The Mutual Information in the Vicinity of Capacity-Achieving Input Distributions

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***Abstract*—The mutual information is bounded from above by a decreasing affine function of the square of the distance between the input distribution and the set of all capacity-achieving input distributions $\Pi_{\mathcal{A}}$, on small enough neighborhoods of $\Pi_{\mathcal{A}}$, using an identity due to Topsøe and the Pinsker's inequality, assuming that the input set of the channel is finite and the constraint set $\mathcal{A}$ is polyhedral, i.e., can be described by (possibly multiple but) finitely many linear constraints. Counterexamples demonstrating nonexistence of such a quadratic bound are provided for the case of infinitely many linear constraints and the case of infinite input sets. Using Taylor's theorem with the remainder term, rather than the Pinsker's inequality and invoking Moreau's decomposition theorem the exact characterization of the slowest decrease of the mutual information with the distance to $\Pi_{\mathcal{A}}$ is determined on small neighborhoods of $\Pi_{\mathcal{A}}$. Corresponding results for classical-quantum channels are established under separable output Hilbert space assumption for the quadratic bound and under finite-dimensional output Hilbert space assumption for the exact characterization. Implications of these observations for the channel coding problem and applications of the proof techniques to related problems are discussed.**

***Index Terms*—Mutual information, Shannon center, polyhedral convexity, Moreau's decomposition theorem, Taylor's theorem, Fisher Information.**

I. Introduction

For a given stationary memoryless channel, for any positive integer n and positive real number ϵ, let $N(n,\epsilon)$ be the largest number of messages that a block code of length n with maximum error probability less than ϵ can have. For discrete memoryless channels (DMCs) by the channel coding theorem and its strong converse, [1], [2], [3], we know that

$$\ln N(n,\epsilon) = Cn + o(n) \qquad \forall \epsilon \in (0,1),$$

where C is the Shannon capacity and $o(n)$ may depend on ϵ.

The Shannon capacity C of a DMC with the transition probability matrix W is equal to the maximum value of the mutual information $I(p;W)$ over all input distributions p, see [2, (4.2.3)] and [3, (3.2)]. The input distributions p satisfying $I(p;W) = C$ are called capacity-achieving. The set of all capacity-achieving input distributions Π is a closed and convex set, as a result of the continuity and the concavity of $I(p;W)$ in p. Although Π may have infinitely many distinct elements, they all induce the same output distribution q_W, called the Shannon center, [2, Theorem 4.5.1], and consequently the gradient of the mutual information at $\bar{p}$ is the same vector for all $\bar{p}$ in Π, see [2, (4.5.5)]. With a slight abuse of notation, we denote this vector by $\nabla I(p;W)|_{\Pi}$.

In his seminal paper [4], Strassen sharpened the results of Shannon in [1] by establishing higher order asymptotic expansions for both source and channel coding problems. In particular, for DMCs Strassen established[1] [4, Theorem 1.2]

$$\ln N(n,\epsilon) = Cn - Q^{-1}(\epsilon)\sqrt{V_{\min} n} + O(\ln n) \quad \forall \epsilon \in \left(0, \tfrac{1}{2}\right),$$

where $Q^{-1}(\cdot)$ is the inverse of the Q function and $V_{\min}$ is the dispersion of the channel, which is defined as the minimum value of a continuous function of p over Π. Starting with [5] and [6], there has been a reviewed interest in sharper characterizations of the optimal performance for both source and channel coding problems in the spirit of [4], see [7], [8], [9], [10], [11], [12], [13], [14], [15], and [16].

In line with standard practice in information theory, Strassen proved two distinct results to establish [4, Theorem 1.2]: an impossibility result establishing an upper bound on $\ln N(n,\epsilon)$ applicable to all codes, and an achievability result establishing a lower bound on $\ln N(n,\epsilon)$ by analyzing the performance of a judiciously chosen code ensemble.

While establishing his impossibility result in [4], Strassen proved for channels with finite input and output sets that there exist positive constants γ and δ for which the mutual information satisfies

$$I(p;W) \le C - \gamma \|p - \bar{p}\|^2 \qquad \text{if } \|p - \bar{p}\| \le \delta \tag{1}$$

[1]Strassen asserts in [4, Theorem 1.2] that the same bound holds with $V_{\max}$ for all $\epsilon \in \left[\tfrac{1}{2}, 1\right)$, as well. That claim, however, is not accurate for exotic channels, see [7, Theorem 45 and §3.4.1].

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where $\overline{p}$ is the projection of p to the set of all capacity-achieving input distributions Π in the underlying Euclidean space, and hence $\|p-\overline{p}\|$ is the distance of p to Π the same space. Strassen's brief and elegant argument relies implicitly on the fact that for any $p \notin \Pi$, the direction $p-\overline{p}$ cannot be simultaneously orthogonal to the gradient of mutual information at $\overline{p}$, i.e., orthogonal to $\nabla I(p;W)|_{\Pi}$, and in the kernel of the linear transformation relating the input distributions to the output distributions, i.e., in $\mathcal{K}_W$. We believe one of the claims in Strassen's proof, which holds trivially for some channels, requires a more nuanced justification to be valid for all channels with finite input and output alphabets. Nevertheless, the claim can be established as is using polyhedral convexity as we discuss in more detail in Appendix A.

One of the claims of Polyanskiy et al. in [6] is to establish (1) with an explicit coefficient γ. They apply an orthogonal decomposition to assert $p-\overline{p}=v_0+v_{\perp}$, where v_0 is the projection of $p-\overline{p}$ to $\mathcal{K}_W$. Then they argue $v_0^T \nabla I(p;W)|_{\Pi} \leq -\Gamma \|v_0\|$ for some $\Gamma > 0$, see [6, (500)]. This claim, however, is wrong for some p's on certain channels as we demonstrate through a particular channel in Appendix B.

In our judgment, the issue overlooked in [6] is the following: the projection of $p-\overline{p}$ to $\mathcal{K}_W$ can have a non-zero component that is also orthogonal to $\nabla I(p;W)|_{\Pi}$ and this component may, in principle, be equal to the projection of $p-\overline{p}$ to $\mathcal{K}_W$ itself. The principle used by Strassen in [4], however, asserts merely that this component cannot be the $p-\overline{p}$ vector itself. This principle can be strengthened using polyhedral convexity to assert that the angle between the $p-\overline{p}$ vector and the subspace of $\mathcal{K}_W$ that is orthogonal to $\nabla I(p;W)|_{\Pi}$ cannot be less than a positive constant, determined by the channel. In §IV, we use this observation together with Pinsker's inequality and an orthogonal decomposition to subspaces, to prove Theorem 1, which implies (1) with explicit expressions for γ and δ for channels with finite input sets and arbitrary output spaces. Replacing the total variation norm with the trace norm, in the proof of Theorem 1, we establish Theorem 3 in §VI-A, which implies (1) with explicit expressions for γ and δ for classical-quantum channels with finite input sets whose density operators are defined on separable Hilbert spaces.

The orthogonal decomposition to a closed convex cone and its polar cone via Moreau's decomposition theorem, rather than the orthogonal decomposition into subspaces, proves to be the more effective use of the orthogonal decomposition idea for the problem at hand. In §V, we employ Moreau's decomposition theorem and Taylor's theorem with the remainder term to prove Theorem 2, which determines the best, i.e., the largest possible, γ coefficient for Strassen's bound in (1) for channels with finite input sets and arbitrary output spaces. Theorem 4 of §VI-B, establishes the corresponding result for classical-quantum channels with finite dimensional Hilbert spaces at the output.

Recently in [17], Cao and Tomamichel presented the first complete proof of (1), in the spirit of [4]. First the cone generated by the vectors $p-\overline{p}$ for $p \notin \Pi$ is proved to be closed, and then a second-order Taylor series expansion for the parametric family of functions $\{I(\overline{p}+\tau(p-\overline{p});W)\}_{p\notin\Pi}$ at $\tau=0$ with a uniform approximation error term for all $p \notin \Pi$ is obtained. Then (1) is established using the extreme value theorem, the fact that $p-\overline{p}$ cannot be an element of $\mathcal{K}_W$ that is orthogonal to $\nabla I(p;W)|_{\Pi}$, and the Taylor series expansion. Cao and Tomamichel, later generalized their analysis to the case with finitely many linear constraints, in [18].

Not only Strassen's [4] but also many other works establishing impossibility results for the channel coding problem since then explicitly relied on (1), e.g., [6], [7], [8], [9], and [10]. Determining explicit expressions for (δ,γ) pairs for which (1) holds might be useful in obtaining non-asymptotic versions of some of these results. In addition, a more complete understanding of the behavior of the mutual information around Π is valuable in and of itself, given the recurrent emergence of this behavior in [6], [7], [8], [9], and [10]. We will consider the constrained version of the problem to shed light on the aspects of the aforementioned behavior that do not emerge in the unconstrained case. Let us finish this introductory discussion with a brief overview of the paper and the main results.

In §II, we review those concepts and results from convex analysis that will be useful in our discussion in the following sections, such as cones, angle between a pair of cones, projections to closed convex sets, polyhedral convexity, and Moreau's decomposition theorem.

In §III, we introduce the channel model we work with in §IV-§V and review certain fundamental observations about the Kullback-Leibler divergence, the mutual information, the Shannon capacity $C_{\mathcal{A}}$, the Shannon center $q_{\mathcal{A}}$, and the capacity achieving input distributions $\Pi_{\mathcal{A}}$ for the case when input distributions p are required to be elements of a closed convex constraint set $\mathcal{A}$. Unlike [4], [6], [17], and [18], we do not assume the channel to have a finite output set, instead we assume the output space of the channel to be a measurable space.

In §IV, we prove Theorem 1 using an orthogonal decomposition to subspaces, Pinsker's inequality, and the minimum angle idea via Lemmas 1 and 2 of §II. Theorem 1 establishes the quadratic decrease of $I(p;W)$ with the distance to $\Pi_{\mathcal{A}}$ for the case when $p \in \mathcal{A}$, with explicit expressions for γ and δ assuming that the input set of the channel is finite and the constraint set $\mathcal{A}$ is polyhedral. Both finite input set assumption and polyhedral constraint set assumption are necessary, as we demonstrate via Examples 1 and 1 in §IV. Theorem 1 is the first result establishing (1) with explicit expressions for γ and δ, except for the corresponding result in the conference paper associated with current work, [19, Theorem 1]

In §V, we prove Theorem 2 using Taylor's theorem with the remainder term and Moreau's decomposition theorem under the hypotheses that the input set of the channel is finite, the constraint set $\mathcal{A}$ is polyhedral, and certain moment, see $\kappa_{\mathcal{A}}$ in (70), is finite. Theorem 2 characterizes the slowest decay of $I(p;W)$ with the distance to $\Pi_{\mathcal{A}}$ by determining the order and the coefficient of the leading non-zero term of its Taylor expansion. The finite $\kappa_{\mathcal{A}}$ hypothesis is not superficial even for channels with finite input sets, see Example 3 in §V-A.

In §VI, we first recall the quantum information-theoretic framework and review certain fundamental observations about quantum information-theoretic quantities in a way analogous to our discussion in §III. Then in §VI-A, assuming that the constraint set is polyhedral, we prove (1) for any classical-quantum channel with a finite input set and a separable Hilbert space using the quantum Pinsker's inequality and the minimum angle idea, similar to §IV. In §VI-B, assuming that the constraint set is polyhedral, we characterize the slowest decay of the quantum mutual information around $\Pi_{\mathcal{A}}$ for a classical-quantum channel with a finite input set and a finite-dimensional Hilbert space, in a way analogous to §VI-B.

In §VII, we discuss our results and their the implications for the channel coding problem and possible applications of the proof techniques to certain related problems.

II. Preliminaries on Convex Analysis

A. The Angle Between a Pair of Cones

A subset $\mathcal{C}$ of the Euclidean space $\mathbb{R}^n$ is said to be a cone iff $\{\tau p : \tau \geq 0\} \subset \mathcal{C}$ for all $p \in \mathcal{C}$. Hence, $\mathbf{0} \in \mathcal{C}$ for any cone by definition and $\{\mathbf{0}\}$ is a cone by convention, where $\mathbf{0}$ is the all zeros vector. A cone $\mathcal{C}$ is closed iff $cl(\mathcal{C}) = \mathcal{C}$, i.e., if its closure is itself. The cone generated by a non-empty set $\mathcal{A} \subset \mathbb{R}^n$ is the set of all conical combination of elements of $\mathcal{A}$, see [20, Definitions A.1.4.5]:

$$cone(\mathcal{A}) := \left\{\sum\nolimits_{\imath=1}^{\jmath} \tau_\imath p_\imath : \tau_\imath \geq 0, p_\imath \in \mathcal{A}, \jmath \in \mathbb{Z}_+\right\}. \quad (2)$$

Its closure is called the closed (convex) conical hull $\mathcal{A}$, see [20, Definition A.1.4.6].

For any two cones $\mathcal{U}$ and $\mathcal{V}$ in $\mathbb{R}^n$, the angle between them is defined as the infimum of the angle between their non-zero elements:

$$\Theta(\mathcal{U},\mathcal{V}) := \begin{cases} \inf_{u\in\mathcal{U}\setminus\{0\}, v\in\mathcal{V}\setminus\{0\}} \angle(u,v) & \text{if } \mathcal{U} \neq \{0\} \text{ and } \mathcal{V} \neq \{0\} \\ \frac{\pi}{2} & \text{otherwise} \end{cases} \quad (3)$$

where the angle $\angle(u,v)$ between any $u,v \in \mathbb{R}^n$ is defined as

$$\angle(u,v) := \begin{cases} \arccos \frac{u^T v}{\|u\|\|v\|} & \text{if } u \neq 0 \text{ and } v \neq 0 \\ \frac{\pi}{2} & \text{otherwise} \end{cases}, \quad (4)$$

where $\|\cdot\|$ is the Euclidean norm (i.e., ℓ^2 norm).

In order to understand why the vector $\mathbf{0}$ is excluded from the infimum in (3), for the case when both $\mathcal{U} \neq \{\mathbf{0}\}$ and $\mathcal{V} \neq \{\mathbf{0}\}$ hold, let us consider the case when $\mathcal{U} = \{\tau u : \tau \geq 0\}$ and $\mathcal{V} = \{\tau u : \tau \leq 0\}$ for some $u \in \mathbb{R}^n \setminus \{\mathbf{0}\}$. Then $\Theta(\mathcal{U},\mathcal{V}) = \pi$ by (3), as expected. However, if the vector $\mathbf{0}$ were not excluded then $\Theta(\mathcal{U},\mathcal{V})$ would have been $\pi/2$ as a result of (4) because $\mathbf{0}$ is an element of any cone by definition. In fact, the maximum possible value of $\Theta(\mathcal{U},\mathcal{V})$ would have been $\pi/2$, if the vector $\mathbf{0}$ were not excluded from the infimum in (3).

If the intersection of two closed (possibly non-convex) cones does not have any non-zero vector, then the angle between them is positive as demonstrated by Lemma 1 in the following.

