PARADOXICAL DECOMPOSITIONS: THE BANACH-TARSKI PARADOX AND OTHERS

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Abstract. These are the lecture notes of a one-week course I taught at the Nesin Mathematics Village in Şirince, İzmir, Turkey during Summer 2018. The aim of the course was to introduce various paradoxical decompositions, prove the Banach-Tarski theorem and introduce amenable groups.

Contents

The main purpose of these lecture notes is to prove one of the most counter-intuitive results in mathematics, namely, the Banach-Tarski paradox[1] which states that

> "A unit ball in $\mathbb{R}^3$ can be decomposed into finitely many disjoint pieces and reassembled into two unit balls using only translations and rotations of $\mathbb{R}^3$."

We shall indeed prove the stronger version of the Banach-Tarski theorem which states that any two bounded subsets of $\mathbb{R}^3$ with non-empty interior can be transformed into each other via rotations and translations after being decomposed into finitely many pieces. We will also learn about various other geometrical "paradoxes". Finally, after an examination of the proof of the Banach-Tarski theorem, we shall introduce a property of groups that does not allow such paradoxical decompositions, namely, the notion of amenability, and prove some basic fact regarding amenable groups.

[1]We would like to warn the reader who is learning about this for the first time that the Banach-Tarski paradox is not really a "paradox" in the sense that it leads to a contradiction. It is called a paradox because of its counter-intuitive nature. In order to avoid any confusion, from now on, we will call it the Banach-Tarski *theorem*.

These lecture notes[2] mainly follow the excellent book *The Banach-Tarski paradox* by Stan Wagon. The reader who is interested in a comprehensive treatment on geometric paradoxes is strongly advised to read Wagon's book.

1. A mathemagical trick

1.1. The Pledge.

What does it mean for a mathematical object to be "decomposed into finitely many pieces and reassembled into another one"? We begin our discussion by giving a precise answer to this question in a mathematical context.

Definition 1. *Let G be a group acting on a set X. Two subsets $A, B \subseteq X$ are said to be G-equidecomposable if there exist sets $A_1, \dots, A_n \subseteq A$ and $B_1, \dots, B_n \subseteq B$ and group elements $g_1, \dots, g_n \in G$ such that*

- $A = \bigcup_{i=1}^{n} A_i$ *and* $B = \bigcup_{i=1}^{n} B_i$,
- $A_i \cap A_j = \emptyset = B_i \cap B_j$ *for all* $1 \leq i < j \leq n$, *and*
- $g_i[A_i] = B_i$ *for all* $1 \leq i \leq n$.

In other words, A and B are G-equidecomposable if A can be partitioned into finitely many pieces whose images under the action of G form a partition of B. Intuitively speaking, the action of the group G is our "toolbox", that is, the set of functions that will be used to move points of the "space" X.

In this terminology, we may restate the Banach-Tarski theorem as follows: A unit ball in $\mathbb{R}^3$ and the union of two disjoint unit balls in $\mathbb{R}^3$ are $\mathcal{G}$-equidecomposable, where $\mathcal{G}$ is the subgroup of the isometry group of $\mathbb{R}^3$ generated by translations and rotations.

Next will be shown that if we have "too many" arbitrary functions in our toolbox, then the notion of G-equidecomposability does not produce interesting results. More specifically, we have the following.

Proposition 1. *Let X be an infinite set. For every $Y \subseteq X$ with $|Y| = |X|$, we have that Y and X are $Sym(X)$-equidecomposable.*

Proof. Let $Y \subseteq X$ be such that $|Y| = |X|$. Since $|Y| = |Y| + |Y|$, we can find disjoint sets $A, B \subseteq Y$ with $|A| = |B| = |Y|$ and $A \sqcup B = Y$. Fix some bijection $f : Y \to X$ and set $C = f[A]$ and $D = f[B]$. It is easily seen that $|X - A| = |X| = |X - C|$ and hence there exists a bijection $\phi : X \to X$ such that $\phi \restriction A = f \restriction A$. Similarly, we have $|X - B| = |X| = |X - D|$ and hence there exists a bijection $\psi : X \to X$ such that $\psi \restriction B = f \restriction B$. Since $A \sqcup B = Y$ and $\phi[A] \sqcup \psi[B] = X$, we have that X and Y are $Sym(X)$-equidecomposable (via 2 pieces.) □

Therefore, the Banach-Tarski theorem would not be interesting if one were allowed to use all functions in $Sym(\mathbb{R}^3)$. What makes the Banach-Tarski theorem interesting (and counter-intuitive) is the fact that it is achieved via rotations and translations of $\mathbb{R}^3$ which are known to preserve length, area and volume.

