AN INTRODUCTION TO INFINITARY COMBINATORICS

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ABSTRACT. These are the lecture notes of a one-week course I taught at the Nesin Mathematics Village in Şirince, İzmir, Turkey during Summer 2017. The aim of the course was to introduce some basic notions of combinatorial set theory and prove some key results regarding partition relations for cardinals and the tree property of cardinals.

CONTENTS

0. PRELUDE

Throughout these notes, we shall use the standard von Neumann construction of ordinal and cardinal numbers. Even though there will be a review of the necessary set theoretic background during the course, it will not be included in these notes. We refer the reader who has not been exposed to axiomatic set theory before to [5], the author's lecture notes located at this hyperlink link or the relevant issues of Matematik Dünyası if the reader knows Turkish.

0.1. **Strangers at a party.** We begin by considering the following question: Given six people, is it always possible to find a group of three people who either all know each other or all not know each other?

The answer to this question turns out to be affirmative. In order to prove this, we first need to make the question mathematically precise. To that end, we shall introduce the notions of colorings and homogeneous sets.

For any set X and any non-zero cardinal μ, we shall write $[X]^\mu$ to denote the set

$$\{Y \subseteq X : |Y| = \mu\}$$

For any non-zero cardinal λ, any function $f : [X]^\mu \to \lambda$ is called a *coloring* of $[X]^\mu$ into λ colors. For any coloring $f : [X]^\mu \to \lambda$, a subset $H \subseteq X$ is said to be *homogeneous* with respect to f if the function f is constant on $[H]^\mu$.

One can represent a group of six people by the set $6 = \{0, 1, 2, 3, 4, 5\}$ and the relationship between these people by the coloring $f : [6]^2 \to 2$ defined by $f(\{i, j\}) = 0$ if and only if the person i and the person j know each other. Having formalized the notions of "person" and "knowing each other" mathematically, we can now rephrase our question as follows: Given a coloring $f : [6]^2 \to 2$, can we always find a homogeneous set of size three?

Theorem 1. *For any function $f : [6]^2 \to 2$, there exists a subset $H \subseteq 6$ such that $|H| = 3$ and f is constant on $[H]^2$.*

Proof. By pigeonhole principle, at least three elements of the set $\{\{0, i\} : 1 \leq i \leq 5\}$ are colored with the same color under f, say $f(\{0, a\}) = f(\{0, b\}) = f(\{0, c\}) = k$ for some distinct $1 \leq a, b, c \leq 5$.

- If f does not take the value k on the set $[\{a, b, c\}]^2$, then we choose H to be the set $\{a, b, c\}$ and f is constant on $[H]^2$ with value $1 - k$.
- If f takes the value k on the set $[\{a, b, c\}]^2$, then we choose H to be the set $\{0, i, j\}$, where $\{i, j\}$ is some element of $[\{a, b, c\}]^2$ such that $f(\{i, j\}) = k$. Then f is constant on $[H]^2$ with value k.

□

Using the terminology of graph theory, we can restate this theorem as follows: If one colors the edges of the complete graph K_6 with two colors, then there exists a subgraph which is isomorphic to K_3 and whose edges are of the same color.

Exercise. Show that there exists a function $f : [5]^2 \to 2$ with no homogeneous set of size three. (In other words, it is possible color the edges of the complete graph K_5 into two colors without having K_3 as a monochromatic subgraph.)
Exercise. Show that for any function $f : [17]^2 \to 3$ there exists a homogeneous set of size three. (In other words, if one colors the edges of the complete graph K_{17} with three colors, then there exists a monochromatic subgraph isomorphic to K_3.)

0.2. **Afterparty at Hilbert's Hotel.** Let us now consider the following more general question: Assume that countably infinitely many people are attending to an afterparty at Hilbert's Hotel. Is it always possible to find countably infinitely many guests who either all know each or all not know each other?

Rephrased in the terminology we introduced, the question becomes the following: Given a 2-coloring $f : [\mathbb{N}]^2 \to \{0, 1\}$ with 2 colors, is it always possible to find an infinite homogeneous set $H \subseteq \mathbb{N}$? The answer to this question is affirmative as shown by the following remarkable theorem, which is usually referred to as the infinite Ramsey theorem.

Theorem 2 (Ramsey)**.** *For every positive integer m, n, every countably infinite set S and every coloring $f : [S]^n \to m$, there exists an infinite homogeneous set $H \subseteq S$.*

Proof. We prove this by induction on $n \geq 1$.

- **Base case**. Given a function $f : [S]^1 \to m$, since finite unions of finite sets are finite, there exists $k \in m$ such that $f^{-1}(k)$ is infinite. Then we have that $\bigcup f^{-1}(k) = H \subseteq S$ is infinite and f is constant on $[H]^1$.
- **Inductive step**. We wish to show that the claim holds for $n+1$ whenever it holds for n. Let $n \geq 1$ and assume that the claim holds for n.

