MATH 501, Analysis, Homework 1

1. (3 pts) Let X be a non-empty set and $\mathcal{M} \subseteq \mathcal{P}(X)$ be a σ-algebra on X generated by $\mathcal{E} \subseteq \mathcal{M}$. Show that

$$\mathcal{M} = \bigcup_{F \in \mathcal{P}_c(\mathcal{E})} \mathcal{M}(F)$$

where $\mathcal{M}(F)$ denotes the σ-algebra generated by F and $\mathcal{P}_c(\mathcal{E})$ denotes the set of countable subsets of $\mathcal{E}$, that is, $\mathcal{P}_c(\mathcal{E}) = \{F \subseteq \mathcal{E} : F \text{ is countable}\}$.

2. (2 pts) Show that the set

$$\{(x, y) \in \mathbb{R}^2 : x^7 + xy + y^7 \notin \mathbb{Q}\}$$

is a Borel subset of $\mathbb{R}^2$. (**Hint.** Recall that the inverse images of open sets are open under continuous mappings.)

3. (2 pts) Let $(X, \mathcal{M})$ be a measurable space and $\mu : \mathcal{M} \to [0, \infty]$ be a *finitely additive* measure on $(X, \mathcal{M})$. Suppose that if $A_1, A_2, \dots \in \mathcal{M}$ with $A_1 \subseteq A_2 \subseteq \dots$, then $\mu(\bigcup_{n=1}^{\infty} A_n) = \lim_{n\to\infty} \mu(A_n)$. Prove that μ is indeed a (σ-additive) measure.

4. (3 pts) Let X be an uncountable set and $\mathcal{M} = \{A \subseteq X : A \text{ or } A^c \text{ is countable}\}$. Recall that the triple $(X, \mathcal{M}, \eta)$ is a measure space where $\eta : \mathcal{M} \to [0, \infty]$ is given by

$$\eta(A) = \begin{cases} 1 & \text{if } A^c \text{ is countable} \\ 0 & \text{if } A \text{ is countable} \end{cases}$$

Consider the relation $\sim$ on $\mathcal{M}$ given by

$$A \sim B \Leftrightarrow \eta(A \triangle B) = 0$$

Show that $\sim$ is an equivalence relation and find the cardinality of $\mathcal{M}/\sim$.

Solutions