Lemma 1: Let $\mathcal{U}$ and $\mathcal{V}$ be closed cones in $\mathbb{R}^n$ such that $\mathcal{U} \cap \mathcal{V} = \{\mathbf{0}\}$ then $\Theta(\mathcal{U},\mathcal{V}) \in (0,\pi]$ and there exists a $u \in \mathcal{U}$ and a $v \in \mathcal{V}$ such that

$$\Theta(\mathcal{U},\mathcal{V}) = \angle(u,v). \quad (5)$$

Furthermore, if the cone $\mathcal{V}$ is also a subspace (i.e., if $\mathcal{V} = -\mathcal{V}$), then $\Theta(\mathcal{U},\mathcal{V}) \in (0, \pi/2]$.

Proof: If either $\mathcal{U} = \{\mathbf{0}\}$ or $\mathcal{V} = \{\mathbf{0}\}$ then $\angle(u,v) = \frac{\pi}{2}$ for all $u \in \mathcal{U}$ and $v \in \mathcal{V}$ by (4). Thus $\Theta(\mathcal{U},\mathcal{V}) = \frac{\pi}{2}$ by (3).

If both $\mathcal{U} \neq \{\mathbf{0}\}$ and $\mathcal{V} \neq \{\mathbf{0}\}$ hold, then

$$\begin{aligned}\Theta(\mathcal{U},\mathcal{V}) &\overset{(a)}{=} \inf_{u\in\mathcal{U}\setminus\{\mathbf{0}\}, v\in\mathcal{V}\setminus\{\mathbf{0}\}} \arccos\left(\tfrac{u^T v}{\|u\|\cdot\|v\|}\right) \\ &\overset{(b)}{=} \inf_{u\in\mathcal{U}:\|u\|=1, v\in\mathcal{V}:\|v\|=1} \arccos(u^T v) \\ &\overset{(c)}{=} \min_{u\in\mathcal{U}:\|u\|=1, v\in\mathcal{V}:\|v\|=1} \arccos(u^T v),\end{aligned}$$

where (a) follows from (4), (b) follows from the definition of a cone, (c) follows from the extreme value theorem and the continuity of the function $\arccos(u^T v)$ in (u,v) because $\{u \in \mathcal{U} : \|u\| = 1\} \times \{v \in \mathcal{V} : \|v\| = 1\}$ is compact. The minimum value is positive, because otherwise there will be a non-zero v such that $v \in \mathcal{U} \cap \mathcal{V}$ and the hypothesis of the lemma a will be violated.

If $\mathcal{V}$ is a subspace and $v \in \mathcal{V}$, then $-v \in \mathcal{V}$, as well; thus $\Theta(\mathcal{U},\mathcal{V}) \leq \pi/2$ by (3) and (4). ◻

B. Projection to a Closed Convex Set

Let $\mathcal{A}$ be a closed convex subset of the Euclidean space $\mathbb{R}^n$. Then by [20, Proposition A.5.2.1], the *tangent cone* of $\mathcal{A}$ at $\overline{p} \in \mathcal{A}$ is a closed convex cone that can be expressed as the closure of the cone generated by $\{p-\overline{p} : p\in\mathcal{A}\}$:

$$\mathcal{T}_{\mathcal{A}}(\overline{p}) = cl(cone(\mathcal{A} - \overline{p})). \quad (6)$$

The *normal cone* of $\mathcal{A}$ at a point $\overline{p} \in \mathcal{A}$ is

$$\mathcal{N}_{\mathcal{A}}(\overline{p}) := \{u \in \mathbb{R}^n : u^T(p - \overline{p}) \leq 0, \ \forall p\in\mathcal{A}\}. \quad (7)$$

Thus the normal cone $\mathcal{N}_{\mathcal{A}}(\overline{p})$ is a closed convex cone, as well. Furthermore,

$$\mathcal{T}_{\mathcal{A}}(\overline{p}) \cap \mathcal{N}_{\mathcal{A}}(\overline{p}) = \{\mathbf{0}\} \qquad \forall \overline{p} \in \mathcal{A} \quad (8)$$

because the normal cone is the polar of the tangent cone, i.e., $\mathcal{N}_{\mathcal{A}}(\overline{p}) = \mathcal{T}_{\mathcal{A}}(\overline{p})^{\circ}$, by [20, Proposition A.5.2.4], where the polar of a convex cone $\mathcal{C}$ is defined as,

$$\mathcal{C}^{\circ} := \{u \in \mathbb{R}^n : u^T v \leq 0, \ \forall v \in \mathcal{C}\}, \quad (9)$$

see [20, Definition A.3.2.1].

Let Π be a closed convex set in $\mathbb{R}^n$, then the *projection* of a point $p\in\mathbb{R}^n$ onto Π is the unique point $P_{\Pi}(p)$ satisfying

$$P_{\Pi}(p) = \arg\min_{\overline{p}\in\Pi} \|p - \overline{p}\| \qquad \forall p \in \mathbb{R}^n,$$

where $\|\cdot\|$ is the Euclidean norm, see [20, p. 46]. Then $\overline{p}$ is $P_{\Pi}(p)$ iff $p - \overline{p} \in \mathcal{N}_{\Pi}(\overline{p})$ for the normal cone defined in (7) by [20, Theorem A.3.1.1], i.e.,

$$\overline{p} = P_{\Pi}(p) \iff (p - \overline{p})^T(u - \overline{p}) \leq 0 \qquad \forall u \in \Pi.$$

On the other hand, if $\Pi \subset \mathcal{A}$ for a closed convex set $\mathcal{A}$, then $p-\overline{p}\in\mathcal{T}_{\mathcal{A}}(\overline{p})$ by (2) and (6) for all $p\in\mathcal{A}$, where $\mathcal{T}_{\mathcal{A}}(\overline{p})$ is the tangent cone of $\mathcal{A}$ at $\overline{p}$. Thus, for all $p\in\mathcal{A}$,

$$\overline{p} = P_{\Pi}(p) \iff p - \overline{p} \in \mathcal{N}_{\Pi}^{\mathcal{A}}(\overline{p}), \quad (10)$$

where $\mathcal{N}_{\Pi}^{\mathcal{A}}(\overline{p})$ is defined for all closed convex sets Π and $\mathcal{A}$ satisfying $\Pi \subset \mathcal{A}$ and $\overline{p} \in \Pi$ as

$$\mathcal{N}_{\Pi}^{\mathcal{A}}(\overline{p}) := \mathcal{T}_{\mathcal{A}}(\overline{p}) \cap \mathcal{N}_{\Pi}(\overline{p}). \quad (11)$$

For all $p \in \mathcal{A}$, a necessary and sufficient condition for $\overline{p}$ to be the projection of p to Π is $p - \overline{p} \in \mathcal{N}_{\Pi}^{\mathcal{A}}(\overline{p})$. This, however, does not ensure the existence of a $p \in \mathcal{A}$ satisfying $p = \overline{p} + \tau v$ for a $\tau > 0$ for all $v \in \mathcal{N}_{\Pi}^{\mathcal{A}}(\overline{p})$ for a $\overline{p} \in \Pi$ because v might not be a feasible direction at $\overline{p}$ for $\mathcal{A}$, i.e., v might not be an element of $cone(\mathcal{A} - \overline{p})$. However, if $\mathcal{T}_{\mathcal{A}}(\overline{p}) = cone(\mathcal{A} - \overline{p})$ for a $\overline{p} \in \Pi$, i.e., iff $cone(\mathcal{A} - \overline{p})$ is closed, then for all $v \in \mathcal{N}_{\Pi}^{\mathcal{A}}(\overline{p})$, there exists $\tau > 0$ satisfying $\overline{p} + \tau v \in \mathcal{A}$. The polyhedral convexity discussed in the following, see §II-C, ensures that $cone(\mathcal{A} - \overline{p})$ is closed and hence $\mathcal{T}_{\mathcal{A}}(\overline{p}) = cone(\mathcal{A} - \overline{p})$ for all $\overline{p} \in \mathcal{A}$.

Let us define $\mathcal{N}_{\Pi}^{\mathcal{A}}$ as the union of all $\mathcal{N}_{\Pi}^{\mathcal{A}}(\overline{p})$'s for $\overline{p} \in \Pi$ defined in (11), i.e.,

$$\mathcal{N}_{\Pi}^{\mathcal{A}} := \bigcup_{\overline{p} \in \Pi} \mathcal{N}_{\Pi}^{\mathcal{A}}(\overline{p}). \tag{12}$$

C. Polyhedral Convexity

Any closed convex set $\mathcal{A}$ in $\mathbb{R}^n$ can be expressed as the intersection closed half spaces, see [20, §A.4.2.b]; when this description can be done with a finitely many half spaces $\mathcal{A}$ is said to be *polyhedral*. In other words, a closed convex set $\mathcal{A} \subset \mathbb{R}^n$ is *polyhedral* iff there exists a finite index set $\mathcal{J}_{\mathcal{A}}$, vectors $\{f_\iota \in \mathbb{R}^n\}_{\iota \in \mathcal{J}_{\mathcal{A}}}$, and constants $\{b_\iota \in \mathbb{R}\}_{\iota \in \mathcal{J}_{\mathcal{A}}}$ such that

$$\mathcal{A} = \{p \in \mathbb{R}^n : p^T f_\iota \leq b_\iota \quad \forall \iota \in \mathcal{J}_{\mathcal{A}}\}. \tag{13}$$

We denote the set of active constraints at $\overline{p}$ by $\mathcal{J}_{\mathcal{A}}(\overline{p})$, i.e.,

$$\mathcal{J}_{\mathcal{A}}(\overline{p}) := \{\iota \in \mathcal{J}_{\mathcal{A}} : \overline{p}^T f_\iota = b_\iota\} \qquad \forall \overline{p} \in \mathcal{A}. \tag{14}$$

Then the tangent cone and the normal cone at any $\overline{p} \in \mathcal{A}$ can be characterized via $\mathcal{J}_{\mathcal{A}}(\overline{p})$ as follows, see [20, p. 67],

$$\mathcal{T}_{\mathcal{A}}(\overline{p}) = \{p \in \mathbb{R}^n : p^T f_\iota \leq 0 \quad \forall \iota \in \mathcal{J}_{\mathcal{A}}(\overline{p})\}, \tag{15}$$

$$\mathcal{N}_{\mathcal{A}}(\overline{p}) = cone(\{f_\iota : \iota \in \mathcal{J}_{\mathcal{A}}(\overline{p})\}). \tag{16}$$

Thus both $\mathcal{T}_{\mathcal{A}}(\overline{p})$ and $\mathcal{N}_{\mathcal{A}}(\overline{p})$ are closed convex polyhedral sets, as well.

$\mathcal{S}$ is an affine subspace iff there exists a finite index set $\mathcal{J}_{\mathcal{S}}$, vectors $\{f_\iota\}_{\iota \in \mathcal{J}_{\mathcal{S}}}$, and constants $\{b_\iota\}_{\iota \in \mathcal{J}_{\mathcal{S}}}$ such that

$$\mathcal{S} = \{p \in \mathbb{R}^n : p^T f_\iota = b_\iota \quad \forall \iota \in \mathcal{J}_{\mathcal{S}}\}. \tag{17}$$

Thus an affine subspace $\mathcal{S}$ can be interpreted as a closed convex polyhedral set for which all constraints are active at all points $\overline{p} \in \mathcal{S}$. Hence, the tangent cone and the normal cone do not change from one point of $\mathcal{S}$ to the next and they can be denoted by $\mathcal{T}_{\mathcal{S}}$ and $\mathcal{N}_{\mathcal{S}}$ instead of $\mathcal{T}_{\mathcal{S}}(\overline{p})$ and $\mathcal{N}_{\mathcal{S}}(\overline{p})$. If $\mathcal{S}$ is non-empty then $\mathcal{T}_{\mathcal{S}}$ and $\mathcal{N}_{\mathcal{S}}$ are

$$\mathcal{T}_{\mathcal{S}} = \{p \in \mathbb{R}^n : p^T f_\iota = 0 \quad \forall \iota \in \mathcal{J}_{\mathcal{S}}\}, \tag{18}$$

$$\mathcal{N}_{\mathcal{S}} = span(\{f_\iota : \iota \in \mathcal{J}_{\mathcal{S}}\}), \tag{19}$$

where $span(\{f_\iota : \iota \in \mathcal{J}_{\mathcal{S}}\})$ is the subspace spanned by f_ι vectors for $\iota \in \mathcal{J}_{\mathcal{S}}$.

Lemma 2: Let $\mathcal{A}$ be a closed convex polyhedral subset of $\mathbb{R}^n$, $\mathcal{S}$ be an affine subspace of $\mathbb{R}^n$, Π be their intersection, i.e., $\Pi := \mathcal{A} \cap \mathcal{S}$. Then $\mathcal{N}_{\Pi}^{\mathcal{A}}$ is a closed cone and

$$\mathcal{T}_{\Pi}(\overline{p}) = \mathcal{T}_{\mathcal{A}}(\overline{p}) \cap \mathcal{T}_{\mathcal{S}} \qquad \forall \overline{p} \in \Pi, \tag{20}$$

$$\mathcal{N}_{\Pi}(\overline{p}) = \mathcal{N}_{\mathcal{A}}(\overline{p}) + \mathcal{N}_{\mathcal{S}} \qquad \forall \overline{p} \in \Pi, \tag{21}$$

$$\mathcal{N}_{\Pi}^{\mathcal{A}}(\overline{p}) \cap \mathcal{T}_{\mathcal{S}} = \{\mathbf{0}\} \qquad \forall \overline{p} \in \Pi, \tag{22}$$

$$\mathcal{N}_{\Pi}^{\mathcal{A}} \cap \mathcal{T}_{\mathcal{S}} = \{\mathbf{0}\}. \tag{23}$$

Furthermore, $\Theta\left(\mathcal{N}_{\Pi}^{\mathcal{A}}(\overline{p}), \mathcal{T}_{\mathcal{S}}\right)$ is uniquely determined by the active constraints at $\overline{p}$ for $\mathcal{A}$ and $\mathcal{S}$, i.e. by $\{f_\iota\}_{\iota \in \mathcal{J}_{\mathcal{A}}(\overline{p})}$ and $\{f_\iota\}_{\iota \in \mathcal{J}_{\mathcal{S}}}$, for all $\overline{p} \in \Pi$. In addition there exists a $\overline{p} \in \Pi$ such that $\Theta\left(\mathcal{N}_{\Pi}^{\mathcal{A}}, \mathcal{T}_{\mathcal{S}}\right) = \Theta\left(\mathcal{N}_{\Pi}^{\mathcal{A}}(\overline{p}), \mathcal{T}_{\mathcal{S}}\right)$.

Proof of Lemma 2: Note that Π is a closed convex polyhedral set because any affine subspace of $\mathbb{R}^n$ is a closed convex polyhedral set and the intersection of two closed convex polyhedral sets is again a closed convex polyhedral set. Furthermore,

$$\mathcal{J}_{\Pi}(\overline{p}) = \mathcal{J}_{\mathcal{A}}(\overline{p}) \cup \mathcal{J}_{\mathrm{S}}(\overline{p}) \qquad \forall \overline{p} \in \Pi. \tag{24}$$

(20) follows from (15), (18), and (24). The identity in (21) follows from (16), (19), and (24). Furthermore, (22) follows from (11) and (20) because $\mathcal{T}_{\Pi}(\overline{p}) \cap \mathcal{N}_{\Pi}(\overline{p}) = \{\mathbf{0}\}$ by (8). (23) follows from (12) and (22).