  Let S be a countably infinite set, m be a positive integer and $f : [S]^{n+1} \to m$ be a function. We need to find an infinite set $H \subseteq S$ such that f is constant on $[H]^{n+1}$.

  For each countably infinite $R \subseteq S$ and $s \in S$, consider the function $F_s : [R - \{s\}]^n \to m$ given by

  $$F_s(X) = f(X \cup \{s\})$$

  By induction hypothesis, for every countably infinite $R \subseteq S$ and every $s \in R$, there exists H_s^R such that F_s is constant on $[H_s^R]^n$.

  Since S is countably infinite, we can linearly order the elements of S by enumerating them, say $s_0 < s_1 < s_2 \cdots <$. By recursion, construct the following sequences $(a_k)_{k\in\mathbb{N}}$ and $(S_k)_{k\in\mathbb{N}}$.
  - $S_0 = S$ and $a_0 = s_0$,
  - $S_{k+1} = H_{a_k}^{S_k} \subseteq S_k - \{a_k\}$ and $a_{k+1} = \min\{s \in S_{k+1} : s > a_k\}$

  for each $k \in \mathbb{N}$. By construction, for each $i \in \mathbb{N}$, since $\{a_j : j > i\} \subseteq S_{i+1}$, the function F_{a_i} is constant on the infinite set $[\{a_j : j > i\}]^n$, say, with the color b_i. The entries of the sequence $(b_i)_{i\in\mathbb{N}}$ are elements of a finite set. It follows that there exists a subsequence $(b_{k_i})_{i\in\mathbb{N}}$ which is constant. Set $H = \{a_{k_i} : i \in \mathbb{N}\}$. It is straightforward to check that the function f is constant on the infinite set $[H]^{n+1}$ with the color b_{k_0}.

□

We shall next prove a finite version of Ramsey's theorem as a corollary to the infinite version. In order to do this, we will need the following lemma whose proof is left to the reader as an exercise.

Lemma 3 (König's tree lemma)**.** *Every infinite finitely branching tree has an infinite branch.*

Proof. Exercise. (**Hint.** After fixing a vertex as the root, observe that, at every level, there exists some vertex above which there are infinitely many vertices.) □

Theorem 4 (Ramsey)**.** *For all positive integers m, n, k, there exists a positive integer $r \geq n$ such that any coloring $f : [r]^n \to m$ admits some homogeneous set of size k.*

Proof. Assume to the contrary that there exist positive integers m, n, k such that for every positive integer $r \geq n$ there exists a coloring $f : [r]^n \to m$ admitting no homogeneous sets of size k.

Fix such m, n, k and consider the set T of colorings $f : [r]^n \to m$ with $r \geq n$ which have no homogeneous set of size k. Then the set $T \cup \{\emptyset\}$ together with the partial order $\preceq$ defined by $f \preceq g \leftrightarrow f \subseteq g$ induces a finitely branching infinite tree, which has an infinite branch by König's lemma, say, the branch $(\emptyset, f_0, f_1, \dots)$. It is easily seen that $\bigcup_{i\in\mathbb{N}} f_i : [\mathbb{N}]^n \to m$ is a coloring with no homogeneous set of size k and hence with no homogeneous infinite set. This contradicts the infinite Ramsey theorem. □

When $n = 2$, the least number r in the statement of this theorem for which the theorem holds is called the Ramsey number $R(k, k, \dots, k)$ where there are m-many k's. For example, we have seen that $R(3, 3) = 6$ and that $R(3, 3, 3) \leq 17$[1]. We refer the reader to any book on combinatorics including a section on Ramsey theory for the general definition of the Ramsey number $R(a_1, a_2, ..., a_n)$. Even though there are known upper and lower bounds for Ramsey numbers, computing exact values of Ramsey numbers turns out to be a notoriously difficult task[2]. Having proven both versions of Ramsey's theorem, we will next see some basic applications of these results.

Theorem 5. *Every infinite sequence of real numbers has a subsequence which is constant, strictly increasing or strictly decreasing.*

Proof. Let $(x_i)_{i \in \mathbb{N}}$ be a sequence of real numbers. Consider $f : [\mathbb{N}]^2 \to \{0, 1, 2\}$ given by

$$f(\{i, j\}) = \begin{cases} 0 & \text{if } i < j \text{ and } x_i < x_j \\ 1 & \text{if } i < j \text{ and } x_i > x_j \\ 2 & \text{if } i < j \text{ and } x_i = x_j \end{cases}$$

It follows from Ramsey's theorem that there exists an infinite homogeneous set $H \subseteq \mathbb{N}$, say, $H = \{n_i\}_{i \in \mathbb{N}}$ with $n_0 < n_1 < \dots$. Then $(x_{n_i})_{i \in \mathbb{N}}$ is either constant, strictly increasing or strictly decreasing. □

Theorem 6. *Let k be a positive integer. There exists a positive integer r such that, among any r points in the plane with no 3 collinear points, one can find k points that form the vertices of a convex k-gon.*

Proof. Let r be some positive integer obtained from the finite Ramsey theorem for the values $n = 3$, $m = 2$ and $k = k$. Assume that we are given r points in the plane with no 3 collinear points. We wish to show that there exist k points among these r points which are the vertices of a convex k-gon.