For the rest of this subsection, let G be a group acting on a set X. Consider the relation $\sim_G$ on the set $\mathcal{P}(X)$ defined by $A \sim_G B$ if and only if A and B are G-equidecomposable. The following is easily observed.

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[2] The intended audience of this course was undergraduate mathematics students with basic background in group theory. While the lectures were self-contained and the necessary background regarding group actions, matrix groups and topology of Euclidean space was introduced in class, it is not included in the lecture notes.

Lemma 1. $\sim_G$ *is an equivalence relation.*

Proof. It is straightforward to check that $\sim_G$ is reflexive and symmetric. We shall only prove that $\sim_G$ is transitive. Let $A, B, C \subseteq X$ such that $A \sim_G B$ and $B \sim_G C$.

Since we have $A \sim_G B$, by definition, there exist partitions $\{A_i : 1 \leq i \leq n\}$ and $\{B_i : 1 \leq i \leq n\}$ of A and B respectively such that $g_i[A_i] = B_i$ for some group elements $g_1, \ldots, g_n \in G$. Similarly, it follows from $B \sim_G C$ that there exists partitions $\{\hat{B}_i : 1 \leq i \leq m\}$ and $\{C_i : 1 \leq i \leq m\}$ of B and C respectively such that $h_i[\hat{B}_i] = C_i$ for some group elements $h_1, \ldots, h_m \in G$.

For each $1 \leq i \leq n$ and $1 \leq j \leq m$, consider the set $\hat{A}_{ij} = g_i^{-1}[B_i \cap \hat{B}_j] \subseteq A$ and the set $\hat{C}_{ij} = h_j[B_i \cap \hat{B}_j] \subseteq C$. We claim that A and C are G-equidecomposable via these pieces. It is straightforward to check that the collections $\{\hat{A}_{ij}\}$ and $\{\hat{C}_{ij}\}$ consist of pairwise disjoint sets whose unions are A and C respectively. Set $k_{ij} = h_j g_i^{-1}$ and $D_{ij} = B_i \cap \hat{B}_j$ for all $1 \leq i \leq n$ and $1 \leq j \leq m$. Then we have that $k_{ij}[\hat{A}_{ij}] = k_{ij}[g_i^{-1}[D_{ij}]] = h_j[D_{ij}] = \hat{C}_{ij}$. Therefore, $A \sim_G C$ and hence $\sim_G$ is transitive. □

It follows from the proof of Lemma 1 that if A and B are G-equidecomposable with n pieces and B and C are G-equidecomposable with m pieces, then A and C are G-equidecomposable with at most[3] nm pieces. Later on, this observation will allow us to count the number of pieces that we use for the Banach-Tarski decomposition.

We will next learn a variant of the Cantor-Schröder-Bernstein theorem for G-equidecomposability first realized by Banach, which will be the main ingredient of the strong form of the Banach-Tarski theorem. For the rest of these notes, let $A \preccurlyeq_G B$ denote that A is G-equidecomposable with a subset of B.

Theorem 2. *If* $A \preccurlyeq_G B$ *and* $B \preccurlyeq_G A$*, then* $A \sim_G B$*.*

Proof. The usual back-and-forth proof of the Cantor-Schröder-Bernstein theorem works if one uses the injections between A and B induced by the decompositions. (A careful unpacking of the proof should reveal that if $A \preccurlyeq_G B$ via n pieces and $B \preccurlyeq_G A$ via m pieces, then $A \sim_G B$ via at most $n + m$ pieces.) □

Before we conclude this subsection, we shall see a basic geometric example of G-equidecomposability.

Theorem 3. *Let* $SO_2(\mathbb{R})$ *act on the unit circle* S^1 *by rotations about the origin. For every countable* $D \subseteq S^1$*, we have that* S^1 *and* $S^1 - D$ *are* $SO_2(\mathbb{R})$*-equidecomposable.*

Proof. Let $\rho_\theta : S^1 \to S^1$ be the bijection induced by the action of the matrix

$$\begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}$$

which rotates S^1 about the origin by θ radians in the counter-clockwise direction. Consider the set

$$W = \{\theta : \exists d \in D \ \exists\ k \in \mathbb{N}^+ \ \rho_\theta^k(d) \in D\}$$

Since D is countable, W is a countable union of countable sets and hence is countable. Therefore there exists some $\gamma \notin W$. Note that, by definition, $\rho_\gamma^k(d) \notin D$ for all $k \in \mathbb{N}^+$ and for all $d \in D$.

[3] Note that some of the sets $\hat{A}_{ij}$ and $\hat{C}_{ij}$ in our construction may be empty, which is actually allowed in our definition of G-equidecomposability.

