M E T U Department of Mathematics

Math 111 Fundamentals of Mathematics Fall 2018 Final Exam 15.01.2019 13:30		
Last Name : Name : Student No :	Section : Duration : 120 *minutes*	
7 QUESTIONS ON 4 PAGES		TOTAL 80 POINTS

1	2	3	4	5	6	7			

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Signature ....

(5+5+5 pts) **1.** The parts of this question are unrelated.

a) Let A, B be sets such that $A \subseteq B$. Show that if B is uncountable and A is countable, then $B - A$ is uncountable.

Assume towards a contradiction that B is uncountable and A is countable, and $B - A$ is countable. Since $A \subseteq B$, we have that $B = A \cup (B - A)$. Because the union of two countable sets is countable and, A and $B - A$ are both countable, we have that B is countable, which is a contradiction.

b) Show that the following statement is false:

For all sets A and B, if A and B are uncountable, then $A \cap B$ is uncountable.

We shall give a counterexample to this statement. Choose $A = (0, 1)$ and $B = (-1, 0)$. Then, we have that $A \sim B \sim \mathbb{R}$ and hence both A and B are uncountable. On the other hand, $A \cap B = \emptyset$, which is countable.

c) Prove that for any set A and B, we have that $\mathcal{P}(A) \cap \mathcal{P}(B) \subseteq \mathcal{P}(A \cap B)$.

Let A and B be sets. We will prove that $\mathcal{P}(A) \cap \mathcal{P}(B) \subseteq \mathcal{P}(A \cap B)$.

Let $S \in \mathcal{P}(A) \cap \mathcal{P}(B)$. Then we have that $S \in \mathcal{P}(A)$ and $S \in \mathcal{P}(B)$, and so $S \subseteq A$ and $S \subseteq B$. It follows from $S \subseteq A$ and $S \subseteq B$ that $S \subseteq A \cap B$, and hence $S \in \mathcal{P}(A \cap B)$.

***(5+5+5 pts)* 2.** Let $X = \{0, 1, 2, 3, 4, \dots\}$, $[0, \infty) = \{x \in \mathbb{R} : 0 \leq x\}$ and $(0, 1) = \{x \in \mathbb{R} : 0 < x < 1\}$. Consider the function

$$f : X \times (0, 1) \to [0, \infty)$$

defined by $f(n, r) = n + r$ for all $(n, r) \in X \times (0, 1)$.

a) Show that the function f is injective.

Let (n, r) and (m, s) be elements of $X \times (0, 1)$. Assume that $f(n, r) = f(m, s)$. Then we have that $n + r = m + s$ and so $n - m = s - r$. Since m and n are integers, $n - m$ is an integer. Since $0 < r, s < 1$, we have that $-1 < s - r = n - m < 1$. This implies that $n - m = 0$ and hence $n = m$. Consequently, $s = r$ and so $(n, r) = (m, s)$. This shows that f is injective.

b) Show that the function f is not surjective.

Notice that $0 \in [0, \infty)$. We claim that there exists no $(n, r) \in X \times (0, 1)$ such that $f(n, r) = 0$ which will show that f is not surjective. Let $(n, r) \in X \times (0, 1)$. Then $f(n, r) = n + r > 0 = 0$. Hence $f(n, r) \neq 0$.

c) You are **given** the fact that there exists an injective function $g : [0, \infty) \to X \times (0, 1)$. Show that $X \times (0, 1)$ and $[0, \infty)$ have the same cardinality.

We have proven in (a) that there exists an injection $f : X \times (0, 1) \to [0, \infty)$ and we are given that there exists an injection $g : [0, \infty) \to X \times (0, 1)$. By Cantor-Schröder-Bernstein theorem, there exists a bijection between $X \times (0, 1)$ and $[0, \infty)$ which shows that these sets have the same cardinality.

***(5 pts)* 3.** Let A, B, C, D be sets and $f : A \to B$ and $g : C \to D$ be functions. Consider the function $h : A \times C \to B \times D$ given by $h(a, c) = (f(a), g(c))$ for all $(a, c) \in A \times C$. Show that if f and g are surjective, then h is surjective.

Assume that f and g are surjective. Let $(b, d) \in B \times D$. Then $b \in B$ and $d \in D$. Since f and g are surjective, there exist $a \in A$ and $c \in C$ such that $f(a) = b$ and $g(c) = d$. Note that $(a, c) \in A \times C$ and $h(a, c) = (f(a), g(c)) = (b, d)$. Thus h is surjective.

(5 pts) **4.** Using **induction**, prove that

$$1 + 3 + 5 + \ldots + (2n - 1) = n^2 \ \text{ for all natural numbers } \ n \geq 2.$$

Base case. We have that $1 + (2 \cdot 2 - 1) = 1 + 3 = 4 = 2^2$, and hence the claim holds for $n = 2$.

Inductive step. Let $n \geq 2$ be an integer. Assume that $1+3+5+\ldots+(2n-1) = n^2$. Then, by adding $2n + 1$ to both sides, we have that

$$1 + 3 + 5 + \ldots + (2n - 1) + (2n + 1) = n^2 + (2n + 1)$$

$$1 + 3 + 5 + \ldots + (2n - 1) + (2(n + 1) - 1) = (n + 1)^2$$

and hence the claim holds for $n+1$. It follows from the principle of induction that $1+3+5+\ldots+(2n-1) = n^2$ for all natural numbers $n \geq 2$.

