MATH 501, Analysis, Homework 4

1. (2 pts) Let $(X, \mathcal{M}, \mu)$ be a finite measure space and let $f, f_n \in L^1(X, \mathcal{M}, \mu)$ be functions such that $f_n \to f$ uniformly. Prove that $\int_X f_n \, d\mu \to \int_X f \, d\mu$.

2. (2+2 pts) Compute the following limits.

- $\displaystyle \lim_{k\to\infty} \int_{[0,1]} \frac{1+kx^2}{(1+x^2)^k} \, d\mathbf{m}$
- $\displaystyle \lim_{n\to\infty} \int_{(0,\infty)} \frac{n\sin(x/n)}{x(1+x/n)^n} \, d\mathbf{m}$

3. (2+2 pts) Let $f : \mathbb{R} \to \mathbb{R}$ be given by

$$f(x) = \begin{cases} \dfrac{1}{\sqrt{x}} & \text{if } x \in (0,1) \\ 0 & \text{otherwise} \end{cases}$$

Let $(r_n)_{n\in\mathbb{N}}$ be an enumeration of $\mathbb{Q}$ and consider the measurable function $g : \mathbb{R} \to \overline{\mathbb{R}}$ given by

$$g(x) = \sum_{n=0}^{\infty} \frac{f(x - r_n)}{2^n}$$

Show that

a. $g \in L^1(\mathbb{R}, \mathfrak{L}, \mathbf{m})$

b. The function g is unbounded on every interval.

Solutions