The angle $\Theta\left(\mathcal{N}_{\Pi}^{\mathcal{A}}(\overline{p}), \mathcal{T}_{\mathcal{S}}\right)$ is determined by the active constraints at $\overline{p}$ for $\mathcal{A}$ and $\mathcal{S}$, i.e. by $\{f_\iota\}_{\iota \in \mathcal{J}_{\mathcal{A}}(\overline{p})}$ and $\{f_\iota\}_{\iota \in \mathcal{J}_{\mathcal{S}}}$, because they determine the active constraints at $\overline{p}$ for Π by (24). Thus they determine not only $\mathcal{T}_{\mathcal{S}}$ by (18), but also $\mathcal{N}_{\Pi}^{\mathcal{A}}(\overline{p})$ by (11), (15), and (16).

On the other hand (3) and (12) imply

$$\Theta\left(\mathcal{N}_{\Pi}^{\mathcal{A}}, \mathcal{T}_{\mathcal{S}}\right) = \inf_{\overline{p} \in \Pi} \Theta\left(\mathcal{N}_{\Pi}^{\mathcal{A}}(\overline{p}), \mathcal{T}_{\mathcal{S}}\right). \tag{25}$$

There are only finitely many distinct possible $\mathcal{T}_{\mathcal{A}}(\overline{p})$ cones for $\overline{p} \in \mathcal{A}$ and finitely many distinct possible $\mathcal{N}_{\Pi}(\overline{p})$ cones for $\overline{p} \in \Pi$ because both $\mathcal{A}$ and Π are polyhedral. Thus there are only finitely many distinct $\mathcal{N}_{\Pi}^{\mathcal{A}}(\overline{p})$ cones for $\overline{p} \in \Pi$ by (11) and hence finite many distinct $\Theta\left(\mathcal{N}_{\Pi}^{\mathcal{A}}(\overline{p}), \mathcal{T}_{\mathcal{S}}\right)$ values for $\overline{p} \in \Pi$. Then $\mathcal{N}_{\Pi}^{\mathcal{A}}$ is a closed cone as a result of (12), because union of a finite collection of closed cones is a closed cone. Furthermore, the infimum in (25) is a minimum and there exists a $\overline{p} \in \Pi$ such that $\Theta\left(\mathcal{N}_{\Pi}^{\mathcal{A}}, \mathcal{T}_{\mathcal{S}}\right) = \Theta\left(\mathcal{N}_{\Pi}^{\mathcal{A}}(\overline{p}), \mathcal{T}_{\mathcal{S}}\right)$. □

D. Projection to a Closed Convex Cone

A linear subspace $\mathcal{S}$ of $\mathbb{R}^n$ and the linear subspace $\mathcal{S}_\perp$ defines an orthogonal decomposition for vectors in $\mathbb{R}^n$. The closed convex cones and their polar cones enjoy an analogous property commonly known as Moreau's decomposition theorem.

Lemma 3 ([20, Theorem A.3.2.5]) Let $\mathcal{C}$ be a closed convex cone. For the three elements v, $\overline{v}$, and v° in $\mathbb{R}^n$, the properties below are equivalent:

(i) $v = \overline{v} + v^\circ$ with $\overline{v} \in \mathcal{C}$, $v^\circ \in \mathcal{C}^\circ$, and $\overline{v}^T v^\circ = 0$;

(ii) $\overline{v} = P_{\mathcal{C}}(v)$ and $v^\circ = P_{\mathcal{C}^\circ}(v)$.

III. Information Theoretic Preliminaries

We denote the set of all probability mass functions on countable subsets of a set $\mathscr{X}$ by $\mathscr{P}(\mathscr{X})$ and the set of probability measures on a measurable space $(\mathscr{Y}, \mathcal{Y})$ by $\mathcal{P}(\mathcal{Y})$. We denote the set of all finite-signed measures on $(\mathscr{Y}, \mathcal{Y})$ by $\mathcal{L}(\mathcal{Y})$. A $w \in \mathcal{L}(\mathcal{Y})$ is absolutely continuous in a σ-finite measures q on $(\mathscr{Y}, \mathcal{Y})$, i.e., $w \prec q$, iff $w(\mathcal{E}) = 0$ for all $\mathcal{E} \in \mathcal{Y}$ satisfying $q(\mathcal{E}) = 0$.

The Kullback-Leibler divergence between two probability measures w and q in $\mathcal{P}(\mathcal{Y})$ is defined as

$$D(w\|q) := \begin{cases} \int \left(\frac{dw}{dq} \ln \frac{dw}{dq}\right) dq & \text{if } w \prec q \\ \infty & \text{if } w \not\prec q \end{cases}. \tag{26}$$

The Kullback-Leibler divergence $D(w\|q)$ is a non-negative and $D(w\|q) = 0$ iff $w = q$. Furthermore, the Kullback-Leibler divergence is bounded from below in terms of the total variation norm via Pinsker's inequality, [21],

$$D(w\|q) \geq \tfrac{1}{2}\|w - q\|_1^2 \tag{27}$$

where $\|\cdot\|_1$ is the total variation norm, which satisfies

$$\|\mu\|_1 = \int \left|\tfrac{d\mu}{d\nu}\right| d\nu \qquad \forall \mu \in \mathcal{L}(\mathcal{Y}),$$

where ν is any σ-finite measure satisfying $\mu \prec \nu$. On the other hand, the Kullback-Leibler divergence is bounded above by χ^2 divergence, see [22, Theorem 5.1], [23, Theorem 5],

$$\chi^2(w\|q) \geq \ln\left(1 + \chi^2(w\|q)\right) \geq D(w\|q) \tag{28}$$

where χ^α divergence is introduced by Vajda, see [24, p. 246], [25], and [26]. For $\alpha > 1$ case χ^α divergence between a finite signed measure w (i.e., $w \in \mathcal{L}(\mathcal{Y})$) and a probability measure q (i.e., $q \in \mathcal{P}(\mathcal{Y})$) is defined as

$$\chi^\alpha(w\|q) := \begin{cases} \int \left|\frac{dw}{dq} - 1\right|^\alpha dq & \text{if } w \prec q \\ \infty & \text{if } w \not\prec q \end{cases}. \tag{29}$$

Note that $\chi^\alpha(w\|q) \geq 0$ and the equality holds iff $w = q$. If $\chi^3(w\|q) < \infty$, then using Taylor's theorem $D(w\|q)$ can be bounded in terms of $\chi^2(w\|q)$ and $\chi^3(w\|q)$, as follows

$$\left|D(w\|q) - \tfrac{1}{2}\chi^2(w\|q)\right| \leq \tfrac{1}{2}\chi^3(w\|q), \tag{30}$$

see Appendix D for a proof.

A channel W is a $\mathcal{P}(\mathcal{Y})$ valued function defined on the input set $\mathscr{X}$, where $\mathcal{Y}$ is the σ-algebra of the output space $(\mathscr{Y}, \mathcal{Y})$, i.e., a channel is a function of the form $W : \mathscr{X} \to \mathcal{P}(\mathcal{Y})$. For any $W : \mathscr{X} \to \mathcal{P}(\mathcal{Y})$, $q \in \mathcal{P}(\mathcal{Y})$, and $p \in \mathscr{P}(\mathscr{X})$, the conditional Kullback-Leibler divergence $D(W\|q \mid p)$ is defined as

$$D(W\|q \mid p) := \sum_x p(x) D(W(x)\|q).$$

For any channel $W : \mathscr{X} \to \mathcal{P}(\mathcal{Y})$ and $p \in \mathscr{P}(\mathscr{X})$, the mutual information $I(p; W)$ is defined as

$$I(p; W) := D\left(W\|q_p \mid p\right), \tag{31}$$

where $q_p \in \mathcal{P}(\mathcal{Y})$ is the output distribution induced by the input distribution p, for any $p \in \mathscr{P}(\mathscr{X})$, which is defined more generally for any $\nu : \mathscr{X} \to \mathbb{R}$ with a countable support satisfying $\sum_x |\nu(x)| < \infty$ as

$$q_\nu := \sum\nolimits_x \nu(x) W(x). \tag{32}$$

The following identity, due to Topsøe [27], can be confirmed by substitution

$$D(W\|q \mid p) = I(p; W) + D\left(q_p\|q\right) \tag{33}$$

for all $p \in \mathscr{P}(\mathscr{X})$ and $q \in \mathcal{P}(\mathcal{Y})$.

For any channel $W : \mathscr{X} \to \mathcal{P}(\mathcal{Y})$ and convex constraint set $\mathcal{A} \subset \mathscr{P}(\mathscr{X})$, let us define the subset $\mathscr{X}_\mathcal{A}$ of the input set $\mathscr{X}$ as

$$\mathscr{X}_\mathcal{A} := \{x \in \mathscr{X} : \exists p \in \mathcal{A} \text{ such that } p(x) > 0\}. \tag{34}$$

Evidently, $\mathcal{A} \subset \mathscr{P}(\mathscr{X}_\mathcal{A})$.

For any convex constraint set $\mathcal{A} \subset \mathscr{P}(\mathscr{X})$, the Shannon capacity $C_\mathcal{A}$ and the set of all capacity-achieving input distributions in $\mathcal{A}$, i.e., $\Pi_\mathcal{A}$, are defined as

$$C_\mathcal{A} := \sup\nolimits_{p \in \mathcal{A}} I(p; W), \tag{35}$$

$$\Pi_\mathcal{A} := \{p \in \mathcal{A} : I(p; W) = C_\mathcal{A}\}. \tag{36}$$

With a slight abuse of notation, we denote $C_{\mathscr{P}(\mathscr{X})}$ and $\Pi_{\mathscr{P}(\mathscr{X})}$ by C and Π.

If $C_\mathcal{A} < \infty$, then by [28] and [29], there exists a unique Shannon center $q_\mathcal{A} \in \mathcal{P}(\mathcal{Y})$ satisfying,

$$D(W\|q_\mathcal{A} \mid p) \leq C_\mathcal{A} \qquad \forall p \in \mathcal{A}. \tag{37}$$

Furthermore, $D\left(q_{\bar{p}}\|q_\mathcal{A}\right) = 0$ for any $\bar{p} \in \Pi_\mathcal{A}$ by (33) and (36). Thus $q_{\bar{p}} = q_\mathcal{A}$ for any $\bar{p} \in \Pi_\mathcal{A}$ by (27); hence for any $\bar{p} \in \Pi_\mathcal{A}$ the identity $D(W\|q_\mathcal{A} \mid \bar{p}) = C_\mathcal{A}$ holds by (31). On the other hand, if both $q_{\bar{p}} = q_\mathcal{A}$ and $D(W\|q_\mathcal{A} \mid \bar{p}) = C_\mathcal{A}$ hold for a $\bar{p} \in \mathcal{A}$, then $\bar{p} \in \Pi_\mathcal{A}$ by (31) and (36). Thus for q_p defined in (32), we have

$$\Pi_\mathcal{A} = \left\{p \in \mathcal{A} : D(W\|q_\mathcal{A} \mid p) = C_\mathcal{A} \text{ and } q_p = q_\mathcal{A}\right\}. \tag{38}$$

For the rest of this section, we assume that $\mathscr{X}_\mathcal{A}$ is finite and the constraint set $\mathcal{A}$ is closed. Then $C_\mathcal{A} < \infty$ because $C_\mathcal{A} \leq \ln|\mathscr{X}_\mathcal{A}|$ and thus a unique Shannon center $q_\mathcal{A}$ exists. Furthermore, as a result of the extreme value theorem, the supremum in (35) is achieved, i.e., $\Pi_\mathcal{A} \neq \emptyset$, because $I(p; W)$ is continuous in p by [29, Lemma 16-(d)] and $\mathcal{A}$ is closed and bounded, i.e., compact. Furthermore, $\Pi_\mathcal{A}$ is a closed set because it is the preimage of a closed set, for a continuous function. These assertions hold both for the total variation norm (i.e., ℓ^1 norm) and the Euclidean norm (i.e., ℓ^2 norm) because these two norms (in fact any norm on $\mathbb{R}^{|\mathscr{X}_\mathcal{A}|}$) induce the same topology on $\mathscr{P}(\mathscr{X}_\mathcal{A})$ when $\mathscr{X}$ is a finite set.

We represent real valued functions on the finite set $\mathscr{X}_\mathcal{A}$ as elements of a Euclidean space $\mathbb{R}^n$ where $n = |\mathscr{X}_\mathcal{A}|$ by choosing an arbitrary but fixed permutation of elements of $\mathscr{X}_\mathcal{A}$. We use $\mathbf{1}$ and $\mathbf{0}$ to represent all ones and and all zeros vectors. For any n-by-n positive semi-definite matrix Λ, the seminorm $\|\cdot\|_\Lambda : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ is defined as

$$\|f\|_\Lambda := \sqrt{f^T \Lambda f} \qquad \forall f \in \mathbb{R}^n.$$

When Λ is the identity matrix, the resulting seminorm is the Euclidean norm (i.e., ℓ^2 norm), which we denote by $\|\cdot\|$.

Under the finite $|\mathscr{X}_\mathcal{A}|$ hypothesis we can rewrite (38) as

$$\Pi_\mathcal{A} = \mathcal{A} \cap \mathcal{S}_\mathcal{A}, \tag{39}$$

where $\mathcal{S}_\mathcal{A}$ is an affine subset of $\mathbb{R}^n$ defined as

$$\mathcal{S}_\mathcal{A} := \left\{\nu \in \mathbb{R}^n : \nu^T D(W\|q_\mathcal{A}) = C_\mathcal{A} \text{ and } q_\nu = q_\mathcal{A}\right\}, \tag{40}$$

where $D(W\|q_\mathcal{A})$ is a column vector whose rows are $D(W(x)\|q_\mathcal{A})$'s for $x \in \mathscr{X}_\mathcal{A}$ and q_ν is defined in (32). Then as a result of (18) the tangent subspace $\mathcal{T}_{\mathcal{S}_{\mathcal{A}}}$ of the affine subspace $\mathcal{S}_{\mathcal{A}}$ satisfies

$$\mathcal{T}_{\mathcal{S}_{\mathcal{A}}} = \mathcal{K}^d_{\mathcal{A}} \cap \mathcal{K}_W, \tag{41}$$

where $\mathcal{K}^d_{\mathcal{A}}$ and $\mathcal{K}_W$ are defined[2] as

$$\mathcal{K}^d_{\mathcal{A}} := \left\{v \in \mathbb{R}^n : v^T D\left(W\|q_{\mathcal{A}}\right) = 0\right\}, \tag{42}$$

$$\mathcal{K}_W := \left\{v \in \mathbb{R}^n : \left\|\sum\nolimits_x v(x)W(x)\right\|_1 = 0\right\}. \tag{43}$$

For any $\delta \geq 0$, we define the δ neighborhood $\Pi^\delta_{\mathcal{A}}$ of the set of all capacity-achieving input distributions $\Pi_{\mathcal{A}}$ as

$$\Pi^\delta_{\mathcal{A}} := \left\{p \in \mathcal{A} : \min_{\bar{p} \in \Pi_{\mathcal{A}}} \|p - \bar{p}\| \leq \delta\right\}. \tag{44}$$

Note that we can use minimum instead of infimum in the definition because $\|\cdot\|$ is a continuous function and $\Pi_{\mathcal{A}}$ is a closed and bounded, i.e., a compact, set.