We claim that the collection $\{\rho_\gamma^k[D] : k \in \mathbb{N}\}$ consists of pairwise disjoint sets. Assume towards a contradiction that there exists $x \in \rho_\gamma^k[D] \cap \rho_\gamma^l[D]$ for some natural numbers $k \neq l$. Without loss of generality, we may assume that $k < l$. Then we would have $\rho_\gamma^{-k}(x) \in D \cap \rho_\gamma^{l-k}[D]$, which is a contradiction. Therefore the sets $D, \rho_\gamma[D], \rho_\gamma^2[D], \dots$ are pairwise disjoint.

Let $A = \bigcup_{k=0}^{\infty} \rho_\gamma^k[D]$ and $B = S^1 - A$. Then we have that $A \sqcup B = S^1$ and $\rho_\gamma[A] \sqcup B = S^1 - D$. Therefore, $S^1 - D$ and S^1 are $SO_2(\mathbb{R})$-equidecomposable via 2 pieces. □

This example shows that any countable subset of S^1 can be "absorbed" via rotations. The generalization of this idea to two dimensions will be central to the proof of the Banach-Tarski theorem. Finally, let us give a special name those sets for which we have the Banach-Tarski phenomenon.

Definition 2. *A non-empty set $E \subseteq X$ is said to be G-paradoxical if for some positive integers n and m, there exist pairwise disjoint sets $A_1, \dots, A_n, B_1, \dots, B_m \subseteq E$ and group elements $g_1, \dots, g_n, h_1, \dots, h_m \in G$ such that*

$$E = \bigcup_{i=1}^{n} g_i[A_i] = \bigcup_{j=1}^{m} h_j[B_j]$$

In other words, E is G-paradoxical if there exist disjoint subsets $A, B \subseteq E$ such that A and B are G-equidecomposable with E. Observe that if E is G-paradoxical with pieces $A, B \subseteq E$, then it follows from $E \sim_G B \preccurlyeq_G E - A \preccurlyeq_G E$ and Theorem 2 that $E - A \sim_G E$. Therefore, the sets $A_1, \dots, A_n, B_1, \dots, B_m \subseteq E$ in the definition above may be chosen to form a partition of E.

In this terminology, the Banach-Tarski theorem can be restated as follows: The unit ball in $\mathbb{R}^3$ is $\mathcal{G}$-paradoxical, where $\mathcal{G}$ is the subgroup of the isometry group of $\mathbb{R}^3$ generated by translations and rotations.

1.2. **The Turn.** We shall now focus on groups. Every group acts freely on itself by left multiplication. We would like to understand those groups that admit paradoxical decompositions which respect to this action. The motivation behind this is the following: When a group G acts on a set X in a certain way, the "geometry" of the group G can be transferred to the set X. Our expectation is to achieve a Banach-Tarski-like decomposition on various groups and transfer this decomposition to various sets they act on.

Definition 3. *A group G is said to be paradoxical if G is G-paradoxical with respect to its action on itself by left multiplication.*

What are examples of paradoxical groups? A moment's thought reveals that no finite group can be paradoxical. It turns out that the most natural examples of paradoxical groups are non-abelian free groups.

Lemma 4. *Let $\mathcal{F}_2$ be the free group on 2 generators. Then $\mathcal{F}_2$ is paradoxical*

Proof. Let σ and τ be the generators of $\mathcal{F}_2$. For each $\rho = \sigma^{\pm 1}, \tau^{\pm 1}$, let $W(\rho)$ denote the set of (reduced) words in $\mathcal{F}_2$ starting with ρ. It is clear that

$$\mathcal{F}_2 = \{e\} \sqcup W(\sigma) \sqcup W(\sigma^{-1}) \sqcup W(\tau) \sqcup W(\tau^{-1})$$

Moreover, we have that $W(\sigma) \cup \sigma W(\sigma^{-1}) = W(\tau) \cup \tau W(\tau^{-1})$ and hence $\mathcal{F}_2$ is $\mathcal{F}_2$-paradoxical. □

It is indeed possible to choose the pieces in this proof to form a partition of $\mathcal{F}_2$. It is an exercise to the reader to check that the pieces

$$\mathcal{F}_2 = (W(\sigma) \cup \Sigma \cup \{e\}) \sqcup (W(\sigma^{-1}) - \Sigma) \sqcup W(\tau) \sqcup W(\tau^{-1})$$

works, where $\Sigma = \{\sigma^{-n} : n \in \mathbb{N}^+\}$. That it suffices to use four pieces to duplicate $\mathcal{F}_2$ shall later be used to find an upper bound for the number of pieces that we are going to use for the Banach-Tarski decomposition of the unit ball.

The next lemma confirms our expectation that the paradoxical decomposition of a group can be transferred to sets it acts on if the action is "nice".

