MATH 501, Analysis, Homework 2

1. (3 pts)[1] Let $\mathcal{A} \subseteq \mathcal{P}(X)$ be an algebra on a set X and $\rho : \mathcal{A} \to [0, +\infty]$ be a premeasure on X. Recall that the map $\mu^* : \mathcal{P}(X) \to [0, \infty]$ given by

$$\mu^*(S) = \inf \left\{ \sum_{i=1}^{\infty} \rho(A_i) : A_i \in \mathcal{A},\ S \subseteq \bigcup_{i=1}^{\infty} A_i \right\}$$

is an outer measure on X. Let $E \subseteq X$ be such that $\mu^*(E) < \infty$. Show that E is μ^*-measurable if and only if there exists a set $A \in \mathcal{M}(\mathcal{A})$ such that $E \subseteq A$ and $\mu^*(A - E) = 0$.

2. (2 pts) Let $F(x) = \lfloor x \rfloor$ be the floor function. Consider the Lebesgue-Stieltjes measure $\mu_F : \mathcal{B}(\mathbb{R}) \to [0, \infty]$ associated to F. Show that there exists a co-countable open set $S \subseteq \mathbb{R}$ which is μ_F-null.

3. (2 pts) Prove that there exists an uncountable closed set $D \subseteq \mathbb{R}$ with the property that D contains no open intervals and $\mathbf{m}(D) = \infty$, where $\mathbf{m}$ is the usual Lebesgue measure on $\mathbb{R}$.

4. (3 pts)[2] Let $A \subseteq \mathbb{R}$ be a Lebesgue measurable set such that $0 < \mathbf{m}(A) < \infty$ where $\mathbf{m}$ is the usual Lebesgue measure on $\mathbb{R}$.

a. Prove that there exist a closed set $F \subseteq A$ and an open set $A \subseteq G$ such that we have $3\mathbf{m}(G) < 4\mathbf{m}(F)$.

b. Let G be as in part a. Prove that there exists an open interval $I \subseteq G$ such that $3\mathbf{m}(I) < 4\mathbf{m}(F \cap I)$.

c. Let I be as in part b. Let $\delta = \mathbf{m}(I)/2$. Show that

$$B(0, \delta) \subseteq A - A = \{a - b : a, b \in A\}$$

(Keep in mind that $(F \cap I) \cup (x + (F \cap I)) \subseteq I \cup (x + I)$. Now try to argue that $F \cap I$ and $x + (F \cap I)$ cannot be disjoint if $|x| < \delta$. Then finish the proof!)

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[1] This is Folland's Exercise 1.18.b. You may want to take a look at the other parts of the question to get a hint.

[2] In this question, we shall prove **Steinhaus's theorem** which states that $A - A$ contains an open interval centered at the origin for any Lebesgue measurable set A with positive measure.

Solutions