(10+5 pts) **5.** Let $\mathbb{N} = \{1, 2, 3, \ldots\}$ be the set of natural numbers. Consider the relation $\sim$ on $\mathbb{N}$ defined by

$$x \sim y \qquad \text{if and only if} \qquad \frac{y}{x} = 2^i \quad \text{for some } i \in \mathbb{Z}$$

for all $x, y \in \mathbb{N}$.

a) Show that $\sim$ is an equivalence relation.

Let $x \in \mathbb{N}$. Then we have that $\frac{x}{x} = 2^0$ and $0 \in \mathbb{Z}$. Thus, by definition, $x \sim x$. Hence, $\sim$ is reflexive.

Let $x, y \in \mathbb{N}$. Assume that $x \sim y$. Then, by definition, $\frac{x}{y} = 2^i$ for some $i \in \mathbb{Z}$. This implies that $\frac{y}{x} = \frac{1}{2^i} = 2^{-i}$ and hence $y \sim x$. Therefore, $\sim$ is symmetric.

Let $x, y, z \in \mathbb{N}$. Assume that $x \sim y$ and $y \sim z$. Then, by definition, $\frac{x}{y} = 2^i$ and $\frac{y}{z} = 2^j$ for some $i, j \in \mathbb{Z}$. It follows that $\frac{x}{z} = \frac{x}{y} \cdot \frac{y}{z} = 2^i \cdot 2^j = 2^{i+j}$. Hence, $x \sim z$ and so $\sim$ is transitive.

a) Construct a bijection from $\mathbb{N}$ to $[1]$.

We have that

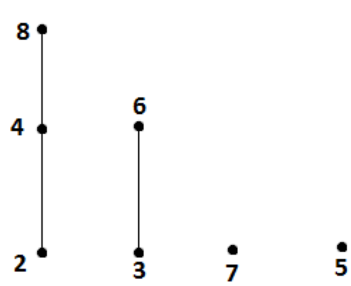
$$[1] = \{n \in \mathbb{N} : 1 \sim n\} = \{n \in \mathbb{N} : \exists i \in \mathbb{Z} \ \ \frac{n}{1} = 2^i\} = \{1, 2, 4, 8, \ldots\}$$

Consider the function $f : \mathbb{N} \to [1]$ given by $f(i) = 2^{i-1}$ for all $i \in \mathbb{N}$. Then f is a bijection.

(5 pts) **6.** Let $X = \{2, 3, 4, 5, 6, 7, 8\}$. Consider the relation $\preceq$ on the set X defined by

$$x \preceq y \qquad \text{if and only if} \qquad \frac{y}{x} = 2^i \quad \text{for some } i \in \mathbb{N} \cup \{0\}$$

for all $x, y \in X$. You are **given** that $\preceq$ is a partial order relation. Draw the Hasse diagram of $\preceq$.



***(5+5+5+5 pts)* 7.** The following proofs are **incorrect**. Briefly explain the mistake in each of the following proofs.

Theorem. Let A, B, C be sets. If $(A \cap B) \subseteq C$, then $(A - C) \cap (B - C) = \emptyset$.
Proof. We shall prove this statement by proving the contrapositive. Assume that $(A \cap B) \nsubseteq C$. Then there exists $a \in A \cap B$ such that $a \notin C$. Since $a \in A \cap B$, we have that $a \in A$ and $a \in B$. It follows from $a \in A$ and $a \notin C$ that $a \in A - C$. Similarly, it follows from $a \in B$ and $a \notin C$ that $a \in B - C$. Therefore, $a \in (A - C) \cap (B - C)$ and hence $(A - C) \cap (B - C) \neq \emptyset$.
Mistake in the proof: The mistake in the proof is that the argument does not prove the contrapositive of the given statement. The contrapositive of the statement is that if $(A - C) \cap (B - C) \neq \emptyset$, then $(A \cap B) \nsubseteq C$.

Theorem. For all $i \in \mathbb{Z}$, there exists $j \in \mathbb{Z}$ such that for all $k \in \mathbb{N}$ we have $2^i \cdot 2^j \leq k + 1$.
Proof. Let $i \in \mathbb{Z}$ and $k \in \mathbb{N}$. Choose $j = -k - i$. Then $j \in \mathbb{Z}$ and

$$2^i \cdot 2^j = 2^i \cdot 2^{-k-i} = 2^{i-k-i} = 2^{-k} = \frac{1}{2^k} \leq 1 \leq k + 1$$

Mistake in the proof: The mistake in the proof is that the integer j cannot depend on the natural number k. For each $i \in \mathbb{Z}$, one should find some $j \in \mathbb{N}$ which works for **every** $k \in \mathbb{N}$ at the same time.

Theorem. Let $\sim$ be the relation on $\mathbb{N} = \{1, 2, 3, \ldots\}$ defined by

$$x \sim y \quad \text{if and only if} \quad \frac{y}{x} = 2^i \quad \text{for some} \quad i \in \mathbb{Z}$$

for all $x, y \in \mathbb{N}$. The relation $\sim$ is reflexive.
Proof. Let $x \in \mathbb{N}$. Assume that $x \sim x$. Then, we have that $\frac{x}{x} = 1 = 2^0$ and $0 \in \mathbb{Z}$. Therefore, the relation is reflexive.
Mistake in the proof: The mistake in the proof is that the argument assumes the statement $x \sim x$ at the beginning. That $x \sim x$ should be the conclusion of the argument, not the assumption.

Theorem. For any set A, if $\mathcal{P}(A)$ is countable, then A is countable.
Proof. Let A be a set. Assume that $\mathcal{P}(A)$ is countable. Since $A \subseteq \mathcal{P}(A)$ and every subset of a countable set is countable, A is countable.
Mistake in the proof: The mistake in the proof is that it is not true in general that $A \subseteq \mathcal{P}(A)$.