Let's wrap up our review of information-theoretic concepts by deriving an expression for mutual information, which serves as the starting point of our analysis. The non-negativity of the mutual information, (33), and (37), imply $D\left(q_p\|q_{\mathcal{A}}\right) \leq C_{\mathcal{A}} < \infty$ for all $p \in \mathcal{A}$. Thus for any $\bar{p} \in \Pi_{\mathcal{A}}$ and $p \in \mathcal{A}$ as a result of (33), we have

$$\begin{aligned} I(p; W) &= D\left(W\|q_{\mathcal{A}} \mid p\right) - D\left(q_p\|q_{\mathcal{A}}\right) \\ &= I(\bar{p}; W) + (p - \bar{p})^T D\left(W\|q_{\mathcal{A}}\right) - D\left(q_p\|q_{\mathcal{A}}\right) \\ &= C_{\mathcal{A}} + (p - \bar{p})^T D\left(W\|q_{\mathcal{A}}\right) - D\left(q_p\|q_{\mathcal{A}}\right), \end{aligned} \tag{45}$$

for q_p defined in (32).

The second term in (45) is non-positive by (37) and its kernel is $\mathcal{K}^d_{\mathcal{A}}$ defined in (42). The third term in (45) is non-positive by (27) and its kernel is the kernel of the channel, i.e., $\mathcal{K}_W$ defined in (43), because $q_p = q_{p-\bar{p}} + q_{\mathcal{A}}$ and the Kullback-Leibler divergence is zero iff its arguments are equal. Thus the intersection of the kernels of the last two terms in (45) is equal to the subspace $\mathcal{T}_{\mathcal{S}_{\mathcal{A}}}$ by (41).

IV. A Simple and General Proof of Quadratic Decay

In this section we bound the mutual information $I(p; W)$ from above by an affine and decreasing function of the square of the distance between the input distribution p and the set of all capacity-achieving input distributions $\Pi_{\mathcal{A}}$, on small enough neighborhoods of $\Pi_{\mathcal{A}}$, using Pinsker's inequality given in (27) together with the fact that the angle between $\mathcal{N}^{\mathcal{A}}_{\Pi_{\mathcal{A}}}$ and $\mathcal{T}_{\mathcal{S}_{\mathcal{A}}}$ is in $(0, \pi/2)$, which follows from (39) and Lemmas 1 and 2 for polyhedral $\mathcal{A}$'s.

First note that for any $p \in \mathcal{A}$ and $\bar{p} \in \Pi_{\mathcal{A}}$, we can bound $I(p; W)$ from above using (27) and (45):

$$I(p; W) \leq C_{\mathcal{A}} + (p - \bar{p})^T D\left(W\|q_{\mathcal{A}}\right) - \frac{1}{2}\left\|q_{(p-\bar{p})}\right\|_1^2, \tag{46}$$

where q_v is defined in (32).

[2]Note that the total variation in (43) can be replaced by any norm on $\mathcal{L}(\mathcal{Y})$, i.e., on the set of all finite-signed measures on the output space $(\mathscr{Y}, \mathcal{Y})$.

We invoke the following bound on $\|q_v\|_1$ in terms of $\|v\|$ to obtain explicit approximation error terms.

$$\begin{aligned} \|q_v\|_1 &= \left\|\sum\nolimits_x v(x)W(x)\right\|_1 \\ &\leq \sum\nolimits_x \|v(x)W(x)\|_1 \\ &= \sum\nolimits_x |v(x)| \cdot \|W(x)\|_1 \\ &= \|v\|_1 \\ &\leq \|v\| \cdot \sqrt{n} \qquad \forall v \in \mathbb{R}^n, \end{aligned} \tag{47}$$

where the first inequality follows from the triangle inequality, and the second inequality follows from the general upper bound on the ℓ^1 norm in terms of the ℓ^2 norm for $\mathbb{R}^n$.

Theorem 1: Let $W : \mathscr{X} \to \mathcal{P}(\mathcal{Y})$ be a channel with a finite input set $\mathscr{X}$ and $\mathcal{A}$ be a closed convex polyhedral subset of $\mathscr{P}(\mathscr{X})$ such that $\mathcal{A} \setminus \Pi_{\mathcal{A}} \neq \emptyset$. Then $\mathcal{K}^d_{\mathcal{A}} \cap \mathcal{N}_{\mathcal{S}_{\mathcal{A}}} \setminus \{\mathbf{0}\} \neq \emptyset$ and

$$I(p; W) \leq C_{\mathcal{A}} - \gamma \left\|p - P_{\Pi_{\mathcal{A}}}(p)\right\|^2 \qquad \forall p \in \Pi^\delta_{\mathcal{A}}, \tag{48}$$

for the set $\Pi^\delta_{\mathcal{A}}$ defined in (44), the angle $\Theta(\cdot, \cdot)$ defined in (3), and positive constants $\beta \in \left(0, \frac{\pi}{2}\right]$, γ, and δ are defined as

$$\beta := \Theta\left(\mathcal{N}^{\mathcal{A}}_{\Pi_{\mathcal{A}}}, \mathcal{T}_{\mathcal{S}_{\mathcal{A}}}\right), \tag{49a}$$

$$\gamma := \tfrac{\sin^2 \beta}{2} \min_{v \in \mathcal{K}^d_{\mathcal{A}} \cap \mathcal{N}_{\mathcal{S}_{\mathcal{A}}} : \|v\|=1} \|q_v\|_1^2, \tag{49b}$$

$$\delta := \left(|\mathscr{X}_{\mathcal{A}}| + \tfrac{\gamma}{\sin^2 \beta}\right)^{-1} \|D\left(W\|q_{\mathcal{A}}\right)\|. \tag{49c}$$

Proof: Let us first prove that $\mathcal{K}^d_{\mathcal{A}} \cap \mathcal{N}_{\mathcal{S}_{\mathcal{A}}} \neq \{\mathbf{0}\}$. As a result of triangle inequality $\left\|v^T W\right\|_1 \geq \left|v^T \mathbf{1}\right|$. Thus any $v \in \mathcal{K}_W$ satisfies $v^T \mathbf{1} = 0$. Then as a result of (18) and (43)

$$\mathcal{K}_W = \left\{v \in \mathbb{R}^n : v^T \mathbf{1} = 0 \text{ and } v^T f_\imath = 0 \quad \forall \imath \in \{1, \ldots, \jmath\}\right\}$$

where $\{\mathbf{1}, f_1, \ldots, f_\jmath\}$ are orthogonal vectors. Note that if $\jmath = 0$ then $W(x)$ has the same value for all $x \in \mathscr{X}_{\mathcal{A}}$. Thus $C_{\mathcal{A}} > 0$, which is implied by $\mathcal{A} \setminus \Pi_{\mathcal{A}} \neq \emptyset$, implies $\jmath \geq 1$. Thus using (41) and (42), we get

$$\mathcal{T}_{\mathcal{S}_{\mathcal{A}}} = \left\{v \in \mathbb{R}^n : v^T D\left(W\|q_{\mathcal{A}}\right) = 0 \text{ and } v^T g_\imath = 0 \quad \forall \imath \in \{1, \ldots, \kappa\}\right\}$$

where $\{D\left(W\|q_{\mathcal{A}}\right), g_1, \ldots, g_\kappa\}$ are orthogonal vectors and κ is a positive integer. Then using (19) we get

$$\begin{aligned} \mathcal{N}_{\mathcal{S}_{\mathcal{A}}} &= span\left(\{D\left(W\|q_{\mathcal{A}}\right), g_1, \ldots, g_\kappa\}\right), \\ \mathcal{K}^d_{\mathcal{A}} \cap \mathcal{N}_{\mathcal{S}_{\mathcal{A}}} &= span\left(\{g_1, \ldots, g_\kappa\}\right). \end{aligned}$$

For any closed convex constraint set $\mathcal{A}$, the set $\Pi_{\mathcal{A}}$ defined in (36) and the affine subspace $\mathcal{S}_{\mathcal{A}}$ defined in (40) satisfy (39). Then the hypotheses of Lemma 2 hold for $(\mathcal{A}, \mathcal{S}, \Pi) \to (\mathcal{A}, \mathcal{S}_{\mathcal{A}}, \Pi_{\mathcal{A}})$ because $\mathcal{A}$ is closed, convex, and polyhedral. Thus $\mathcal{N}^{\mathcal{A}}_{\Pi_{\mathcal{A}}} \cap \mathcal{T}_{\mathcal{S}_{\mathcal{A}}} = \{\mathbf{0}\}$ by (23) and $\mathcal{N}^{\mathcal{A}}_{\Pi_{\mathcal{A}}}$ is a closed cone. Then the angle between $\mathcal{N}^{\mathcal{A}}_{\Pi_{\mathcal{A}}}$ and $\mathcal{T}_{\mathcal{S}_{\mathcal{A}}}$ (i.e., β defined in (49a)) is in $(0, \frac{\pi}{2}]$ by Lemma 1. Consequently,

$$\left\|P_{\mathcal{T}_{\mathcal{S}_{\mathcal{A}}}}(v)\right\| \leq \|v\| \cdot \cos\beta \qquad \forall v \in \mathcal{N}^{\mathcal{A}}_{\Pi_{\mathcal{A}}}.$$

On the other hand $v = P_{\mathcal{T}_{\mathcal{S}_{\mathcal{A}}}}(v) + (v - P_{\mathcal{T}_{\mathcal{S}_{\mathcal{A}}}}(v))$ forms an orthogonal decomposition because $\mathcal{T}_{\mathcal{S}_{\mathcal{A}}}$ is a subspace. Thus

$$\left\|v - P_{\mathcal{T}_{\mathcal{S}_{\mathcal{A}}}}(v)\right\| \geq \|v\| \cdot \sin\beta \qquad \forall v \in \mathcal{N}^{\mathcal{A}}_{\Pi_{\mathcal{A}}}. \tag{50}$$

The subspaces $\mathcal{T}_{\mathcal{S}_{\mathcal{A}}}$, $\mathcal{K}^d_{\mathcal{A}} \cap \mathcal{N}_{\mathcal{S}_{\mathcal{A}}}$, and $\{\tau D(W\|q_{\mathcal{A}}) : \tau \in \mathbb{R}\}$ are orthogonal to one another by (41) and (42). Furthermore,

$$span\left(\mathcal{T}_{\mathcal{S}_{\mathcal{A}}}, \mathcal{K}^d_{\mathcal{A}} \cap \mathcal{N}_{\mathcal{S}_{\mathcal{A}}}, D(W\|q_{\mathcal{A}})\right) = \mathbb{R}^n, \tag{51}$$

by (41) and (42). Then any $v \in \mathbb{R}^n$ can be decomposed into three orthogonal vectors as follows

$$v = v_1 + v_2 + v_3, \tag{52}$$

where v_1, v_2, and v_3 are projections of v to the subspaces $\mathcal{T}_{\mathcal{S}_{\mathcal{A}}}$, $\mathcal{K}^d_{\mathcal{A}} \cap \mathcal{N}_{\mathcal{S}_{\mathcal{A}}}$, and $\{\tau D(W\|q_{\mathcal{A}}) : \tau \in \mathbb{R}\}$, respectively:

$$v_1 := P_{\mathcal{T}_{\mathcal{S}_{\mathcal{A}}}}(v), \tag{53a}$$

$$v_2 := P_{\mathcal{K}^d_{\mathcal{A}} \cap \mathcal{N}_{\mathcal{S}_{\mathcal{A}}}}(v), \tag{53b}$$

$$v_3 := \tfrac{v^T D(W\|q_{\mathcal{A}})}{\|D(W\|q_{\mathcal{A}})\|^2} D(W\|q_{\mathcal{A}}). \tag{53c}$$

Let $v \in \mathbb{R}^n$ be

$$v := p - P_{\Pi_{\mathcal{A}}}(p). \tag{54}$$

Then the upper bound on $I(p; W)$ for any $p \in \mathcal{A}$ in (46) is

$$I(p; W) \le C_{\mathcal{A}} + v^T D(W\|q_{\mathcal{A}}) - \tfrac{1}{2}\|q_v\|_1^2. \tag{55}$$

Let us proceed with bounding the terms in (55). Note that the sign of the inner product $v^T D(W\|q_{\mathcal{A}})$ cannot be positive because otherwise (37) would be violated. Thus

$$\begin{aligned} v^T D(W\|q_{\mathcal{A}}) &= v_3^T D(W\|q_{\mathcal{A}}) \\ &= -\|v_3\| \cdot \|D(W\|q_{\mathcal{A}})\|. \end{aligned} \tag{56}$$

On the other hand, since $\mathcal{T}_{\mathcal{S}_{\mathcal{A}}} \subset \mathcal{K}_W$, we have

$$\begin{aligned} \|q_v\|_1^2 &= \left\|q_{v_2} + q_{v_3}\right\|_1^2 \\ &\overset{(a)}{\ge} \left(\left\|q_{v_2}\right\|_1 - \left\|q_{v_3}\right\|_1\right)^2 \\ &\ge \left\|q_{v_2}\right\|_1^2 - 2\left\|q_{v_2}\right\|_1 \cdot \left\|q_{v_3}\right\|_1 \\ &\overset{(b)}{\ge} \left\|q_{v_2}\right\|_1^2 - 2|\mathscr{X}_{\mathcal{A}}| \cdot \|v_2\| \cdot \|v_3\| \\ &\overset{(c)}{\ge} \tfrac{2\gamma}{\sin^2\beta}\|v_2\|^2 - 2|\mathscr{X}_{\mathcal{A}}| \cdot \|v_2\| \cdot \|v_3\| \\ &= \tfrac{2\gamma}{\sin^2\beta}\|v_2 + v_3\|^2 - 2\left(\tfrac{\gamma \cdot \|v_3\|}{\sin^2\beta} + |\mathscr{X}_{\mathcal{A}}| \cdot \|v_2\|\right)\|v_3\| \\ &\overset{(d)}{\ge} \tfrac{2\gamma}{\sin^2\beta}\|v_2 + v_3\|^2 - 2\|v\|\tfrac{\|D(W\|q_{\mathcal{A}})\|}{\delta}\|v_3\| \\ &\overset{(e)}{\ge} 2\gamma\|v\|^2 - 2\|v\|\tfrac{\|D(W\|q_{\mathcal{A}})\|}{\delta}\|v_3\|, \end{aligned} \tag{57}$$

where (a) follows from the triangle inequality, (b) follows from (47), (c) follows from the definition of γ given in (49b), (d) follows from (49c) and $\|v_2\| \vee \|v_3\| \le \|v\|$, and (e) follows from $\|v_2 + v_3\| \ge \|v\| \sin\beta$ which is implied by (10), (50), (53), and (54).

(48) holds for all $p \in \Pi^\delta_{\mathcal{A}}$ as a result of (55), (56), and (57).