Lemma 5. *Let G be a paradoxical group with a free action on a set X. Then X is G-paradoxical.*

Proof. Since G is paradoxical, there exists a partition $\{A_1, \ldots, A_n, B_1, \ldots, B_m\}$ such that $G = \bigcup_{i=1}^n g_i A_i = \bigcup_{j=1}^m h_j B_j$ for some $g_1, \ldots, g_n, h_1, \ldots, h_m \in G$. By the axiom of choice, there exists a set $T \subseteq X$ which is a transversal for the orbit equivalence relation of the action $G \curvearrowright X$, that is, T contains exactly one element from each orbit of the action $G \curvearrowright X$. Since the action $G \curvearrowright X$ is free, the collection $\{g[T] : g \in G\}$ is a partition of X. For each $1 \leq i \leq n$ and $1 \leq j \leq m$, let

$$\hat{A}_i = \bigcup_{g \in A_i} g[T] \text{ and } \hat{B}_j = \bigcup_{g \in B_j} g[T]$$

It is not difficult to check that these sets are pairwise disjoint and that

$$X = \bigcup_{i=1}^n g_i[\hat{A}_i] = \bigcup_{j=1}^m h_j[\hat{B}_j]$$

Thus X is G-paradoxical with the same number of pieces witnessing that G is paradoxical. $\square$

An immediate corollary of this lemma is that the property of being paradoxical passes from subgroups to supergroups.

Corollary 6. *If $H \leqslant G$ and H is paradoxical, then G is paradoxical.*

Proof. The action of H on G by left multiplication is free. Thus, by Lemma 5, G is H-paradoxical and hence is G-paradoxical. $\square$

Having proven Lemma 4 and Lemma 5, we shall next seek isomorphic copies of non-abelian free groups in the isometry group of $\mathbb{R}^3$ hoping to transfer their paradoxical decompositions to various subsets of $\mathbb{R}^3$. It turns out that the rotation group $SO_3(\mathbb{R})$ already contains a copy of $\mathcal{F}_2$.

Lemma 7. *There exists a subgroup $G \leqslant SO_3(\mathbb{R})$ isomorphic to $\mathcal{F}_2$.*

Proof. See [1, Theorem 2.1] for a construction of such a subgroup inside $SO_3(\mathbb{Q})$. $\square$

We are now in a position to prove the Hausdorff paradox which, besides being an ingredient of the Banach-Tarski theorem, is surprising on its own.

Theorem 8 (The Hausdorff paradox)**.** *There exists a countable subset $D \subseteq S^2$ such that $S^2 - D$ is $SO_3(\mathbb{R})$-paradoxical.*

Proof. Let $G \leqslant SO_3(\mathbb{R})$ be an isomorphic copy of $\mathcal{F}_2$ which exists by Lemma 7. Consider the set

$$D = \{p \in S^2 : \exists g \in G - \{e\} \; g \cdot p = p\}$$

that is, D is the set of all points fixed by some non-identity element in G. As every non-identity element in G fixes two points on S^2, we have that D is countable. By the definition of D, it is clear that the action of G on $S^2 - D$ is free.

Since G is paradoxical and acting freely on $S^2 - D$, it follows from Lemma 5 that $S^2 - D$ is G-paradoxical and hence, is $SO_3(\mathbb{R})$-paradoxical. □

Recall by Theorem 3 that we were able to "absorb" a countable subset of S^1 via rotations. By imitating the proof, one can pull off the same trick with S^2.

Lemma 9. *Let $SO_3(\mathbb{R})$ act on the unit sphere S^2 by rotations about the lines passing through the origin. For every countable $D \subseteq S^2$, we have that S^2 and $S^2 - D$ are $SO_3(\mathbb{R})$-equidecomposable.*

Proof sketch. Let $D \subseteq S^2$ be a countable set. Then, D being countable, we can find a line l through the origin which does not intersect D. It is easily seen that the set of angles θ for which the rotation about the line l through $n\theta$ radians (with respect to some fixed orientation) takes some point of D to D for some $n > 0$ is countable.

It follows that there exists some angle γ for which the rotation ρ_γ^n about the line l through $n\theta$ radians moves points of D to $S^2 - D$ for every $n > 0$. In this case, the sets $\{\rho_\gamma^k[D] : k \in \mathbb{N}\}$ are pairwise disjoint. Let $A = \bigcup_{k=0}^{\infty} \rho_\gamma^k[D]$ and $B = S^2 - A$. Then we have that $A \sqcup B = S^2$ and $\rho_\gamma[A] \sqcup B = S^2 - D$. Therefore, $S^2 - D$ and S^2 are $SO_3(\mathbb{R})$-equidecomposable via 2 pieces. □

It is a relatively easy exercise for the reader to check that, given a group G acting on a set X, if $A, B \subseteq X$ are G-equidecomposable and A is G-paradoxical, then B is G-paradoxical. This observation, together with Lemma 9 and the Hausdorff paradox, imply the following result.