Let us first enumerate these points, say, the points are in $P = \{p_1, p_2, \dots, p_r\}$. We shall color $[P]^3$ with two colors $\{0, 1\}$ as follows: The set $\{p_a, p_b, p_c\}$ with $a < b < c$ is colored to 0 if and only if we move in the counter-clockwise direction as we move on the union of the line segments $p_a p_b$ and $p_b p_c$ from the point p_a to the point p_c. It follows from Ramsey's theorem that there exists a homogeneous subset of P of size k. It can be checked that the k-gon whose vertices are this homogeneous set is convex. □

1. Partition Relations for Cardinals

In this section, we shall focus on the following question: To what extent can Ramsey's theorem be generalized for uncountable sets? For example, if there are κ-many guests at a party for an infinite cardinal κ, is it always possible to find κ-many guests who either all know each or all not know each other? If not, how many such guests can be chosen?

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[1] Indeed, we have that $R(3, 3, 3) = 17$.

[2] According to Joel Spencer's book *Ten Lectures on the Probabilistic Method: Second Edition*, "Erdös asks us to imagine an alien force, vastly more powerful than us, landing on Earth and demanding the value of R(5,5) or they will destroy our planet. In that case, he claims, we should marshal all our computers and all our mathematicians and attempt to find the value. But suppose, instead, that they ask for R(6,6). In that case, he believes, we should attempt to destroy the aliens."

1.1. **The arrow notation and its basic properties.** We begin our investigation by introducing a notation, due to Erdös and Rado, which is usually called the arrow notation. Given non-zero cardinals $\kappa, \lambda, \mu, \nu$, we shall write

$$\kappa \to (\lambda)^{\mu}_{\nu}$$

if for every coloring $f : [\kappa]^{\mu} \to \nu$ with ν colors, there exists a homogeneous set $H \subseteq \kappa$ with $|H| = \lambda$. From now on, any statement of the form $\kappa \to (\lambda)^{\mu}_{\nu}$ will be called a partition relation. Let us restate what we have learned so far using this notation.

- $5 \nrightarrow (3)^2_2$
- $6 \to (3)^2_2$
- $17 \to (3)^2_3$
- For all positive integers m, n, we have $\omega \to (\omega)^n_m$
- For all positive integers m, n, k, there exists an integer r such that $r \to (k)^n_m$

Before we proceed, we would like to mention some restrictions regarding the arrow notation. First, we would like both the set $[\kappa]^{\mu}$ to be colored and the homogeneous part $[\lambda]^{\mu}$ of this set to be non-empty. Consequently, we need to assume $\kappa \geq \lambda \geq \mu$ when we use the arrow notation.

Second, we will assume that the exponent μ is always a positive natural number. The reason for this seemingly strong assumption is that μ cannot be infinite if we are to obtain non-trivial partition relations as shown by the next theorem, which is due to Erdös and Rado.

Theorem 7. *For any infinite cardinal κ, $\kappa \nrightarrow (\omega)^{\omega}_2$.*

Proof. Let κ be an infinite cardinal. It follows from the axiom of choice[3] that there exists a (strict) well-order relation $\prec$ on $[\kappa]^{\omega}$. Consider the coloring $f : [\kappa]^{\omega} \to 2$ given by

$$f(x) = \begin{cases} 0 & \text{if } x \prec y \text{ for every countably infinite } y \subsetneq x \\ 1 & \text{otherwise} \end{cases}$$

We claim that no countably infinite subset of κ is homogeneous with respect to f. Assume to the contrary that there exists a countably infinite homogeneous $x \subseteq \kappa$. Then the restriction of $\prec$ onto $[x]^{\omega}$ is a well-order relation as well and hence there exists $y \in [x]^{\omega}$ which is minimal with respect to $\prec$. Since y is $\prec$-minimal, we have that $f(y) = 0$. Let $(a_i)_{i \in \mathbb{N}}$ be an enumeration of y. Then $y_0 \subsetneq y_1 \subsetneq y_2 \subsetneq \cdots \subsetneq y$ where $y_m = \{a_0, a_2, \ldots, a_{2m}, a_1, a_3, a_5, \ldots\}$ for each $m \in \mathbb{N}$. It follows from the homogeneity of x and $y \subseteq x$ that $f(y_m) = 0$ for all $m \in \mathbb{N}$. This implies that $\cdots \prec y_2 \prec y_1 \prec y_0$ which contradicts that $\prec$ is a well-order relation. □

Finally, we note that if the number of colors ν exceeds κ, then we do not get non-trivial partition relations[4]. Thus we shall only consider partition relations $\kappa \to (\lambda)^n_{\nu}$ where κ, λ, ν are non-zero cardinals and n is a positive natural number with $\kappa \geq \lambda \geq n$ and $\kappa > \nu$.

