MATH 501, Analysis, Homework 3

1. (2 pts) Let $(X, \mathcal{M}, \mu)$ be a measure space and let $f_n : X \to \mathbb{R}$ be measurable for all $n \in \mathbb{N}$. Prove that $\{x \in X : \lim_{n\to\infty} f_n(x) \text{ exists } \} \in \mathcal{M}$.

Hint. Recall that for a sequence $(a_n)_{n\in\mathbb{N}}$ of real numbers, $\lim_{n\to\infty} a_n$ exists if and only if $(a_n)_{n\in\mathbb{N}}$ is Cauchy.

2. (2 pts) Find an index set I and a family $\{f_i : \mathbb{R} \to \mathbb{R} \mid i \in I\}$ of functions such that

- f_i is Borel measurable for all $i \in I$,
- The function $f : \mathbb{R} \to \mathbb{R}$ given by

$$f(x) = \sup_{i\in I} f_i(x) \quad \text{for all } x \in \mathbb{R}$$

  is *not* Borel measurable.

3. (3+3 pts) Let $(X, \mathcal{M}, \mu)$ be a measure space and fix a map $f \in L^+(X, \mathcal{M}, \mu)$. Consider the map $\eta : \mathcal{M} \to [0, +\infty]$ given by

$$\eta(E) = \int_E f \, d\mu$$

a) Prove that η is a measure on the measurable space $(X, \mathcal{M})$.
b) Prove that, for every $g \in L^+(X, \mathcal{M}, \eta)$, we have that

$$\int g \, d\eta = \int fg \, d\mu$$

Hint. Apply the "usual trick". First, show that this holds for simple functions and try to extend your result via appropriate tools.

Solutions