Corollary 10. *S^2 is $SO_3(\mathbb{R})$-paradoxical.*

In other words, we can partition the sphere $S^2 = A \sqcup B$ in such a way that both of the parts A and B can be separately decomposed into finitely many pieces which, after the application of various rotations of $SO_3(\mathbb{R})$, form S^2. Having proven this amazing result, we are now ready for the grand finale.

1.3. **The Prestige.** Having a paradoxical decomposition of S^2, we shall "radially stretch" the pieces of this decomposition and "glue" these pieces together in order to obtain a paradoxical decomposition of the unit ball minus the origin. By Corollary 10, S^2 is $SO_3(\mathbb{R})$-paradoxical and hence there exist a partition of $\{A_1, \ldots, A_n, B_1, \ldots, B_m\}$ of S^2 and rotations $g_1, \ldots, g_n, h_1, \ldots, h_m \in SO_3(\mathbb{R})$ such that

$$S^2 = \bigcup_{i=1}^{n} g_i[A_i] = \bigcup_{j=1}^{m} h_j[B_j]$$

Let B denote the unit ball in $\mathbb{R}^3$ and $\mathbf{0}$ denote the origin. It is straightforward to verify that

$$\begin{aligned} B - \{\mathbf{0}\} &= \bigsqcup_{0<\alpha\leq 1} \{(\alpha x, \alpha y, \alpha z) : x^2 + y^2 + z^2 = 1\} \\ &= \bigcup_{i=1}^{n} g_i \cdot \left[\bigsqcup_{0<\alpha\leq 1} \{(\alpha x, \alpha y, \alpha z) : (x,y,z) \in A_i\} \right] \\ &= \bigcup_{i=j}^{m} h_j \cdot \left[\bigsqcup_{0<\alpha\leq 1} \{(\alpha x, \alpha y, \alpha z) : (x,y,z) \in B_j\} \right] \end{aligned}$$

Therefore, $B - \{\mathbf{0}\}$ is $SO_3(\mathbb{R})$-paradoxical with the same number pieces that a paradoxical decomposition of S^2 uses. We have *almost* proven what we had set out to prove. The only problem is that the set we duplicated is off by one point. What are we going to do about the origin $\mathbf{0}$?

Recall that $\mathcal{G}$ denotes the subgroup of the isometry group of $\mathbb{R}^3$ generated by rotations and translations. Consider the circle

$$C = \{(x,y,z) : (x - 1/4)^2 + y^2 = 1/16, z = 0\}$$

which contains the origin $\mathbf{0}$. By imitating the proof of Theorem 3, one can show that $C \sim_{\mathcal{G}} C - \{\mathbf{0}\}$. It then follows that

$$\begin{aligned} B = (B - C) \sqcup C &\sim_{\mathcal{G}} (B - C) \sqcup (C - \{\mathbf{0}\}) \\ &\sim_{\mathcal{G}} B - \{\mathbf{0}\} \end{aligned}$$

Combining this observation with what we have proven so far, we have the following.

Theorem 11 (The Banach-Tarski theorem)**.** *The unit ball B is $\mathcal{G}$-paradoxical.*

We would like to note that, by allowing α to range in $(0,\infty)$ instead of $(0,1]$ while "gluing" the pieces of the paradoxical decomposition of S^2 together, one can also prove that $\mathbb{R}^3 - \{\mathbf{0}\}$ is $SO_3(\mathbb{R})$-paradoxical. Since a single point can be absorbed via some rotation, we have that $\mathbb{R}^3$ is $\mathcal{G}$-paradoxical.

Let us now estimate how many pieces are needed in order to duplicate the unit ball B with our construction. We know that S^2 and $S^2 - D$ are $SO_3(\mathbb{R})$-equidecomposable via 2 pieces where D is the fixed points of the action of $\mathcal{F}_2$ on S^2. Moreover, there exists a 4-piece paradoxical decomposition of $\mathcal{F}_2$ which induces a 4-piece paradoxical decomposition of $S^2 - D$. Therefore, it takes $2 \times 4 = 8$ pieces to duplicate S^2. Indeed, each copy of S^2 is created via 2×2 pieces since both parts of the 4-piece decomposition of $\mathcal{F}_2$ consists of 2 pieces.