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[3]Even though it is beyond the scope of this course, we would like to mention that this theorem does not hold if one rejects the axiom of choice. Indeed, Donald Martin proved that $\omega_1 \to (\omega_1)^{\omega_1}_2$ under the axiom of determinacy.

[4]More precisely, we have that $\kappa \to (n)^n_{\kappa}$ for every positive integer n. The reason is that the cardinality of $[\kappa]^n$ is κ and consequently, we can color the elements of $[\kappa]^n$ to distinct colors in which case any homogeneous set with respect to this coloring is of size n.

The following lemma shows that, given a partition relation, we can increase the size of the set whose subsets we are coloring; and decrease the number of colors and the size of the homogeneous set.

Lemma 8. *Let $\kappa, \kappa', \lambda, \lambda', \nu, \nu'$ be non-zero cardinals and n be a positive integer with $\kappa' \geq \kappa \geq n$, $n \leq \lambda' \leq \lambda$ and $\nu' \leq \nu \leq \kappa$. If $\kappa \to (\lambda)^n_\nu$, then $\kappa' \to (\lambda')^n_{\nu'}$.*

Proof. Trivial. □

Exercise. Prove that if $\kappa \to (\lambda)^n_\nu$, then $\kappa \to (\lambda)^m_\nu$ for every $0 < m < n$.

1.2. **The Erdös-Rado theorem.** We have seen in the previous subsection that the number of colors we use should be less than the size of the set whose subsets we are coloring.

An obvious question to ask is the following: If we are coloring the two-element subsets of a set with κ colors, how many elements should there be to guarantee the existence of an infinite homogeneous set? We know that there must be at least κ^+ elements. The following theorem shows that 2^κ elements are not sufficient.

Theorem 9 (Gödel, Erdös-Kakutani)**.** *For any cardinal κ, $2^\kappa \nrightarrow (3)^2_\kappa$.*

Proof. Recall that 2^κ is the cardinality of the set ${}^\kappa 2$, the set of functions from κ to 2. Thus it suffices to color $[{}^\kappa 2]^2$ with κ colors so that there exists no homogeneous set of size 3. Consider the coloring $f : [{}^\kappa 2]^2 \to \kappa$ given by

$$f(\{\varphi, \psi\}) = \min\{\alpha \in \kappa : \varphi(\alpha) \neq \psi(\alpha)\}$$

It is straightforward to check that there exists no homogeneous set of size 3 with respect to the coloring f as three distinct functions from κ to 2 cannot simultaneously disagree for the first time at the same ordinal α. □

It follows that if we are to guarantee an infinite homogeneous set for κ colors, then the set whose two-element subsets we are coloring must have size at least $(2^\kappa)^+$. That the cardinal $(2^\kappa)^+$ suffices follows from the following famous theorem known as the Erdös-Rado theorem.

Theorem 10 (Erdös-Rado)**.** *Let κ be an infinite cardinal. For every natural number n, we have that $\beth_n(\kappa)^+ \to (\kappa^+)^{n+1}_\kappa$.*

Proof. We prove this induction on n. For the base case $n = 0$, the partition relation $\kappa^+ \to (\kappa^+)^1_\kappa$ holds since κ^+ is a regular cardinal. For the inductive step, let n be a natural number and assume that the partition relation $\beth_n(\kappa)^+ \to (\kappa^+)^{n+1}_\kappa$ holds. We wish to prove that $\beth_{n+1}(\kappa)^+ \to (\kappa^+)^{n+2}_\kappa$. Let $f : [\Lambda]^{n+2} \to \kappa$ be a coloring where Λ denotes the set $\beth_{n+1}(\kappa)^+$. For every $a \in \Lambda$, let $f_a : [\Lambda - \{a\}]^{n+1} \to \kappa$ be the coloring given by

$$f_a(X) = f(X \cup \{a\})$$

Assume for the moment that

Claim. There exists $A \subseteq \Lambda$ of size $\beth_{n+1}(\kappa)$ such that for every $B \subseteq A$ of size $\beth_n(\kappa)$ and $b \in \Lambda - B$ there exists $a \in A - B$ such that $f_a \restriction [B]^{n+1} = f_b \restriction [B]^{n+1}$.