We are left with establishing the positivity of γ. First note that γ is achieved by some v_* in $\{v \in \mathcal{K}^d_{\mathcal{A}} \cap \mathcal{N}_{\mathcal{S}_{\mathcal{A}}} : \|v\| = 1\}$ and the use of a minimum rather than an infimum in (49b), is justified as a result of the extreme value theorem because $\{v \in \mathcal{K}^d_{\mathcal{A}} \cap \mathcal{N}_{\mathcal{S}_{\mathcal{A}}} : \|v\| = 1\}$ is a closed and bounded set (i.e., a compact set) and $\|q_v\|_1^2$ is continuous in v by (47) and the triangle inequality. If the minimum value in (49b) is zero then $v_* \in \mathcal{K}_W \setminus \{\mathbf{0}\}$ by (32) and (43); on the other hand $v_* \in \mathcal{K}^d_{\mathcal{A}}$, for $\mathcal{K}^d_{\mathcal{A}}$ defined in (42), by hypothesis. Thus $v_* \in \mathcal{T}_{\mathcal{S}_{\mathcal{A}}} \setminus \{\mathbf{0}\}$ by (41). This, however, is a contradiction because $v_* \in \mathcal{N}_{\mathcal{S}_{\mathcal{A}}}$ by hypothesis. Hence γ is positive. □

Theorem 1 assumes $\mathcal{A}$ to be polyhedral and input set $\mathscr{X}$ to be finite; both of these assumptions are necessary to establish a quadratic bound on the worst case decrease of the mutual information with the Euclidean distance to $\Pi_{\mathcal{A}}$. Example 1 in the following describes a channel with a finite input set and a convex constraint set $\mathcal{A}$ that is not polyhedral for which the decrease of the mutual information with the distance to $\Pi_{\mathcal{A}}$ is proportional to the fourth power of the distance to $\Pi_{\mathcal{A}}$, which is much slower. Example 2 describes a channel with countably infinite input set and finite output set for which, if

$$I(p; W) \le C - f(\|p - P_\Pi(p)\|) \qquad \forall p \in \Pi^\delta, \tag{58}$$

for some $f : \mathbb{R}_{\ge 0} \to \mathbb{R}_{\ge 0}$, then $f(z) = 0$ for all $z \in [0, 1 \wedge \delta]$.

Example 1: Let the channel W with three input letters and two output letters and convex constraint set $\mathcal{A}$ be

$$W = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 1 \end{bmatrix} \qquad \begin{aligned} \mathcal{A} &= \left\{p \in \mathscr{P}(\mathscr{X}) : \|p - u\| \le \tfrac{\sqrt{6}}{12}\right\} \\ u &= \left[\tfrac{2}{3} \ \tfrac{1}{6} \ \tfrac{1}{6}\right]^T \end{aligned}$$

Then $C_{\mathcal{A}} = \ln 2$, $\Pi_{\mathcal{A}} = \left\{\left[\tfrac{1}{2} \ \tfrac{1}{4} \ \tfrac{1}{4}\right]^T\right\}$, and $q_{\mathcal{A}} = \left[\tfrac{1}{2} \ \tfrac{1}{2}\right]$. Furthermore, the boundary of $\mathcal{A}$ can be described by a parametric family of input distributions as

$$\partial\mathcal{A} = \{p_\tau : \tau \in (-\pi, \pi]\},$$

where p_τ is given by

$$p_\tau = \tfrac{1}{12}\begin{bmatrix} 8 - 2\cos\tau \\ 2 + \cos\tau + \sqrt{3}\sin\tau \\ 2 + \cos\tau - \sqrt{3}\sin\tau \end{bmatrix} \qquad \forall \tau \in (-\pi, \pi].$$

Then $\Pi_{\mathcal{A}} = \{p_0\}$, and the distance between p_τ and $\Pi_{\mathcal{A}}$ is

$$\left\|p_\tau - P_{\Pi_{\mathcal{A}}}(p_\tau)\right\| = \tfrac{\sqrt{6}}{6}\left|\sin\tfrac{\tau}{2}\right|.$$

Furthermore, the corresponding parametric expressions for the output distribution and the mutual information are

$$\begin{aligned} q_{p_\tau} &= \left[\tfrac{4-\cos\tau}{6} \ \ \tfrac{2+\cos\tau}{6}\right], \\ &= q_{\mathcal{A}} + \left[\tfrac{1}{3}\sin^2\tfrac{\tau}{2} \ -\tfrac{1}{3}\sin^2\tfrac{\tau}{2}\right], \\ I(p_\tau; W) &= C_{\mathcal{A}} - D\left(q_{p_\tau}\|q_{\mathcal{A}}\right), \\ D\left(q_{p_\tau}\|q_{\mathcal{A}}\right) &= \tfrac{1}{3}\left(\sin^2\tfrac{\tau}{2}\right)\ln\left(1 + \tfrac{4\sin^2\frac{\tau}{2}}{3-2\sin^2\frac{\tau}{2}}\right) \\ &\quad + \tfrac{1}{2}\ln\left(1 - \tfrac{4}{9}\sin^4\tfrac{\tau}{2}\right). \end{aligned}$$

Thus

$$\lim_{\tau \downarrow 0} \tfrac{C_{\mathcal{A}} - I(p_\tau; W)}{\left\|p_\tau - P_{\Pi_{\mathcal{A}}}(p_\tau)\right\|^4} = 8.$$

Hence, the decrease of mutual information with the distance from $\Pi_{\mathcal{A}}$ is proportional to the forth power of the distance, rather than the second power for the points on $\partial\mathcal{A}$, i.e., on the boundary of $\mathcal{A}$. Thus (48) of Theorem 1 does not hold for any positive constants γ and δ.

Example 2: Let the channel W whose input set is the set of all integers and whose output set has only two elements, be

$$W(x) = \begin{cases} \begin{bmatrix} 1 & 0 \end{bmatrix} & \text{if } x = 1, \\ \begin{bmatrix} \frac{1+\tanh x}{2} & \frac{1-\tanh x}{2} \end{bmatrix} & \text{if } x \in \mathbb{Z} \setminus \{-1, 1\}, \\ \begin{bmatrix} 0 & 1 \end{bmatrix} & \text{if } x = -1, \end{cases}$$

Then $C = \ln 2$ and $\Pi = \{p_1\}$ where $p_\imath$ is the uniform distribution on the input letters $\imath$ and $-\imath$ for all $\imath \in \mathbb{Z}_+$. Then $\left\| p_\imath - P_\Pi(p_\imath) \right\| = 1$ for all $\imath \in \mathbb{Z}_{>1}$ and $I(p_\imath; W) \uparrow C$ as $\imath \uparrow \infty$. The concavity of the mutual information in the input distribution, and the Jensen's inequality imply for all $\imath \in \mathbb{Z}_+$ and $\tau \in [0, 1]$

$$I((1-\tau)p_1 + \tau p_\imath; W) \geq (1-\tau)C + \tau I(p_\imath; W).$$

On the other hand, the fact that $\Pi = \{p_1\}$ imply

$$\left\| (1-\tau)p_1 + \tau p_\imath - P_\Pi((1-\tau)p_1 + \tau p_\imath) \right\| = \tau$$

Thus (58) holds for a $\delta > 0$ iff $f(z) = 0$ for all $z \in [0, 1 \wedge \delta]$.

V. An Exact Characterization of the Slowest Decay

In the previous section we bounded $I(p; W)$ from above by an affine and decreasing function of the square of the distance between p and $\Pi_{\mathcal{A}}$ on $\Pi_{\mathcal{A}}^\delta$ for small enough δ. However, the decrease of $I(p; W)$ is a linear function of the distance between p and $\Pi_{\mathcal{A}}$ for certain constraint sets $\mathcal{A}$, up to quadratic error terms.

In this section, we qualitatively characterize the slowest decay of $I(p; W)$ as a function of the distance between p and $\Pi_{\mathcal{A}}$ for polyhedral constraint sets $\mathcal{A}$ for channels with finite input set, by showing that the slowest decrease can be proportional to either the first or the second power of the distance between p and $\Pi_{\mathcal{A}}$, and determining the necessary and sufficient conditions for each case. In addition, we will determine the exact coefficient of the leading term in both cases, see Theorem 2.

As was the case in §IV, the starting point of our analysis will be (45). Instead of using Pinsker's inequality given in (27), however, we will use (30) to bound $D(q_p \| q_{\mathcal{A}})$ for p in $\Pi_{\mathcal{A}}^\delta$, see Lemma 4 in §V-A. In §V-B, instead of invoking Lemmas 1 and 2 to prove $\left\| v - P_{\mathcal{T}_{\mathcal{S}_{\mathcal{A}}}}(v) \right\| \geq \|v\| \sin\beta$ for all $v \in \mathcal{N}_{\Pi_{\mathcal{A}}}^{\mathcal{A}}$ for some fixed $\beta \in \left(0, \frac{\pi}{2}\right]$, we will use Moreau's decomposition theorem, i.e., Lemma 3. These changes will allow us to determine the exact coefficient of the leading term of the slowest decay of $I(p; W)$ with the distance between p and $\Pi_{\mathcal{A}}$ in §V-B. We will use Lemmas 1 and 2 in §V-B to obtain definite approximation error terms.

A. A Taylor's Theorem for $D(q_p \| q_{\mathcal{A}})$

Note that (26) and (37) imply $W(x) \prec q_{\mathcal{A}}$, and hence the existence of the Radon-Nikodym derivative $\frac{dW(x)}{dq_{\mathcal{A}}}$ for all x in $\mathscr{X}_{\mathscr{A}}$. Let $\Lambda_{\mathcal{A}} : \mathscr{X}_{\mathcal{A}} \times \mathscr{X}_{\mathcal{A}} \to [-1, \infty]$ be

$$\Lambda_{\mathcal{A}}(x, z) := \int \left(\frac{dW(x)}{dq_{\mathcal{A}}} - 1 \right) \left(\frac{dW(z)}{dq_{\mathcal{A}}} - 1 \right) dq_{\mathcal{A}} \qquad \forall x, z \in \mathscr{X}_{\mathcal{A}}. \tag{59}$$

Since $W(x) \in \mathcal{P}(\mathcal{Y})$ for all $x \in \mathscr{X}_{\mathcal{A}}$ and $q_{\mathcal{A}} \in \mathcal{P}(\mathcal{Y})$,

$$\Lambda_{\mathcal{A}}(x, z) = \int \left(\frac{dW(x)}{dq_{\mathcal{A}}} \right) \left(\frac{dW(z)}{dq_{\mathcal{A}}} \right) dq_{\mathcal{A}} - 1 \qquad \forall x, z \in \mathscr{X}_{\mathcal{A}}. \tag{60}$$

This, however, does not ensure the finiteness of $\Lambda_{\mathcal{A}}(x, z)$, see Example 3 for a channel with a finite input set, countable output set, for which $\Lambda_{\mathcal{A}}(x, x) = \infty$ for some $x \in \mathscr{X}_{\mathcal{A}}$.

The Cauchy-Schwarz inequality and (59) imply[3]

$$|\Lambda_{\mathcal{A}}(x, z)| \leq \sqrt{\Lambda_{\mathcal{A}}(x, x) \Lambda_{\mathcal{A}}(z, z)} \qquad \forall x, z \in \mathscr{X}_{\mathcal{A}}. \tag{62}$$

Furthermore, if $|\mathscr{X}_{\mathcal{A}}| = n$ for an $n \in \mathbb{Z}_+$ and $\Lambda_{\mathcal{A}}(x, x) < \infty$ for all $x \in \mathscr{X}_{\mathcal{A}}$ then $\Lambda_{\mathcal{A}}$ can be represented by a symmetric n-by-n matrix as a result of (59) and (62). The corresponding matrix for $\Lambda_{\mathcal{A}}$ is a Fisher information matrix, see Appendix C for a brief discussion. In addition $\Lambda_{\mathcal{A}}$ is positive semi-definite, because (32) and (59) imply

$$v^T \Lambda_{\mathcal{A}} v = \int \left(\frac{dq_v}{dq_{\mathcal{A}}} - v^T \mathbf{1} \right)^2 dq_{\mathcal{A}} \qquad \forall v \in \mathbb{R}^n. \tag{63}$$

Thus $\Lambda_{\mathcal{A}}$ defines a seminorm on $\mathbb{R}^n$ and $v^T \Lambda_{\mathcal{A}} v > 0$ unless $\frac{dq_v}{dq_{\mathcal{A}}} = v^T \mathbf{1}$ holds $q_{\mathcal{A}}$-a.s. Furthermore, if $q_v = \gamma q_{\mathcal{A}}$ for a $v \in \mathbb{R}^n$ and a $\gamma \in \mathbb{R}$, then $\gamma = v^T \mathbf{1}$ by (32) and thus $\Lambda_{\mathcal{A}} v = \mathbf{0}$. Therefore,

$$\Lambda_{\mathcal{A}} v = 0 \qquad \text{iff} \qquad \exists \gamma \in \mathbb{R} \text{ s.t. } q_v = \gamma q_{\mathcal{A}}. \tag{64}$$

Hence, if a $v \in \mathbb{R}^n$ satisfies $q_v = \gamma q_{\mathcal{A}}$ for some $\gamma \in \mathbb{R}$, then

$$\|p\|_{\Lambda_{\mathcal{A}}}^2 = \|p - v\|_{\Lambda_{\mathcal{A}}}^2 \qquad \forall p \in \mathbb{R}^n. \tag{65}$$

On the other hand, if $p^T \mathbf{1} \neq 0$ then the square of seminorm $\|p\|_{\Lambda_{\mathcal{A}}}$ is proportional to the χ^2 divergence defined in (29):

$$\|p\|_{\Lambda_{\mathcal{A}}}^2 = \left(p^T 1\right)^2 \chi^2 \left(q_{\frac{p}{p^T 1}} \| q_{\mathcal{A}} \right). \tag{66}$$

Thus using (65) and (66), for any $p \in \mathbb{R}^n$ satisfying $p^T \mathbf{1} = 1$ and any $\overline{p} \in \mathbb{R}^n$ satisfying $q_{\overline{p}} = \gamma q_{\mathcal{A}}$ for some $\gamma \in \mathbb{R}$, we have

$$\|p - \overline{p}\|_{\Lambda_{\mathcal{A}}}^2 = \chi^2 \left(q_p \| q_{\mathcal{A}} \right). \tag{67}$$

On the other hand, for all $\overline{p} \in \mathbb{R}^n$ satisfying $q_{\overline{p}} = q_{\mathcal{A}}$ and $p \in \mathbb{R}^n$, the Cauchy-Schwarz inequality implies

$$\begin{aligned} \left| \frac{dq_p}{dq_{\mathcal{A}}} - 1 \right| &= \left| \sum\nolimits_{x \in \mathscr{X}_{\mathcal{A}}} (p(x) - \overline{p}(x)) \frac{dW(x)}{dq_{\mathcal{A}}} \right|, \\ &\leq \|p - \overline{p}\| \cdot \sqrt{\sum\nolimits_{x \in \mathscr{X}_{\mathcal{A}}} \left(\frac{dW(x)}{dq_{\mathcal{A}}} \right)^2}. \end{aligned} \tag{68}$$

Thus for all $\overline{p} \in \mathbb{R}^n$ satisfying $q_{\overline{p}} = q_{\mathcal{A}}$ and $p \in \mathbb{R}^n$ we have

$$\chi^3 \left(q_p \| q_{\mathcal{A}} \right) \leq \kappa_{\mathcal{A}} \cdot \|p - \overline{p}\|^3, \tag{69}$$

where $\kappa_{\mathcal{A}}$ is defined as follows

$$\kappa_{\mathcal{A}} := \int \left(\sum\nolimits_{x \in \mathscr{X}_{\mathcal{A}}} \left(\frac{dW(x)}{dq_{\mathcal{A}}} \right)^2 \right)^{3/2} dq_{\mathcal{A}}. \tag{70}$$

Applying (30) for $w = q_p$ and $q = q_{\mathcal{A}}$, and invoking (67) and (69), we get the following lemma.