It is now time to duplicate the unit ball B. We first decompose B into two pieces, namely, $\{\mathbf{0}\}$ and $B - \{\mathbf{0}\}$. Since S^2 can be duplicated by $4 + 4$ pieces, we can duplicate $B - \{\mathbf{0}\}$ via $4 + 4$ pieces by our construction. We can then translate the piece $\{\mathbf{0}\}$ to be the center of one of the copies of the unit ball we created. In order to "create" a center for the other ball, we use the fact that a missing point on a circle can be retrieved back by a 2-piece decomposition. This means that, in order to create the center of the other ball, the number of pieces we need is twice the number of pieces required for its creation. Therefore, we need $4+1+4\times 2 = 13$ pieces to duplicate the unit ball B.

We should warn the reader that this number is far from the optimal. Indeed, it is well-known that the Banach-Tarski paradox can be achieved with 5 pieces but cannot be achieved with 4 pieces.

Finally, before we conclude this section, we shall prove the strong form of the Banach-Tarski theorem which, as a corollary, implies not only a ball can be duplicated but also a pea can be chopped up into finitely many pieces and reassembled into the Sun using rotations and translations only!

Theorem 12 (The Banach-Tarski theorem, Strong Form)**.** *Let A and B be bounded subsets of $\mathbb{R}^3$ with non-empty interior. Then A and B are $\mathcal{G}$-equidecomposable.*

Proof. Since A has non-empty interior, there exists an open ball $B_\epsilon \subseteq A$. Notice that the proof of the Banach-Tarski theorem can easily be modified so that it holds arbitrary open balls in $\mathbb{R}^3$. Consequently, we have that B_ϵ is $\mathcal{G}$-equidecomposable with the union of n-copies of B_ϵ for every $n \in \mathbb{N}^+$. Since B is bounded, one can cover B using finitely many translates of B_ϵ and it follows that $B \preccurlyeq_{\mathcal{G}} A$. Arguing symmetrically, one can also obtain that $A \preccurlyeq_{\mathcal{G}} B$. Hence, by Theorem 2, we have that $A \sim_{\mathcal{G}} B$. □

2. Some variations

At this point, the Banach-Tarski theorem may not (and indeed, should not) seem as surprising and counter-intuitive as it had seemed in the first place. All in all, the real reason behind this phenomenon is that the isometry group of $\mathbb{R}^3$ contains free non-abelian subgroups which, not surprisingly, have paradoxical decompositions. Having realized this, mathematicians have come up with various constructions of similar paradoxical nature. In this section, we shall learn some of these variations.

The first variation we shall learn indeed predates the Banach-Tarski theorem and is due to Mazurkiewicz and Sierpiński. Although the isometry group of $\mathbb{R}^2$ does not contain a non-abelian free group, it does contain a non-abelian free *semigroup*. Using ideas similar to those in the previous section, one can prove the following.

Theorem 13 (Sierpiński-Mazurkiewicz paradox)**.** *There exists a subset of $\mathbb{R}^2$ which is $Isom(\mathbb{R}^2)$-paradoxical.*

After the publication of the Banach-Tarski theorem, John von Neumann realized that the group of area-preserving affine transformations of the plane contains a non-abelian free group of rank 2 and hence one can imitate the proof of the Banach-Tarski theorem to obtain the following.

Theorem 14 (von Neumann paradox)**.** *Any two bounded subsets of $\mathbb{R}^2$ with non-empty interiors are $SA_2(\mathbb{R})$-equidecomposable.*

The next variation takes place in the hyperbolic plane $\mathbb{H}^2$ and is due to Mycielski and Wagon. As before, it based on the existence of a free action of a non-abelian free group of rank 2 on $\mathbb{H}^2$ by isometries.

Theorem 15 (Mycielski-Wagon paradox)**.** *$\mathbb{H}^2$ is $Isom(\mathbb{H}^2)$-paradoxical.*

Since this course is not intended to be a comprehensive treatment of paradoxical geometric constructions, we shall not attempt to prove any of these results. For further information, we refer the reader to [1].

3. Paradoxes and Amenability

The reader may have realized that the Banach-Tarski theorem and some of its variations are all based on the fact that $\mathcal{F}_2$ is a paradoxical group. One may ask whether or not being paradoxical can be characterized without explicitly appealing to paradoxical decompositions.

Soon after the Banach-Tarski theorem was proven, John von Neumann introduced an abstract property of groups that do *not* allow such paradoxical decompositions, namely, the property of being amenable. In this section, we shall learn some basic facts regarding amenability.

Although the notion of amenability can be defined for locally compact Hausdorff topological groups in the most general setting, we shall only restrict our attention to countable (discrete) groups in this course.