holds. Let $A \subseteq \Lambda$ be some set as in the claim and choose $a \in \Lambda - A$. By transfinite recursion, we can construct a transfinite sequence $(x_\alpha)_{\alpha \in \beth_n(\kappa)^+}$ of distinct elements of A such that

$$f_{x_\alpha} \restriction [\{x_\beta : \beta < \alpha\}]^{n+1} = f_a \restriction [\{x_\beta : \beta < \alpha\}]^{n+1}$$

for every $\alpha < \beth_n(\kappa)^+$. Consider the coloring $f_a \restriction [\{x_\alpha : \alpha < \beth_n(\kappa)^+\}]^{n+1}$. By the induction assumption, there exists a set

$$H \subseteq \{x_\alpha : \alpha < \beth_n(\kappa)^+\}$$

of size κ^+ which is homogeneous with respect to this coloring with some color $\theta \in \kappa$. We claim that H is also homogeneous with respect to the coloring f with the color θ. To see this, let $x_{\alpha_0}, x_{\alpha_1}, \ldots, x_{\alpha_{n+1}}$ be elements of H with $\alpha_0 < \alpha_1 < \cdots < \alpha_{n+1}$. Then

$$f(\{x_{\alpha_0}, \ldots, x_{\alpha_{n+1}}\}) = f_{x_{\alpha_{n+1}}}(\{x_{\alpha_0}, \ldots, x_{\alpha_n}\}) = f_a(\{x_{\alpha_0}, \ldots, x_{\alpha_n}\}) = \theta$$

It remains to prove that the claim holds.

Proof of the claim. By transfinite recursion, we shall construct a sequence of sets $(A_\alpha)_{\alpha < \beth_n(\kappa)^+}$ of size $\beth_{n+1}(\kappa)$ as follows:

- Let $A_0 \subseteq \Lambda$ be an arbitrary set of size $\beth_{n+1}(\kappa)$.
- Let $\alpha < \beth_n(\kappa)^+$ and assume that the set A_α has been constructed. We wish to construct a set $A_{\alpha+1}$. Notice that for each subset $B \subseteq A_\alpha$ of size $\beth_n(\kappa)$, the cardinality of the set
$$\{f_b \restriction [B]^{n+1} : b \in \Lambda - B\}$$
is at most
$$\kappa^{\beth_n(\kappa)} = 2^{\beth_n(\kappa)} = \beth_{n+1}(\kappa)$$
Thus, for each subset $B \subseteq A_\alpha$ of size $\beth_n(\kappa)$, we can choose a set $U_B \subseteq \Lambda - B$ of size $\beth_{n+1}(\kappa)$ such that
$$\{f_b \restriction [B]^{n+1} : b \in \Lambda - B\} = \{f_b \restriction [B]^{n+1} : b \in U_B\}$$
Set
$$A_{\alpha+1} = A_\alpha \cup \bigcup_{B \subseteq A_\alpha,\ |B| = \beth_n(\kappa)} U_B$$
Since the number of subsets of A_α of size $\beth_n(\kappa)$ is at most
$$\beth_{n+1}(\kappa)^{\beth_n(\kappa)} = 2^{\beth_n(\kappa)} = \beth_{n+1}(\kappa)$$
the set $A_{\alpha+1}$ has size $\beth_{n+1}(\kappa)$ provided that A_α has size $\beth_{n+1}(\kappa)$.
- Let $\gamma < \beth_n(\kappa)^+$ be a limit ordinal and assume that the set A_α of size $\beth_{n+1}(\kappa)$ has been constructed for each $\alpha < \gamma$. Set $A_\gamma = \bigcup_{\alpha < \gamma} A_\alpha$

A straightforward transfinite induction argument shows that A_α is of size $\beth_{n+1}(\kappa)$ for all $\alpha < \beth_n(\kappa)^+$ and hence $A = \bigcup_{\alpha < \beth_n(\kappa)^+} A_\alpha$ is of size $\beth_{n+1}(\kappa)$. Moreover, the set A satisfies conditions of the claim since any subset of A of size at most $\beth_n(\kappa)$ is contained in some A_α. □

Theorem 9 and 10 together show that the partition relation $(2^\kappa)^+ \to (\kappa^+)^2_\kappa$ is optimal. Erdös and Rado showed that for each $n \geq 1$ the partition relation $\beth_n(\kappa)^+ \to (\kappa^+)^{n+1}_\kappa$ is indeed optimal. However, we shall not include a proof of this more general fact.

1.3. **A theorem of Sierpiński.** In the previous subsection, we learned the relationship between κ and ν if the partition relation $\kappa \to (\omega)^2_\nu$ is to hold. In this subsection, we shall focus on the relationship between κ and λ if the partition relation $\kappa \to (\lambda)^2_2$ is to hold. More specifically, we are aiming to prove a theorem of Sierpiński stating that $2^\kappa \nrightarrow (\kappa^+)^2_2$. We shall need the following lemma.