[3] One can use the Cauchy-Schwarz inequality and (60) to prove

$$0 \leq \Lambda_{\mathcal{A}}(x, z) + 1 \leq \sqrt{(1 + \Lambda_{\mathcal{A}}(x, x))(1 + \Lambda_{\mathcal{A}}(z, z))} \tag{61}$$

holds for all $x, z \in \mathscr{X}_{\mathcal{A}}$.

Lemma 4: For any $W : \mathscr{X} \to \mathcal{P}(\mathcal{Y})$ with a finite input set $\mathscr{X}$ and a closed convex constraint set $\mathcal{A} \subset \mathscr{P}(\mathscr{X})$ satisfying $\kappa_{\mathcal{A}} < \infty$, for all $p \in \mathcal{A}$ and $\overline{p} \in \Pi_{\mathcal{A}}$ we have

$$\left| D\left(q_p \| q_{\mathcal{A}}\right) - \tfrac{1}{2}\|p - \overline{p}\|^2_{\Lambda_{\mathcal{A}}} \right| \le \tfrac{\kappa_{\mathcal{A}}}{2}\|p - \overline{p}\|^3. \tag{71}$$

Using Jensen's inequality and the convexity of function $z^{3/2}$ in z, we can bound $\kappa_{\mathcal{A}}$ from below by $(n + \mathrm{Tr}[\Lambda_{\mathcal{A}}])^{3/2}$, where $\mathrm{Tr}[\Lambda_{\mathcal{A}}]$ is the trace of the matrix $\Lambda_{\mathcal{A}}$. On the other hand, we can bound $\kappa_{\mathcal{A}}$ from above using the general bound on the ℓ^2 norm in terms of the ℓ^3 norm, i.e., $\|v\|_2 \le n^{1/6}\|v\|_3$ for all $v \in \mathbb{R}^n$. Thus

$$n^{\frac{1}{2}} \sum\nolimits_{x \in \mathscr{X}_{\mathcal{A}}} \int \left(\frac{\mathrm{d}W(x)}{\mathrm{d}q_{\mathcal{A}}}\right)^3 \mathrm{d}q_{\mathcal{A}} \ge \kappa_{\mathcal{A}} \ge (n + \mathrm{Tr}[\Lambda_{\mathcal{A}}])^{\frac{3}{2}}. \tag{72}$$

For channels with finite input and output sets $\kappa_{\mathcal{A}} < \infty$, i.e., the hypothesis of Lemma 4 is always satisfied. For channels with a finite input set and an infinite output set, however, even $\Lambda_{\mathcal{A}}(x,x)$ can be infinite for some $x \in \mathscr{X}_{\mathcal{A}}$.

Example 3: Let the discrete channel $W : \mathscr{X} \to \mathcal{P}(\mathbb{Z})$ with the finite input set $\mathscr{X} = \{0, 1, \ldots, (n-1)\}$ be

$$W(y|x) = \begin{cases} \frac{|y|^{-2} \mathbb{1}_{\{y<0\}}}{\zeta(2)} & \text{if } x = 0 \\ \frac{\mathbb{1}_{\{y=x\}}}{2} + \frac{|y|^{-3}\mathbb{1}_{\{y<0\}}}{2\zeta(3)} & \text{if } x \in \{1, 2, \ldots, (n-1)\} \end{cases}$$

where $\zeta(s) := \sum_{y \in \mathbb{Z}_+} y^{-s}$, i.e., the Riemann zeta function. If $\mathcal{A} = \mathscr{P}(\mathscr{X})$ and $n \ge 1 + \left(\frac{2\zeta(3)}{\zeta(2)}\right)^2 e^{2\sum_{y \in \mathbb{Z}_+} \frac{y^{-2}}{\zeta(2)} \ln y}$, then

$$\begin{aligned} C_{\mathcal{A}} &= \ln \sqrt{n-1}, \\ q_{\mathcal{A}}(y) &= \tfrac{\mathbb{1}_{\{1 \le y \le n-1\}}}{2(n-1)} + \tfrac{|y|^{-3}\mathbb{1}_{\{(y<0)\}}}{2\zeta(3)}, \\ D\left(W(\cdot \mid 0) \| q_{\mathcal{A}}\right) &= \ln \tfrac{2\zeta(3)}{\zeta(2)} + \sum\nolimits_{y \in \mathbb{Z}_+} \tfrac{y^{-2}}{\zeta(2)} \ln y \quad \le C_{\mathcal{A}}. \end{aligned}$$

The diagonal entry of the matrix $\Lambda_{\mathcal{A}}$ corresponding to the input letter 0 is infinite:

$$\begin{aligned} \Lambda_{\mathcal{A}}(0,0) &= -1 + \sum\nolimits_{y \in \mathbb{Z}} q_{\mathcal{A}}(y) \left(\tfrac{W(y|0)}{q_{\mathcal{A}}(y)}\right)^2 \\ &\ge -1 + \tfrac{2\zeta(3)}{(\zeta(2))^2} \sum\nolimits_{y \in \mathbb{Z}_+} \tfrac{1}{y} \\ &= \infty. \end{aligned}$$

Then $\kappa_{\mathcal{A}} = \infty$, as well because $\kappa_{\mathcal{A}} \ge (n + \mathrm{Tr}[\Lambda_{\mathcal{A}}])^{3/2}$. Thus for this channel Lemma 4 is mute[4].

In our analysis we will need an operator-norm bound analogous to (47). We bound $\|v\|_{\Lambda_{\mathcal{A}}}$ from above by the product of $\|v\|$ and the trace of $\Lambda_{\mathcal{A}}$ using the Cauchy-Schwarz inequality:

$$\begin{aligned} \|v\|^2_{\Lambda_{\mathcal{A}}} &= \int \left[\sum\nolimits_x v(x)\left(\tfrac{\mathrm{d}W(x)}{\mathrm{d}q_{\mathcal{A}}} - 1\right)\right]^2 \mathrm{d}q_{\mathcal{A}} \\ &\le \int \|v\|^2 \cdot \left[\sum\nolimits_{x \in \mathcal{X}_{\mathcal{A}}} \left(\tfrac{\mathrm{d}W(x)}{\mathrm{d}q_{\mathcal{A}}} - 1\right)^2\right] \mathrm{d}q_{\mathcal{A}} \\ &= \|v\|^2 \cdot \mathrm{Tr}[\Lambda_{\mathcal{A}}]. \end{aligned} \tag{73}$$

B. Exact Characterization via Moreau's Decomposition Theorem

The positivity of the minimum angle between the cone of directions pointing away from $\Pi_{\mathcal{A}}$ and towards points in $\mathcal{A} \setminus \Pi_{\mathcal{A}}$, and the subspace of the intersection of the kernels of the gradient of mutual information and the channel, i.e., the positivity of $\Theta\left(\mathcal{N}^{\mathcal{A}}_{\Pi_{\mathcal{A}}}, \mathcal{K}^d_{\mathcal{A}} \cap \mathcal{K}_W\right)$, is sufficient to establish an upper bound on the mutual information that is decreasing linearly with the square of the distance to $\Pi_{\mathcal{A}}$, as we have seen in §IV. One can even determine whether the slowest decay is linear or quadratic in the distance to $\Pi_{\mathcal{A}}$ using the extreme value theorem and the fact that $\mathcal{N}^{\mathcal{A}}_{\Pi_{\mathcal{A}}}$ is closed. To determine the tightest coefficient in the case when the decrease is linear with the square of the distance, however, the positivity of the angle $\Theta\left(\mathcal{N}^{\mathcal{A}}_{\Pi_{\mathcal{A}}}, \mathcal{K}^d_{\mathcal{A}} \cap \mathcal{K}_W\right)$ by itself is not sufficient; projections to closed convex cones via Moreau's decomposition theorem rather than projections to subspaces need to be considered.

If $\mathscr{X}$ is a finite set and $\mathcal{A}$ is a closed convex polyhedral subset of $\mathscr{P}(\mathscr{X})$, then $\mathcal{N}^{\mathcal{A}}_{\Pi_{\mathcal{A}}}(\overline{p})$ is a closed convex polyhedral cone for all $\overline{p} \in \Pi_{\mathcal{A}}$ and $\mathcal{N}^{\mathcal{A}}_{\Pi_{\mathcal{A}}}$, defined in (12) as the union of $\mathcal{N}^{\mathcal{A}}_{\Pi_{\mathcal{A}}}(\overline{p})$'s for $\overline{p} \in \Pi_{\mathcal{A}}$, is a closed cone by Lemma 2. However, $\overline{p} \in \Pi_{\mathcal{A}}$, is not necessarily convex because the union of two or more convex cones is not necessarily convex. Hence, we can apply Moreau's decomposition theorem, i.e., Lemma 3, to each $\mathcal{N}^{\mathcal{A}}_{\Pi_{\mathcal{A}}}(\overline{p})$ separately, but not necessarily to $\mathcal{N}^{\mathcal{A}}_{\Pi_{\mathcal{A}}}$ itself.

We will employ the minimum angle idea by invoking Lemmas 1 and 2 in our analysis in this section too, though in a more nuanced manner. Let $\Upsilon_{\mathcal{A}}(\overline{p})$ be

$$\Upsilon_{\mathcal{A}}(\overline{p}) := \mathcal{N}^{\mathcal{A}}_{\Pi_{\mathcal{A}}}(\overline{p}) \cap \mathcal{K}^d_{\mathcal{A}} \qquad \forall \overline{p} \in \Pi_{\mathcal{A}}. \tag{74}$$

Then $\Upsilon_{\mathcal{A}}(\overline{p})$ is a closed convex cone because it is the intersection of two closed convex cones. Thus, any $v \in \mathcal{N}^{\mathcal{A}}_{\Pi_{\mathcal{A}}}(\overline{p})$ can be decomposed into two orthogonal components $\overline{v} = P_{\Upsilon_{\mathcal{A}}(\overline{p})}(v)$ and $v^{\circ} = P_{\Upsilon_{\mathcal{A}}(\overline{p})^{\circ}}(v)$ by Lemma 3, even if $\Upsilon_{\mathcal{A}}(\overline{p}) = \{\mathbf{0}\}$.

On the other hand the hypothesis of Lemma 2 is satisfied for $(\mathcal{A}, \mathcal{S}, \Pi) \to (\mathcal{N}^{\mathcal{A}}_{\Pi_{\mathcal{A}}}(\overline{p}), \mathcal{K}^d_{\mathcal{A}}, \Upsilon_{\mathcal{A}}(\overline{p}))$ because $\mathcal{N}^{\mathcal{A}}_{\Pi_{\mathcal{A}}}(\overline{p})$ is a closed convex polyhedral set and $\mathcal{K}^d_{\mathcal{A}}$ is an affine subspace of $\mathbb{R}^n$. Thus (23) of Lemma 2 implies

$$\mathcal{N}^{\mathcal{N}^{\mathcal{A}}_{\Pi_{\mathcal{A}}}(\overline{p})}_{\Upsilon_{\mathcal{A}}(\overline{p})} \cap \mathcal{T}_{\mathcal{K}^d_{\mathcal{A}}} = \{\mathbf{0}\} \qquad \forall \overline{p} \in \Pi_{\mathcal{A}}, \tag{75}$$

where $\mathcal{N}^{\mathcal{B}}_{\mathcal{D}}(\overline{p})$ and $\mathcal{N}^{\mathcal{B}}_{\mathcal{D}}$ are defined for any closed convex set $\mathcal{B} \subset \mathcal{D}$ and $\overline{p} \in \mathcal{B}$ in (11) and (12).

As a subspace of $\mathbb{R}^n$, $\mathcal{K}^d_{\mathcal{A}}$ is not only affine but also linear; thus $\mathcal{T}_{\mathcal{K}^d_{\mathcal{A}}} = \mathcal{K}^d_{\mathcal{A}}$. Then the hypothesis of Lemma 1 is satisfied for $(\mathcal{U}, \mathcal{V}) \to (\mathcal{N}^{\mathcal{N}^{\mathcal{A}}_{\Pi_{\mathcal{A}}}(\overline{p})}_{\Upsilon_{\mathcal{A}}(\overline{p})}, \mathcal{K}^d_{\mathcal{A}})$ by (75). Thus $\phi_{\mathcal{A}}(\overline{p}) \in \left(0, \frac{\pi}{2}\right]$, where $\phi_{\mathcal{A}}(\overline{p})$ is defined as

$$\phi_{\mathcal{A}}(\overline{p}) := \Theta\left(\mathcal{N}^{\mathcal{N}^{\mathcal{A}}_{\Pi_{\mathcal{A}}}(\overline{p})}_{\Upsilon_{\mathcal{A}}(\overline{p})}, \mathcal{K}^d_{\mathcal{A}}\right). \tag{76}$$

Remark 1: If $\Upsilon_{\mathcal{A}}(\overline{p}) = \{\mathbf{0}\}$ then $\Upsilon_{\mathcal{A}}(\overline{p})^{\circ} = \mathbb{R}^n$, $\overline{v} = \mathbf{0}$, $v^{\circ} = v$, $\mathcal{N}^{\mathcal{N}^{\mathcal{A}}_{\Pi_{\mathcal{A}}}(\overline{p})}_{\Upsilon_{\mathcal{A}}(\overline{p})} = \mathcal{N}^{\mathcal{N}^{\mathcal{A}}_{\Pi_{\mathcal{A}}}(\overline{p})}_{\Upsilon_{\mathcal{A}}(\overline{p})}(\mathbf{0})$, and $\mathcal{N}^{\mathcal{N}^{\mathcal{A}}_{\Pi_{\mathcal{A}}}(\overline{p})}_{\Upsilon_{\mathcal{A}}(\overline{p})}(\mathbf{0}) = \mathcal{N}^{\mathcal{A}}_{\Pi_{\mathcal{A}}}(\overline{p})$. Thus, $\phi_{\mathcal{A}}(\overline{p}) = \Theta\left(\mathcal{N}^{\mathcal{A}}_{\Pi_{\mathcal{A}}}(\overline{p}), \mathcal{K}^d_{\mathcal{A}}\right)$ for the angle $\phi_{\mathcal{A}}(\overline{p})$ defined in (76). Furthermore, $\phi_{\mathcal{A}}(\overline{p}) \in \left(0, \frac{\pi}{2}\right]$ by Lemma 1, because $\mathcal{N}^{\mathcal{A}}_{\Pi_{\mathcal{A}}}(\overline{p})$ and $\mathcal{K}^d_{\mathcal{A}}$ are closed and $\mathcal{N}^{\mathcal{A}}_{\Pi_{\mathcal{A}}}(\overline{p}) \cap \mathcal{K}^d_{\mathcal{A}} = \{\mathbf{0}\}$ by (74) because $\Upsilon_{\mathcal{A}}(\overline{p}) = \{\mathbf{0}\}$.