Definition 4. *Let G be a countable group. Then G is said to be amenable if there exists a finitely additive G-invariant probability measure on $\mathcal{P}(G)$, that is, a function $\mu : \mathcal{P}(G) \to [0,1]$ such that*

- $\mu(G) = 1$,
- $\mu(A \cup B) = \mu(A) + \mu(B)$ *for all disjoint* $A, B \subseteq G$, *and*
- $\mu(gA) = \mu(A)$ *for all* $g \in G$ *and* $A \subseteq G$.

The following proposition easily follows from the definition.

Proposition 2. *Let G be a countable group. If G is amenable, then G is not paradoxical.*

Proof. Assume that G is amenable, say, $\mu : \mathcal{P}(G) \to [0,1]$ is a finitely additive G-invariant probability measure. Assume towards a contradiction that G is paradoxical, say, we have a partition $\{A, B\}$ of G such that both A and B are G-equidecomposable with G. Since μ is G-invariant, any two G-equidecomposable sets have the same measure. Consequently, we have that $1 = \mu(G) = \mu(A \cup B) = \mu(A) + \mu(B) = 1 + 1 = 2$, which is a contradiction. Therefore, G is not paradoxical. □

It turns out that the converse implication also holds. However, we shall skip the proof of this beautiful fact.

Theorem 16 (Tarski)**.** *Let G be a countable group. Then G is paradoxical if and only if it is not amenable.*

Tarski's theorem together with Corollary 6 imply that if G is non-amenable, then every supergroup of G is also non-amenable. Consequently, any group containing the free group $\mathcal{F}_2$ is non-amenable. That this is the only obstruction for amenability is known as the von Neumann conjecture[4]. More precisely, the von Neumann conjecture is the statement that if G is non-amenable, then there exists a subgroup of G isomorphic to $\mathcal{F}_2$. This conjecture was disproven in 1980 by Ol'shanskii.

Theorem 17 (Ol'shanskii)**.** *There exists a non-amenable group which does not contain an isomorphic copy of $\mathcal{F}_2$.*

[4]Although von Neumann's name is attached to the conjecture, it turns out that the conjecture is first explicitly state by Day in 1957.

We have seen examples of non-amenable groups such as non-abelian free groups. What groups *are* amenable? Every finite group is clearly amenable for the map $\mu(G) \to [0,1]$ given by $\mu(A) = |A|/|G|$ for all $A \subseteq G$ can easily be checked to be a finitely additive G-invariant probability measure. Are there any infinite amenable groups?

It turns out that the group $\mathbb{Z}$ of integers is amenable[5]. The rest of the notes will be devoted to a sketch of the proof of the fact that $\mathbb{Z}$ is amenable.

How can one define a finitely additive $\mathbb{Z}$-invariant probability measure on $\mathcal{P}(\mathbb{Z})$? Let us try some naive ideas. We know how to measure the subsets of a finite group. We simply consider the *density* of the subset in the whole group. One can try to generalize this idea by trying to measure a subset of $\mathbb{Z}$ via its *asymptotic density.* More precisely, we can define $\mu(A)$ to be

$$\lim_{n\to\infty} \frac{|A \cap [-n,n]|}{2n+1}$$

It is not difficult to show that the map μ is indeed $\mathbb{Z}$-invariant and that $\mu(\mathbb{Z}) = 1$ and $\mu(A \cup B) = \mu(A) + \mu(B)$ whenever these values exist. Unfortunately, the domain of μ is not $\mathcal{P}(\mathbb{Z})$, that is, there are subsets of $\mathbb{Z}$ which do not have asymptotic density. Our naive idea could have worked if we were able to extend this function to all of $\mathcal{P}(\mathbb{Z})$ while keeping its nice properties. How are we going to fix the problem?

4. Ultrafilters to the rescue

Let X be a non-empty set. A set $\mathcal{U} \subseteq \mathcal{P}(X)$ consisting of subsets of X is said to be a *filter* on X if we have

- $\emptyset \notin \mathcal{U}$ and $X \in U$,
- $A \cap B \in \mathcal{U}$ whenever $A \in \mathcal{U}$ and $B \in \mathcal{U}$, and
- $B \in \mathcal{U}$ whenever $A \in \mathcal{U}$ and $A \subseteq B$.

An example of a filter on an infinite set X is the filter $\mathcal{U} = \{A \subseteq X : |X - A| < \aleph_0\}$ consisting of cofinite sets. A filter $\mathcal{U}$ on a set X is said to be an *ultrafilter* on X if for every $A \subseteq X$ we have either $A \in \mathcal{U}$ or $X - A \in \mathcal{U}$. One should think of ultrafilters as measuring the "bigness" of subsets of the underlying set. Indeed, if $\mathcal{U}$ is an ultrafilter on a set X, then the map

$$\chi(A) = \begin{cases} 1 & \text{if } A \in \mathcal{U} \\ 0 & \text{if } A \notin \mathcal{U} \end{cases}$$

is a $\{0,1\}$-valued finitely additive probability measure on $\mathcal{P}(X)$. What are examples of ultrafilters?

