MATH 501, Analysis, Homework 5

1. (3 pts) Let $\mathcal{C} \subseteq [0,1]$ be the Cantor set and let $f : [0,1] \to \mathbb{R}$ be given by

$$f(x) = \begin{cases} 0 & \text{if } x \in \mathcal{C} \\ x & \text{if } x \notin \mathcal{C} \end{cases}$$

Show that f is Riemann integrable and find $\int_0^1 f(x)dx$.

2. (2 pts) Let $(\mathbf{X}, \mathcal{M}, \mu)$ be a measure space. Let $f, g \in L(\mathbf{X}, \mathcal{M}, \mu)$ and let $(f_n)_{n\in\mathbb{N}}$ and $(g_n)_{n\in\mathbb{N}}$ be sequences of functions in $L(\mathbf{X}, \mathcal{M}, \mu)$. Prove that if $f_n \longrightarrow f$ in measure and $g_n \longrightarrow g$ in measure, then $f_n + g_n \longrightarrow f + g$ in measure.

3. (2 pts) Consider the measure space $(\mathbb{N}, \mathcal{P}(\mathbb{N}), \mu)$ where μ is the counting measure. Let $f \in L(\mathbb{N}, \mathcal{P}(\mathbb{N}), \mu)$ and let $(f_n)_{n\in\mathbb{N}}$ be a sequence of functions in $L(\mathbb{N}, \mathcal{P}(\mathbb{N}), \mu)$. Show that if f_n's and f are **integer valued** and $f_n \to f$ in L^1, then there exists $k \in \mathbb{N}$ such that for all $n \geq k$ we have $f_n = f$.

4. (3 pts) Let $(\mathbf{X}, \mathcal{M}, \mu)$ be a probability space. Consider the function

$$\rho : L(\mathbf{X}, \mathcal{M}, \mu) \times L(\mathbf{X}, \mathcal{M}, \mu) \to [0,1]$$

given by

$$\rho(f, g) = \int_X \min\{|f - g|, 1\}\ d\mu$$

Let $f \in L(\mathbf{X}, \mathcal{M}, \mu)$ and let (f_n) be a sequence of functions in $L(\mathbf{X}, \mathcal{M}, \mu)$. Show that if $f_n \to f$ in measure, then for all $\epsilon \in \mathbb{R}^+$ there exists $k \in \mathbb{N}$ such that for all $n \in \mathbb{N}$ with $n \geq k$, we have that $\rho(f_n, f) < \epsilon$.

Solutions