Lemma 11. *Let κ be an infinite cardinal and $\prec$ be the (strict) linear order relation on ${}^\kappa 2$ given by*

$$f \prec g \iff f(\theta) < g(\theta) \text{ where } \theta = \min\{\alpha \in \kappa : f(\alpha) \neq g(\alpha)\}$$

Then there exists no sequence $(f_\alpha)_{\alpha<\kappa^+}$ of elements of ${}^\kappa 2$ of length κ^+ which is monotone with respect to $\prec$.

Proof. We shall prove that there exists no such increasing sequence. Assume towards a contradiction that there exists an increasing sequence of length κ^+ in the linearly ordered set $({}^\kappa 2, \prec)$.

Let $\gamma \leq \kappa$ be the least ordinal such that there exists an increasing sequence of length κ^+ in $({}^\gamma 2, \prec)$ and let $(x_\alpha)_{\alpha<\kappa^+}$ be such a sequence. By definition, for every $\alpha < \kappa^+$, we have that $x_\alpha \prec x_{\alpha+1}$. Hence, for every $\alpha < \kappa^+$, there exists $\epsilon_\alpha < \gamma$ such that

- $x_\alpha(\epsilon_\alpha) = 0$
- $x_{\alpha+1}(\epsilon_\alpha) = 1$
- $x_\alpha \restriction \epsilon_\alpha = x_{\alpha+1} \restriction \epsilon_\alpha$

Since $\kappa^+ = \bigcup_{\epsilon<\gamma}\{\alpha : \epsilon_\alpha = \epsilon\}$ and κ^+ is a regular cardinal, there exists $\epsilon < \gamma$ such that $\mathcal{X} = |\{x_\alpha : \epsilon_\alpha = \epsilon\}| = \kappa^+$.

Let x_ξ and x_η be two distinct elements of this set. If it were the case that $x_\xi \restriction \epsilon = x_\eta \restriction \epsilon$, then, by the definition of ϵ, we would have $x_\xi(\epsilon) = x_\eta(\epsilon) = 0$ and $x_{\xi+1}(\epsilon) = x_{\eta+1}(\epsilon) = 1$, in which case we have $x_\xi \prec x_{\eta+1}$ and $x_\eta \prec x_{\xi+1}$ contradicting that x_ξ and x_η are distinct. Therefore, for every distinct x_ξ and x_η in the set $\mathcal{X}$, we have that $x_\xi \restriction \epsilon \neq x_\eta \restriction \epsilon$. However, this implies that there exists an increasing sequence of length κ^+ in the linearly ordered set $({}^\epsilon 2, \prec)$, namely, the sequence that consists of x_η's in the set $\mathcal{X}$ indexed in the obvious manner. This contradicts the minimality of γ.

Hence there exists no increasing sequence of length κ^+ in $({}^\kappa 2, \prec)$. Carrying out this proof symmetrically, one can show that no decreasing sequence of length κ^+ in $({}^\kappa 2, \prec)$. □

We are now ready to prove the main theorem of this subsection.

Theorem 12 (Sierpiński)**.** *For every infinite cardinal κ, we have that $2^\kappa \nrightarrow (\kappa^+)^2_2$.*

Proof. As before, it suffices to color $[{}^\kappa 2]^2$ with 2 colors so that there exists no homogeneous set of size κ^+. Let $(g_\alpha)_{\alpha<2^\kappa}$ be an enumeration of elements of ${}^\kappa 2$ and $f : [{}^\kappa 2]^2 \to 2$ be the coloring given by

$$f(\{g_\alpha, g_\beta\}) = 0 \iff (\alpha < \beta \Leftrightarrow g_\alpha \prec g_\beta)$$

where $\prec$ is the lexicographic order relation defined before. It is easily seen that any homogeneous subset of ${}^\kappa 2$ of size θ induces a monotone sequence in $({}^\kappa 2, \prec)$ of length θ. Thus, there cannot be any homogeneous set of size κ^+ by the previous lemma. □

Since $2^\kappa \geq \kappa^+$ for every cardinal κ, the following corollary immediately follows from Sierpiński's theorem.

Corollary 13. *For all infinite cardinals κ, $\kappa^+ \nrightarrow (\kappa^+)^2_2$.*

Corollary 13 shows that the partition relation $\kappa \to (\kappa)^2_2$, the natural generalization of Ramsey's theorem, does not hold for successor cardinals. Can it hold for limit cardinals? Does there exist an uncountable limit cardinal κ such that the partition relation $\kappa \to (\kappa)^2_2$ holds?

1.4. **Weakly compact cardinals.** Any uncountable cardinal κ such that $\kappa \to (\kappa)^2_2$ is called a *weakly compact cardinal*[5]. The following is an important consequence of Sierpiński's theorem.

Theorem 14. *Weakly compact cardinals are inaccessible.*

Proof. Let κ be a weakly compact cardinal. We wish to show that κ is inaccessible, i.e. κ is uncountable, regular and for all cardinals $\theta < \kappa$ we have $2^\theta < \kappa$. That κ is uncountable follows from the definition.