Theorem 2: For a channel $W : \mathscr{X} \to \mathcal{P}(\mathcal{Y})$ with a finite input set $\mathcal{A} \subset \mathscr{P}(\mathscr{X})$ and a closed convex polyhedral

[4] $\Lambda_{\mathcal{A}}(0,1) = 0$, $\Lambda_{\mathcal{A}}(1,1) = -1 + n/2$, $\Lambda_{\mathcal{A}}(1,1) = -1/2$.

constraint set $\mathcal{A} \subset \mathscr{P}(\mathscr{X})$ satisfying both $\mathcal{A} \setminus \Pi_{\mathcal{A}} \neq \emptyset$ and $\kappa_{\mathcal{A}} < \infty$ for $\kappa_{\mathcal{A}}$ defined in (70), let γ_1 be

$$\gamma_1 := \min_{v \in \mathcal{N}_{\Pi_{\mathcal{A}}}^{\mathcal{A}} : \|v\|=1} -v^T D(W\|q_{\mathcal{A}}) \tag{77}$$

for $\mathcal{N}_{\Pi_{\mathcal{A}}}^{\mathcal{A}}$ defined in (12). Then $\mathcal{N}_{\Pi_{\mathcal{A}}}^{\mathcal{A}} \setminus \{\mathbf{0}\} \neq \emptyset$ and

$$I(p; W) \le C_{\mathcal{A}} - \gamma_1 \left\|v_p\right\| \qquad \forall p \in \mathcal{A}, \tag{78}$$

where $v_p := p - P_{\Pi_{\mathcal{A}}}(p)$ and there exists a $p \in \mathcal{A} \setminus \Pi_{\mathcal{A}}$ satisfying

$$I(p(\tau); W) \ge C_{\mathcal{A}} - \gamma_1 \left\|v_p\right\| \tau - \mathrm{Tr}\left[\Lambda_{\mathcal{A}}\right] \cdot \left\|v_p\right\|^2 \tau^2 \tag{79}$$

for all $\tau \in [0,1]$, where $p(\tau) := P_{\Pi_{\mathcal{A}}}(p) + \tau v_p$ and $\Lambda_{\mathcal{A}}$ is defined in (59). Furthermore, if $\gamma_1 = 0$, then $\mathcal{N}_{\Pi_{\mathcal{A}}}^{\mathcal{A}} \cap \mathcal{K}_{\mathcal{A}}^d \setminus \{\mathbf{0}\} \neq \emptyset$ and

$$I(p; W) \le C_{\mathcal{A}} - \gamma_2 \left\|v_p\right\|^2 + \tfrac{\kappa_{\mathcal{A}}}{2} \left\|v_p\right\|^3 \qquad \forall p \in \Pi_{\mathcal{A}}^{\delta} \tag{80}$$

for positive constants γ_2 and δ defined in terms of $\mathcal{N}_{\Pi_{\mathcal{A}}}^{\mathcal{A}}(\overline{p})$ and $\phi_{\mathcal{A}}(\overline{p})$ defined in (11) and (76), as follows

$$\gamma_2 := \tfrac{1}{2} \min_{v \in \mathcal{N}_{\Pi_{\mathcal{A}}}^{\mathcal{A}} \cap \mathcal{K}_{\mathcal{A}}^d : \|v\|=1} \|v\|_{\Lambda_{\mathcal{A}}}^2, \tag{81a}$$

$$\delta(\overline{p}) := \tfrac{\sin(\phi_{\mathcal{A}}(\overline{p}))}{\mathrm{Tr}[\Lambda_{\mathcal{A}}] + \gamma_2} \|D(W\|q_{\mathcal{A}})\|, \tag{81b}$$

$$\delta := \min_{\overline{p} \in \Pi_{\mathcal{A}}} \delta(\overline{p}), \tag{81c}$$

and there exists a $p \in \mathcal{A} \setminus \Pi_{\mathcal{A}}$ satisfying

$$I(p(\tau); W) \ge C_{\mathcal{A}} - \gamma_2 \left\|v_p\right\|^2 \tau^2 - \tfrac{\kappa_{\mathcal{A}}}{2} \left\|v_p\right\|^3 \tau^3 \quad \forall \tau \in [0,1]. \tag{82}$$

Proof: First note that $\mathcal{A} \setminus \Pi_{\mathcal{A}} \neq \emptyset$ hypothesis implies $\mathcal{N}_{\Pi_{\mathcal{A}}}^{\mathcal{A}} \neq \{\mathbf{0}\}$. Furthermore, (77) can be stated as a minimum rather than an infimum by the extreme value theorem because $\mathcal{N}_{\Pi_{\mathcal{A}}}^{\mathcal{A}}$ is closed by Lemma 2 and thus the minimization in (77) is that of a continuous function over a closed and bounded (i.e., compact) set. Thus we can use minimum instead of an infimum.

Note that $v_p \in \mathcal{N}_{\Pi_{\mathcal{A}}}^{\mathcal{A}}(\overline{p})$ by (10) where $\overline{p}$ is the projection of a $p \in \mathcal{A}$ onto $\Pi_{\mathcal{A}}$, i.e., $\overline{p} = P_{\Pi_{\mathcal{A}}}(p)$. Hence $\mathcal{N}_{\Pi_{\mathcal{A}}}^{\mathcal{A}} \cap \mathcal{K}_{\mathcal{A}}^d \neq \emptyset$ because $\mathcal{A} \setminus \Pi_{\mathcal{A}} \neq \emptyset$. Furthermore, (27) and (45) imply

$$\begin{aligned} I(p; W) &\le C_{\mathcal{A}} + v_p^T D(W\|q_{\mathcal{A}}) \\ &\le C_{\mathcal{A}} + \left\|v_p\right\| \max_{v \in \mathcal{N}_{\Pi_{\mathcal{A}}}^{\mathcal{A}}(\overline{p}) : \|v\|=1} v^T D(W\|q_{\mathcal{A}}), \end{aligned} \tag{83}$$

for all $p \in \mathcal{A}$. Then (78) holds by (12) and the extreme value theorem because $\mathcal{N}_{\Pi_{\mathcal{A}}}^{\mathcal{A}}$ is closed by Lemma 2.

Let $v_\star$ be a minimizer for the minimization defining γ_1 in (77). Then there exists a $\overline{p_\star} \in \Pi_{\mathcal{A}}$ satisfying $v_\star \in \mathcal{N}_{\Pi_{\mathcal{A}}}^{\mathcal{A}}(\overline{p_\star})$ by (12). Furthermore, there exists a $p_\star \in \mathcal{A} \setminus \Pi_{\mathcal{A}}$ such that $P_{\Pi_{\mathcal{A}}}(p_\star) = \overline{p_\star}$ by (10) and (11) because the polyhedral convexity of $\mathcal{A}$ implies $\mathcal{T}_{\mathcal{A}}(\overline{p_\star}) = cone(\mathcal{A} - \overline{p_\star})$. Let $p_\star(\tau)$ be $p_\star(\tau) = \overline{p_\star} + \tau v_\star$, then

$$\begin{aligned} I(p_\star(\tau); W) &= C_{\mathcal{A}} + \tau v_\star^T D(W\|q_{\mathcal{A}}) - D\left(q_{p_\star(\tau)}\|q_{\mathcal{A}}\right) && \text{by (45)}, \\ &= C_{\mathcal{A}} - \gamma_1 \|\tau v_\star\| - D\left(q_{p_\star(\tau)}\|q_{\mathcal{A}}\right) && \text{by (77)}, \\ &\ge C_{\mathcal{A}} - \gamma_1 \|\tau v_\star\| - \chi^2\left(q_{p_\star(\tau)}\|q_{\mathcal{A}}\right) && \text{by (28)}, \\ &= C_{\mathcal{A}} - \gamma_1 \|\tau v_\star\| - \|\tau v_\star\|_{\Lambda_{\mathcal{A}}}^2 && \text{by (67)}, \\ &\ge C_{\mathcal{A}} - \gamma_1 \|\tau v_\star\| - \mathrm{Tr}\left[\Lambda_{\mathcal{A}}\right] \cdot \|\tau v_\star\|^2 && \text{by (73)}. \end{aligned}$$

Then (79) holds for $p = p_\star$ because $\|\tau v_\star\| = \tau \|v_\star\|$.

Let us proceed with the claims for $\gamma_1 = 0$ case. First note that, $\mathcal{N}_{\Pi_{\mathcal{A}}}^{\mathcal{A}} \cap \mathcal{K}_{\mathcal{A}}^d \setminus \{\mathbf{0}\} \neq \emptyset$ because $v_\star^T D(W\|q_{\mathcal{A}}) = 0$. On the other hand $\mathcal{N}_{\Pi_{\mathcal{A}}}^{\mathcal{A}} \cap \mathcal{K}_{\mathcal{A}}^d$ is closed because $\mathcal{N}_{\Pi_{\mathcal{A}}}^{\mathcal{A}}$ and $\mathcal{K}_{\mathcal{A}}^d$ are closed. Thus (81a) can be stated as a minimum rather than an infimum by the extreme value theorem. Let $v_\dagger$ be the minimizer of (81a). Then there exists a $\overline{p_\dagger} \in \Pi_{\mathcal{A}}$ satisfying $v_\dagger \in \mathcal{N}_{\Pi_{\mathcal{A}}}^{\mathcal{A}}(\overline{p_\dagger}) \cap \mathcal{K}_{\mathcal{A}}^d$ by (12). Furthermore, there exists a $p_\dagger \in \mathcal{A} \setminus \Pi_{\mathcal{A}}$ such that $P_{\Pi_{\mathcal{A}}}(p_\dagger) = \overline{p_\dagger}$ by (10) and (11) because $\mathcal{T}_{\mathcal{A}}(\overline{p_\dagger}) = cone(\mathcal{A} - \overline{p_\dagger})$ as a result of polyhedral convexity of $\mathcal{A}$. Let $p_\dagger(\tau)$ be $p_\dagger(\tau) = \overline{p_\dagger} + \tau v_\dagger$, then

$$\begin{aligned} I(p_\dagger(\tau); W) &= C_{\mathcal{A}} + \tau v_\dagger^T D(W\|q_{\mathcal{A}}) - D\left(q_{p_\dagger(\tau)}\|q_{\mathcal{A}}\right) && \text{by (45)}, \\ &= C_{\mathcal{A}} - D\left(q_{p_\dagger(\tau)}\|q_{\mathcal{A}}\right) && \text{by } v_\dagger \in \mathcal{K}_{\mathcal{A}}^d, \\ &\ge C_{\mathcal{A}} - \chi^2\left(q_{p_\dagger(\tau)}\|q_{\mathcal{A}}\right) && \text{by (28)}, \\ &= C_{\mathcal{A}} - \left\|\tau v_\dagger\right\|_{\Lambda_{\mathcal{A}}} && \text{by (67)}, \\ &= C_{\mathcal{A}} - 2\gamma_2 \left\|\tau v_\dagger\right\|^2 && \text{by (81a)}, \\ &= C_{\mathcal{A}} - 2\gamma_2 \left\|v_\dagger\right\|^2 \tau^2. \end{aligned}$$

Then γ_2 is positive because otherwise $p_\dagger \in \Pi_{\mathcal{A}}$ would hold, but $p_\dagger \in \mathcal{A} \setminus \Pi_{\mathcal{A}}$ by construction. Invoking (71) instead of (28) we get

$$\begin{aligned} I\left(p_\dagger(\tau); W\right) &\ge C_{\mathcal{A}} - \tfrac{1}{2}\left\|\tau v_\dagger\right\|_{\Lambda_{\mathcal{A}}} - \tfrac{\kappa_{\mathcal{A}}}{2}\left\|\tau v_\dagger\right\|^3 \\ &= C_{\mathcal{A}} - \gamma_2 \left\|\tau v_\dagger\right\|^2 - \tfrac{\kappa_{\mathcal{A}}}{2}\left\|\tau v_\dagger\right\|^3 \qquad \text{by (81a)}. \end{aligned}$$

Then (82) holds for $p = p_\dagger$ because $\left\|\tau v_\dagger\right\| = \tau \left\|v_\dagger\right\|$.

Furthermore, $\delta(\overline{p})$ is positive for all $\overline{p} \in \Pi_{\mathcal{A}}$ by definition because $\phi_{\mathcal{A}}(\overline{p})$ is positive for $\Upsilon_{\mathcal{A}}(\overline{p})$ defined in (74). On the other hand there are only finitely many distinct $\mathcal{N}_{\Pi_{\mathcal{A}}}^{\mathcal{A}}(\overline{p})$ cones, and hence only finitely many distinct $\Upsilon_{\mathcal{A}}(\overline{p})$ cones and $\delta(\overline{p})$ values, for $\overline{p} \in \Pi_{\mathcal{A}}$. Thus the minimization defining δ given in (81c) can be written as a minimum rather than an infimum and δ is positive whenever $\gamma_1 = 0$, as well.

Since $\Upsilon_{\mathcal{A}}(\overline{p})$ is a closed convex cone the projection on $\Upsilon_{\mathcal{A}}(\overline{p})$ and the projection on its polar cone $\Upsilon_{\mathcal{A}}(\overline{p})^\circ$ form an orthogonal decomposition by Lemma 3, i.e.,

$$v_p = \overline{v_p} + v_p^\circ \qquad \text{and} \qquad \overline{v_p}^T v_p^\circ = 0, \tag{84}$$

where $\overline{v_p}$ and v_p° are

$$\overline{v_p} := \mathrm{P}_{\Upsilon_{\mathcal{A}}(\overline{p})}\left(v_p\right), \qquad \text{and} \qquad v_p^\circ := \mathrm{P}_{\Upsilon_{\mathcal{A}}(\overline{p})^\circ}\left(v_p\right). \tag{85}$$

Note that $(\overline{v_p})^T D(W\|q_{\mathcal{A}}) = 0$ because $\overline{v_p} \in \Upsilon_{\mathcal{A}}(\overline{p}) \subset \mathcal{K}_{\mathcal{A}}^d$ by construction. Thus (84) implies

$$\begin{aligned} v_p^T D(W\|q_{\mathcal{A}}) &= (v_p^\circ)^T D(W\|q_{\mathcal{A}}) \\ &\le -\left\|v_p^\circ\right\| \cdot \|D(W\|q_{\mathcal{A}})\| \cdot \sin(\phi_{\mathcal{A}}(\overline{p})), \end{aligned} \tag{86}$$

where $\phi_{\mathcal{A}}(\overline{p})$ is the angle between $\mathcal{K}_{\mathcal{A}}^d$ and $\mathcal{N}_{\Upsilon_{\mathcal{A}}(\overline{p})}^{\mathcal{N}_{\Pi_{\mathcal{A}}}^{\mathcal{A}}(\overline{p})}$, defined in (76). To see why the last inequality holds first note that $v_p^\circ \in \mathcal{N}_{\Upsilon_{\mathcal{A}}(\overline{p})}^{\mathcal{N}_{\Pi_{\mathcal{A}}}^{\mathcal{A}}(\overline{p})}$. Thus the angle between v_p° and $\mathcal{K}_{\mathcal{A}}^d$ is bounded below by $\phi_{\mathcal{A}}(\overline{p}) \in \left(0, \frac{\pi}{2}\right]$. Then the angle between v_p° and $D(W\|q_{\mathcal{A}})$ lies either in $\left[0, \frac{\pi}{2} - \phi_{\mathcal{A}}(\overline{p})\right]$ or in $\left[\frac{\pi}{2} + \phi_{\mathcal{A}}(\overline{p}), \pi\right]$. On the other hand, $v_p^T D(W\|q_{\mathcal{A}}) \le 0$ by (37); thus the angle between v_p° and $D(W\|q_{\mathcal{A}})$ has to lie in $\left[\frac{\pi}{2}, \pi\right]$. Thus the angle between v_p° and $D(W\|q_{\mathcal{A}})$ lies in $\left[\frac{\pi}{2} + \phi_{\mathcal{A}}(\overline{p}), \pi\right]$ and its cosine is bounded from above by $-\sin(\phi_{\mathcal{A}}(\overline{p}))$.

