MATH 501, Analysis, Homework 6

1. (2 pts) Let $(\mathbf{X}, \mathcal{M}, \mu)$ and $(\mathbf{Y}, \mathcal{N}, \eta)$ be measure spaces with $\mathcal{N} \neq \mathcal{P}(\mathbf{Y})$ and $\mu(A) = 0$ for some non-empty $A \in \mathcal{M}$. Show that the product measure space $(\mathbf{X} \times \mathbf{Y}, \mathcal{M} \otimes \mathcal{N}, \mu \times \eta)$ is not complete.

2. (3 pts) Consider the measure spaces $([0,1], \mathcal{B}([0,1]), \mathbf{m})$ and $([0,1], \mathcal{B}([0,1]), \eta)$ where $\mathbf{m}$ is the Lebesgue measure and η is the counting measure. Let

$$D = \{(x,x) \in [0,1] \times [0,1] : x \in [0,1]\}$$

Prove that

$$\int_{[0,1]\times[0,1]} \chi_D(x,y)\ d(\mathbf{m} \times \eta) = \infty$$

3. (2+3 pts)

a) Consider the measure space $(\mathbb{N}, \mathcal{P}(\mathbb{N}), \mu)$ where the measure $\mu : \mathcal{P}(\mathbb{N}) \to [0,\infty]$ is given by

$$\mu(S) = \sum_{n \in S} \frac{1}{3^n}$$

Let $f : \mathbb{N} \to \mathbb{R}$ be any integrable function. Find a sequence of simple functions $f_n : \mathbb{N} \to \mathbb{R}$ such that $\lim_{n\to\infty} f_n(k) = f(k)$ for all $k \in \mathbb{N}$. Then express the integral

$$\int_{\mathbb{N}} f(k)\ d\mu$$

in terms of $f(k)$'s.

b) Now consider the product space

$$(\mathbb{R} \times \mathbb{N}, \mathcal{B}(\mathbb{R}) \otimes \mathcal{P}(\mathbb{N}), \mathbf{m} \times \mu)$$

where m is the usual Lebesgue measure on $\mathbb{R}$. You are given that the function $g : \mathbb{R} \times \mathbb{N} \to \mathbb{R}$ defined by $g(x,k) = \dfrac{2^k}{1+x^2}$ is measurable. Evaluate the integral

$$\int_{\mathbb{R}\times\mathbb{N}} \frac{2^k}{1+x^2}\ d(m \times \mu)$$

Justify each step of your calculation by referring to the relevant theorems.

Solutions