To show that κ is regular, assume towards a contradiction that $cf(\kappa) = \gamma < \kappa$, say $(\delta_\xi)_{\xi<\gamma}$ is an increasing sequence of ordinals such that $sup\{\delta_\xi : \xi < \gamma\} = \kappa$. Consider the coloring $f : [\kappa]^2 \to 2$ given by

$$f(\{\alpha, \beta\}) = 1 \iff \exists \xi < \gamma \quad \alpha, \beta \in [\delta_\xi, \delta_{\xi+1})$$

It is straightforward to check that any homogeneous set with respect to this coloring has size $< \kappa$ which contradicts that $\kappa \to (\kappa)^2_2$. Therefore κ is regular.

To show that κ is strong limit, assume towards a contradiction that there exists $\theta < \kappa$ such that $\kappa \leq 2^\theta$. Since $\theta^+ \leq \kappa$ and $\kappa \to (\kappa)^2_2$, we have that $2^\theta \to (\kappa)^2_2$ and hence $2^\theta \to (\theta^+)^2_2$ which contradicts Sierpiński's theorem. Therefore κ is an inaccessible cardinal. □

Weakly compact cardinals are a special type of *large cardinals*[6]. On the one hand, that there are no weakly compact cardinals is relatively consistent[7] with ZFC. On the other hand, the existence of inaccessible cardinals (and hence, of weakly compact cardinals) implies that ZFC has a set-model. It follows from Gödel's completeness theorem and second incompleteness theorem that this statement cannot be proven in ZFC and cannot be even shown in ZFC to be relatively consistent with ZFC, provided that ZFC is consistent. It is widely believed among set theorists that the existence of weakly compact cardinals is relatively consistent with ZFC.

Before we conclude this section, we would like to mention the following fact which shows that weakly compact cardinals indeed satisfy all partition relations they can possibly satisfy.

Fact 15. [7, Theorem 7.8] *Let κ be an uncountable cardinal. Then κ is weakly compact if and only if $\kappa \to (\kappa)^n_\lambda$ for every $n \in \omega$ and every cardinal $\lambda < \kappa$.*

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[5]The reason for this terminology is that, for any such κ, the infinitary logic $\mathcal{L}_{\kappa,\kappa}$ satisfies an analogue of the famous compactness theorem of first-order logic.

[6]There are truly marvelous facts about large cardinals, which this margin is too narrow to contain. We refer the reader to [7], the most comprehensive treatment of the subject.

[7]A sentence φ is said to be relatively consistent with a theory T if the consistency of T implies the consistency of the theory $T + \varphi$.

2. The tree property

In this section, we turn our attention to mathematical objects that are seemingly unrelated to those we have been dealing with: Trees.

2.1. **Set-theoretic trees.**

In mathematics, there are various different notions of a tree. For example, from the point of view of graph theory, a tree is a connected acyclic graph, i.e. a graph whose distinct vertices are connected by a unique path.

From the point of view of set theory, a *tree* is a partially ordered set $(T, <)$ with a least element such that the set $pred_T(x) = \{y \in T : y < x\}$ of predecessors of x is well-ordered by $<$ for every $x \in T$. Let $(T, <)$ be a tree. Then

- The least element of a tree $(T, <)$ is called its *root*.
- For each $x \in T$, the *height* of the element x is the order-type $ht(x)$ of the well-ordered set $pred_T(x)$.
- For each ordinal α, the α-th level of the tree $(T, <)$ is the set
$$T_\alpha = \{x \in T : ht(x) = \alpha\}$$
- The *height* of the tree $(T, <)$ is the ordinal $ht(T) = \min\{\alpha : T_\alpha = \emptyset\}$.
- A *branch* of the tree $(T, <)$ of is a maximally linearly ordered subset of T.
- A branch $S \subseteq T$ of the tree $(T, <)$ is said to be of *length* α if the order type of the well-ordered set $(S, <)$ is α.

One can visually represent trees using their Hasse diagrams: We place a vertex for every point in the tree $(T, <)$ and connect every vertex v to its successor vertex $v^+ = \min\{w \in T : v < w\}$. It is easily seen that no two vertices can be joined via two distinct paths in the corresponding diagram, which justifies the usage of the word "tree".

Exercise. Prove that any graph-theoretic tree with a distinguished vertex naturally induces a set-theoretic tree of height at most ω; and any set-theoretic tree of height at most ω naturally induces a graph-theoretic tree.
Exercise. Prove that there exists a tree of height of ω_1 and of size $2^{\aleph_0}$.