It is straightforward to check that if $x \in X$, then the collection $\{A \subseteq X : x \in A\}$ is an ultrafilter. Such ultrafilters are known as *principal* ultrafilters. Are there non-principal ultrafilters?

An application of Zorn's lemma to the partially ordered set of proper filters on a set X containing the cofinite filter shows that non-principal ultrafilters do exist.

Let $\mathcal{U}$ be a non-principal ultrafilter on $\mathbb{N}$. Given a sequence $(x_i)_{i\in\mathbb{N}}$ of real numbers and a real number L, we shall say that the *ultralimit* of the sequence

[5]More generally, solvable groups are amenable and the class of amenable groups is closed under group extensions, quotients and direct limits. Using these operations, one can construct various amenable groups. Unfortunately, we are not going to cover any of these facts in this course.

$(x_i)_{i\in\mathbb{N}}$ along $\mathcal{U}$ is L and write

$$\lim_{\mathcal{U}} x_i = L$$

if we have that

$$\forall \epsilon > 0 \ \{i \in \mathbb{N} : |x_i - L| < \epsilon\} \in \mathcal{U}$$

Intuitively speaking, the ultralimit of a sequence is the usual limit of the sequence on indices which form a "big" subset of $\mathbb{N}$. It is an exercise[6] to the reader to check that if $\lim_{n\to\infty} x_i = L$, then $\lim_{\mathcal{U}} x_i = L$. Therefore, the notion of ultralimit generalizes the usual notion of limit.

One of the marvelous properties of the notion of ultralimit is that, if one is working with sequences taking values in a compact topological space such as $[0,1]$, then *every* sequence has an ultralimit along an ultrafilter.

Theorem 18. *Let $\mathcal{U}$ be a non-principal ultrafilter on $\mathbb{N}$ and $(x_i)_{i\in\mathbb{N}}$ be a sequence such that $x_i \in [0,1]$ for all $i \in \mathbb{N}$. Then $\lim_{\mathcal{U}} x_i$ exists.*

Proof. Assume to the contrary that $\lim_{\mathcal{U}} x_i$ does not exist. Then, by the definition of ultralimit, for every $c \in [0,1]$, we can find some ϵ_c such that

$$\{i \in \mathbb{N} : x_i \notin B(c, \epsilon_c)\} \in \mathcal{U}$$

where $B(c,\epsilon_c)$ is the open ball in $[0,1]$ centered around c with radius ϵ_c. Since $\bigcup_{c\in[0,1]} B(c,\epsilon_c)$ is an open covering of $[0,1]$ and $[0,1]$ is compact, there exist finitely many $c_1, \ldots, c_n$ such that $[0,1] = B(c_1,\epsilon_{c_1}) \cup \cdots \cup B(c_n,\epsilon_{c_n})$. As filters are closed under finite intersections, we have

$$\bigcap_{j=1}^{n} \{i \in \mathbb{N} : x_i \notin B(c_j, \epsilon_{c_j})\} \in \mathcal{U}$$

$$\{i \in \mathbb{N} : x_i \in \bigcap_{j=1}^{n} \left(B(c_j, \epsilon_{c_j})\right)^C\} \in \mathcal{U}$$

$$\emptyset = \{i \in \mathbb{N} : x_i \in [0,1]^C\} \in \mathcal{U}$$

which is a contradiction. Therefore, $\lim_{\mathcal{U}} x_i$ exists. □

Having show that the ultralimit of any sequence exists, we shall try to modify our previous idea of measuring subsets via their asymptotic densities.

Theorem 19. *$\mathbb{Z}$ is amenable.*

Proof sketch. Fix a non-principal ultrafilter $\mathcal{U}$ on $\mathbb{N}$ and define $\mu : \mathcal{P}(\mathbb{Z}) \to [0,1]$ by

$$\mu(A) = \lim_{\mathcal{U}} \frac{|A \cap [-n,n]|}{2n+1}$$

for all $A \subseteq \mathbb{Z}$. It is a (slightly difficult) exercise to the reader to check that μ is a finitely additive $\mathbb{Z}$-invariant probability measure. □

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[6]**Hint.** First observe that any non-principal ultrafilter on an infinite set contains the cofinite filter.

References

[1] Grzegorz Tomkowicz and Stan Wagon, *The Banach-Tarski paradox*, second ed., Encyclopedia of Mathematics and its Applications, vol. 163, Cambridge University Press, New York, 2016, With a foreword by Jan Mycielski. MR 3616119

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