C$ implies that there exists **some** element $w \in A$ with $w \notin C$. However, since we picked $x \in A$ to be arbitrary, we cannot deduce that $x \notin C$.

Theorem. Let A, B, C be sets. If $C - A \subseteq B$, then $C \subseteq A \cup B$.

Proof. Assume that $C - A \subseteq B$. Let $x \in C - A$. Then, since $C - A \subseteq B$, we have that $x \in B$. On the other hand, by properties of union, $B \subseteq A \cup B$ and so, $x \in B$ implies that $x \in A \cup B$. This means that $C \subseteq A \cup B$.

Answer: The proof is incorrect. The mistake in the proof is that, because we picked an element x from $C - A$ and showed that $x \in B \cup C$, we can only deduce $C - A \subseteq A \cup B$ at the end, not that $C \subseteq A \cup B$. In a correct proof, we should have picked an element x from C and then argued that $x \in A \cup B$.

Theorem. Let $\backsim$ be the relation on $\mathbb{R}^2$ given by

$$(x_1, y_1) \backsim (x_2, y_2) \text{ if and only if } x_1 \cdot y_1 = x_2 \cdot y_2$$

for all $(x_1, y_1), (x_2, y_2) \in \mathbb{R}^2$. Then $\backsim$ is symmetric.

Proof. Let $(x, y), (\hat{x}, \hat{y}) \in \mathbb{R}^2$. Assume that $(x, y) \backsim (\hat{x}, \hat{y})$ and $(\hat{x}, \hat{y}) \backsim (x, y)$. Then, by definition, we have $x \cdot y = \hat{x} \cdot \hat{y}$ since $(x, y) \backsim (\hat{x}, \hat{y})$. Similarly, we have $\hat{x} \cdot \hat{y} = x \cdot y$ as $(\hat{x}, \hat{y}) \backsim (x, y)$. Thus $\backsim$ is symmetric.

Answer: The proof is incorrect. The mistake in the proof is that, at the beginning, the proof starts with both $(x, y) \backsim (\hat{x}, \hat{y})$ and $(\hat{x}, \hat{y}) \backsim (x, y)$ as assumptions. However, in order to show symmetry, we should only assume $(x, y) \backsim (\hat{x}, \hat{y})$ and then **conclude** that $(\hat{x}, \hat{y}) \backsim (x, y)$.

(5 pts) **8.** The following proof is a **correct** proof of a theorem whose statement is left incomplete. Complete the statement of the theorem appropriately.

Theorem. Let X, Y, Z be sets. ...... If $Y - X = \emptyset$, then $X \nsubseteq Z$ or $Y \subseteq Z$. ......

Proof. We shall prove this statement by contrapositive. Suppose that $X \subseteq Z$ and $Y \nsubseteq Z$. By the assumption that $Y \nsubseteq Z$, there exists $y \in Y$ such that $y \notin Z$. If it were the case that $y \in X$, then, as $X \subseteq Z$, we would have $y \in Z$, which leads to a contradiction. Therefore $y \notin X$. Since $y \in Y$ and $y \notin X$, we have that $y \in Y - X$. Hence $Y - X \neq \emptyset$.