2.2. **Aronszajn trees.**

Let κ be an infinite cardinal. A tree $(T, <)$ is said to be a *κ-Aronszajn tree* if

- $ht(T) = \kappa$,
- $|T_\alpha| < \kappa$ for every ordinal α, and
- $(T, <)$ has no branch of length κ

An ω_1-Aronszajn tree is usually called an *Aronszajn tree.* An infinite cardinal κ is said to have the *tree property* if there exist no κ-Aronszajn trees. Notice that we can restate König's tree lemma in this terminology as follows.

Theorem 16 (König's lemma, revisited)**.** *ω has the tree property.*

The obvious question to ask at this point is the following: Do other cardinals have the tree property? The following exercise shows that the tree property fails at singular cardinals.

Exercise. Prove that there exists a κ-Aronszajn tree for every infinite singular cardinal κ.

What about regular cardinals? Can the tree property hold at a regular infinite cardinal other than ω? Nachman Aronszajn proved that the answer to this question is negative for the first uncountable cardinal ω_1.

Theorem 17 (Aronszajn). *ω_1 does not have the tree property.*

Proof. We refer the reader to [8, Theorem III.5.9] or [6, Theorem 9.16] for two different constructions. □

Whether or not the tree property holds at larger regular cardinals turns out to be a difficult question whose answer is beyond the limits of ZFC and consequently, of this course[8].

2.3. **Tree property vs Weak compactness.** In this subsection, we shall focus on the exact relationship between the tree property and weak compactness.

Theorem 18. *Let κ be an uncountable cardinal. Then κ is weakly compact if and only if it is inaccessible and has the tree property.*

Proof. We shall only prove the left-to-right direction of this theorem. The reader is referred to [6, Lemma 9.26] for the right-to-left direction of this theorem. Let κ be a weakly compact cardinal. By Theorem 14, κ is inaccessible. Hence we only need to prove that κ has the tree property.

Let $(T, <_T)$ be a tree with $ht(T) = \kappa$ and $|T_\alpha| < \kappa$ for every $\alpha < \kappa$. We wish to prove that T has a branch of length κ.

Since κ is regular, $ht(T) = \kappa$ and $|T_\alpha| < \kappa$ together imply that $|T| = \kappa$ and hence, by relabelling the elements of T, we can assume without loss of generality that $T = \kappa$. Let $\prec$ be the (strict) linear order relation on T given as follows:

- $\xi \prec \zeta$ whenever $\xi <_T \zeta$.
- $\xi \prec \zeta$ whenever ξ and ζ are distinct and incomparable elements of T such that $\min(pred_T(\xi) - pred_T(\zeta)) < \min(pred_T(\zeta) - pred_T(\xi))$

We skip the details of checking that $\prec$ is indeed a linear order relation. Consider the coloring $f : [\kappa]^2 \to 2$ given by

$$f(\{\xi, \zeta\}) = 1 \iff (\xi < \zeta \Leftrightarrow \xi \prec \zeta)$$

It follows from weak compactness of κ that there exists a homogeneous set $H \subseteq \kappa$ of size κ. Consider the set

$$K = \{\xi \in T : |\{\zeta \in H : \xi <_T \zeta\}| = \kappa\}$$

Observe that each level of T contains some element of K. Consequently, K is a branch of length κ if every two elements of K are comparable with respect to $<_T$.

Assume towards a contradiction that there exist $<_T$-incomparable elements ξ and ζ in K. Without loss of generality, we may assume that $\xi \prec \zeta$. It is easily seen that we can find ordinals $\alpha < \beta < \gamma$ in H such that $\xi <_T \alpha$, $\zeta <_T \beta$ and $\xi <_T \gamma$. It follows from the definition of $\prec$ that $\alpha \prec \beta$ and $\gamma \prec beta$. On the other

[8]For example, it is known that the theory ZFC+"ω_2 has the tree property" is consistent if and only if the theory ZFC+"there exists a weakly compact cardinal" is consistent. Assuming the existence of appropriate large cardinals, it is relatively consistent with ZFC that the tree property holds at ω_n for each $2 \leq n < \omega$ and at $\omega_{\omega+1}$. Since the study of the tree property is an active research area in set theory, the author, not being an expert in this area, is almost certain that there have been stronger results in the literature of which he is unaware. The curious reader is referred to Google for more information.

hand, $f(\{\alpha,\beta\}) = 0 \neq 1 = f(\{\gamma,\beta\})$, which contradicts the homogeneity of H. Therefore every two elements of K are comparable with respect to $<_T$ and hence K is a branch of length κ. This completes the proof. □

3. Coda

This one-week course intended to serve as a brief introduction to combinatorial set theory by providing some key results. The reader who has been seduced by the intrinsic beauty of the subject and of the results is strongly suggested to read the relevant parts of the books listed in the references.

Acknowledgements. The author would like to thank the staff of Nesin Mathematics Village for their hospitality and Ali Nesin for having built the village. The author also wishes to express his appreciation for Ludwig van Beethoven who composed most of the great music that accompanied typing of these lecture notes.

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