Furthermore,

$$\begin{aligned}\left\|v_p\right\|_{\Lambda_{\mathcal{A}}}^2 &= \left\|\overline{v_p}+v_p^{\circ}\right\|_{\Lambda_{\mathcal{A}}}^2,\\ &\overset{(a)}{\geq}\left(\left\|\overline{v_p}\right\|_{\Lambda_{\mathcal{A}}}-\left\|v_p^{\circ}\right\|_{\Lambda_{\mathcal{A}}}\right)^2,\\ &\geq \left\|\overline{v_p}\right\|_{\Lambda_{\mathcal{A}}}^2-2\left\|\overline{v_p}\right\|_{\Lambda_{\mathcal{A}}}\cdot\left\|v_p^{\circ}\right\|_{\Lambda_{\mathcal{A}}},\\ &\overset{(b)}{\geq}\left\|\overline{v_p}\right\|_{\Lambda_{\mathcal{A}}}^2-2\operatorname{Tr}[\Lambda_{\mathcal{A}}]\cdot\left\|\overline{v_p}\right\|\cdot\left\|v_p^{\circ}\right\|,\\ &\overset{(c)}{\geq}2\gamma_2\left\|\overline{v_p}\right\|^2-2\operatorname{Tr}[\Lambda_{\mathcal{A}}]\cdot\left\|\overline{v_p}\right\|\cdot\left\|v_p^{\circ}\right\|,\\ &\overset{(d)}{=}2\gamma_2\cdot\left\|v_p\right\|^2-2\left(\operatorname{Tr}[\Lambda_{\mathcal{A}}]\left\|\overline{v_p}\right\|+\gamma_2\left\|v_p^{\circ}\right\|\right)\left\|v_p^{\circ}\right\|,\\ &\overset{(e)}{\geq}2\gamma_2\left\|v_p\right\|^2-2\frac{\left\|v_p\right\|\cdot\left\|D(W\|q_{\mathcal{A}})\right\|\sin(\phi_{\mathcal{A}}(\overline{p}))\left\|v_p^{\circ}\right\|}{\delta(\overline{p})},\end{aligned} \tag{87}$$

where (a) follows from the triangle inequality, (b) follows from (73), (c) follows from (81a), (d) follows from (84), (e) follows from $\left\|\overline{v_p}\right\|\vee\left\|v_p^{\circ}\right\|\leq\left\|v_p\right\|$ and the definition of $\delta(\overline{p})$ given in (81b). On the other hand (45) and (71) imply

$$I(p;W)\leq C_{\mathcal{A}}+v_p^T D(W\|q_{\mathcal{A}})-\tfrac{1}{2}\left\|v_p\right\|_{\Lambda_{\mathcal{A}}}^2+\tfrac{\kappa_{\mathcal{A}}}{2}\left\|v_p\right\|^3 \qquad \forall p\in\mathcal{A}.$$

Then (80) follows from (81c), (86), and (87). □

Remark 2: Using (83) and (86) together with the observations in Remark 1, one can improve $\delta(\overline{p})$ value for the case when $\Upsilon_{\mathcal{A}}(\overline{p})=\{\mathbf{0}\}$ slightly to get

$$\delta(\overline{p}):=\begin{cases}\frac{\sin(\phi_{\mathcal{A}}(\overline{p}))}{\gamma_2}\left\|D(W\|q_{\mathcal{A}})\right\| & \text{if } \Upsilon_{\mathcal{A}}(\overline{p})=\{\mathbf{0}\}\\ \frac{\sin(\phi_{\mathcal{A}}(\overline{p}))}{\operatorname{Tr}[\Lambda_{\mathcal{A}}]+\gamma_2}\left\|D(W\|q_{\mathcal{A}})\right\| & \text{if } \Upsilon_{\mathcal{A}}(\overline{p})\neq\{\mathbf{0}\}\end{cases}. \tag{88}$$

VI. Quantum Mutual Information in the Vicinity of the Capacity-Achieving Input Distribution

In this section, we present the analysis for the quantum mutual information between the input and the output of a classical-quantum channel with a finite input set. We will first introduce the quantum information-theoretic framework and quantities. In §VI-Aa, we extend the analysis in §IV to establish the quadratic decay for the quantum mutual information on classical-quantum channels whose Hilbert spaces at the output are separable. In §VI-B, we characterize the slowest decay of quantum mutual information with the distance to the capacity-achieving input distributions on classical-quantum channels with finite-dimensional output Hilbert spaces.

Let $\mathcal{H}$ be a separable Hilbert space, i.e., a complete inner product space that has a countable orthonormal basis. We denote the set of all bounded operators on $\mathcal{H}$, i.e., all continuous linear mappings of the form $T:\mathcal{H}\to\mathcal{H}$, by $\mathcal{L}(\mathcal{H})$. The operator absolute value $|T|\in\mathcal{L}(\mathcal{H})$ of a bounded linear operator T is defined in terms of its adjoint operator T^* as

$$|T|:=\sqrt{T^*T} \qquad \forall T\in\mathcal{L}(\mathcal{H}). \tag{89}$$

An operator T is self-adjoint iff $T^*=T$. We denote a non-commutative quotient for self-adjoint operator T and positive definite operator M as

$$\frac{T}{M}:=M^{-\frac{1}{2}}TM^{-\frac{1}{2}}. \tag{90}$$

Subsequently, we recall the fact of that $T(\cdot)T^*$ is a positive-preserving map for all $T\in\mathcal{L}(\mathcal{H})$, see e.g., [30, §4], i.e.,

$$\text{“}M\geq 0 \quad\Rightarrow\quad TMT^*\geq 0\text{”} \qquad \forall T\in\mathcal{L}(\mathcal{H}). \tag{91}$$

A gentle introduction to separable Hilbert spaces can be found in [30, Chapter 1].

We denote the set of all density operators, i.e., positive semi-definite operators with unit trace, on a separable Hilbert space $\mathcal{H}$ by $\mathcal{S}(\mathcal{H})$. The eigenvalues of a density operator in $\mathcal{S}(\mathcal{H})$ correspond to a probability mass function, [30, Theorem 2.5]. The *quantum relative entropy*, a quantum generalization of the Kullback-Leibler divergence, $D(\rho\|\sigma)$ is defined for any $\rho,\sigma\in\mathcal{S}(\mathcal{H})$ as, see [31],

$$D(\rho\|\sigma):=\begin{cases}\operatorname{Tr}\left[\rho(\ln\rho-\ln\sigma)\right] & \text{if } \rho\prec\sigma\\ \infty & \text{if } \rho\not\prec\sigma\end{cases}, \tag{92}$$

where Tr is the standard trace, and $\rho\prec\sigma$ means that the support of ρ is contained in that of σ. Furthermore, the quantum relative entropy, is bounded from below in terms of the trace-norm via quantum Pinsker's inequality [32, Theorem 3.1]:

$$D(\rho\|\sigma)\geq\tfrac{1}{2}\left\|\rho-\sigma\right\|_1^2, \tag{93}$$

where $\|\cdot\|_1$ is the trace-norm, i.e., the trace of the operator absolute value of a bounded operator:

$$\|T\|_1:=\operatorname{Tr}[|T|] \qquad \forall T\in\mathcal{L}(\mathcal{H}). \tag{94}$$

On the other hand, the quantum relative entropy is bounded above by the quantum χ^2 divergence, see [33, Lemma 2.2 and Remark 2.3] for a proof for finite-dimensional Hilbert spaces,

$$\chi^2(\rho\|\sigma)\geq D(\rho\|\sigma) \tag{95}$$

where χ^{α} divergence is defined for $\alpha>1$ as,

$$\chi^{\alpha}(\rho\|\sigma):=\begin{cases}(\alpha-1)\int_0^{\infty}\operatorname{Tr}\left[\left|\frac{\rho-\sigma}{\sigma+s\mathbf{I}}\right|^{\alpha}\right]\mathrm{d}s & \text{if } \rho\prec\sigma\\ \infty & \text{if } \rho\not\prec\sigma\end{cases}, \tag{96}$$

where $\mathbf{I}$ stands for the identity operator on $\mathcal{H}$.

When ρ and σ commute, i.e., when they have the same set of eigenvectors, the definition in (96) reduces to the one in (29) for countable $\mathcal{Y}$ case, as expected. If $\chi^3(\rho\|\sigma)<\infty$, then we can bound $D(\rho\|\sigma)$ in terms of $\chi^2(\rho\|\sigma)$ and $\chi^3(\rho\|\sigma)$ using Taylor's theorem, as we did in (30) for the case when ρ and σ commute, as follows

$$\left|D(\rho\|\sigma)-\tfrac{1}{2}\chi^2(\rho\|\sigma)\right|\leq\tfrac{1}{2}\chi^3(\rho\|\sigma), \tag{97}$$

see Appendix E for a proof.

A classical-quantum channel $W:\mathcal{X}\to\mathcal{S}(\mathcal{H})$ maps letters of the input alphabet $\mathcal{X}$ to a density operator on the output Hilbert space $\mathcal{H}$. For any $W:\mathcal{X}\to\mathcal{S}(\mathcal{H})$ and $p\in\mathcal{P}(\mathcal{X})$, the mutual information $I(p;W)$ is defined as

$$I(p;W):=\sum\nolimits_x p(x)D\left(W(x)\|\sigma_p\right), \tag{98}$$

where $\sigma_p\in\mathcal{S}(\mathcal{H})$ is the output density operator induced by the input distribution p, for any $p\in\mathcal{P}(\mathcal{X})$, which is defined more generally for any $v:\mathcal{X}\to\mathbb{R}$ with a countable support satisfying $\sum_x|v(x)|<\infty$ as

$$\sigma_v:=\sum\nolimits_x v(x)W(x). \tag{99}$$

Note that (33) can be confirmed for the quantum case by substitution using (99), instead of (32). Furthermore, all of the properties of the Shannon capacity and center discussed in §III hold for the classical-quantum channels, as well, see for example [34, Theorem 2]discussing the case of image-additive quantum channels, which covers as a special case the classical to quantum channels, with a finite-dimensional $\mathcal{H}$. Thus, (45) holds for classical-quantum channels, i.e., for any $\overline{p}\in\Pi_{\mathcal{A}}$ and $p\in\mathcal{A}$,

$$I(p;W)=C_{\mathcal{A}}+(p-\bar{p})^{T}D\left(W\|\sigma_{\mathcal{A}}\right)-D\left(\sigma_{p}\|\sigma_{\mathcal{A}}\right), \quad (100)$$

where σ_p is defined in (99) and $\sigma_{\mathcal{A}}\in\mathcal{S}(\mathcal{H})$ is the Shannon center for the classical-quantum channel W for the convex constraint set $\mathcal{A}$, satisfying $\sigma_{\mathcal{A}}=\sigma_{\overline{p}}$ for all $\overline{p}\in\Pi_{\mathcal{A}}$.

Without loss of generality, we assume that $\mathcal{S}(\mathcal{H})$ equals to the union of the supports of all the channel outputs $W(x)$'s for $x\in\mathscr{X}_{\mathcal{A}}$ and $\mathscr{X}_{\mathcal{A}}$ defined in (34); otherwise, we may restrict the underlying Hilbert space to this union. Such a consideration ensures the Shannon center $\sigma_{\mathcal{A}}$ to have full support.

A. A Simple and General Proof of Quadratic Decay

Using (93) and (100), we can confirm that (46) holds for classical-quantum channels, as well. Thus for any $p\in\mathcal{A}$ and $\overline{p}\in\Pi_{\mathcal{A}}$, we have

$$I(p;W)\leq C_{\mathcal{A}}+(p-\bar{p})^{T}D\left(W\|\sigma_{\mathcal{A}}\right)-\tfrac{1}{2}\left\|\sigma_{(p-\bar{p})}\right\|_{1}^{2}. \quad (101)$$

On the other hand, by following the reasoning as in (47), we can translate the trace-norm on the quantum output space back to the ℓ^2 norm on the classical input space:

$$\|\sigma_v\|_1\leq\|v\|\cdot\sqrt{n} \qquad \forall v\in\mathbb{R}^n. \quad (102)$$

We can follow the argument in the proof of Theorem 1 given in §IV by invoking (101) and (102) in place of (46) and (47) to get the following result on the quadratic decay of the quantum mutual information for a classical-quantum channel.

Theorem 3: Let $W:\mathscr{X}\to\mathcal{S}(\mathcal{H})$ be a classical-quantum channel with a finite input set $\mathscr{X}$ and separable Hilbert space $\mathcal{H}$, and $\mathcal{A}$ be a closed convex polyhedral subset of $\mathscr{P}(\mathscr{X})$ such that $\mathcal{A}\setminus\Pi_{\mathcal{A}}\neq\emptyset$. Then $\mathcal{K}^{d}_{\mathcal{A}}\cap\mathcal{N}_{\mathcal{S}_{\mathcal{A}}}\neq\{\mathbf{0}\}$ and

$$I(p;W)\leq C_{\mathcal{A}}-\gamma\left\|p-P_{\Pi_{\mathcal{A}}}(p)\right\|^{2} \qquad \forall p\in\Pi^{\delta}_{\mathcal{A}}, \quad (103)$$

for the set $\Pi^{\delta}_{\mathcal{A}}$ defined in (44), the angle $\Theta(\cdot,\cdot)$ defined in (3), and positive constants $\beta\in\left(0,\frac{\pi}{2}\right]$, γ, and δ defined in (49).

B. An Exact Characterization of the Slowest Decay

We define the *Bogoliubov-Kubo–Mori inner product* with respect to some positive definite operator $\sigma\in\mathcal{L}(\mathcal{H})$ on bounded operator space $\mathcal{L}(\mathcal{H})$ over field $\mathbb{C}$ [35, §7.5] as

$$\langle\rho,\omega\rangle^{\sigma}_{\text{BKM}}:=\int_{0}^{\infty}\mathrm{Tr}\left[\frac{\rho^{*}}{\sigma+s\mathbf{I}}\frac{\omega}{\sigma+s\mathbf{I}}\right]\mathrm{d}s \quad \forall\rho,\omega\in\mathcal{L}(\mathcal{H}). \quad (104)$$

For any classical-quantum channel $W:\mathscr{X}\to\mathcal{S}(\mathcal{H})$ with a finite-dimensional $\mathcal{H}$ and convex constraint set $\mathcal{A}\subset\mathscr{P}(\mathscr{X})$, we define the set $\mathscr{X}_{\mathcal{A}}$ using (34) and the extended real valued function $\Lambda_{\mathcal{A}}:\mathscr{X}_{\mathcal{A}}\times\mathscr{X}_{\mathcal{A}}\to[-1,\infty]$ via the Bogoliubov-Kubo–Mori inner product:

$$\begin{aligned}\Lambda_{\mathcal{A}}(x,z)&:=\langle W(x)-\sigma_{\mathcal{A}},W(z)-\sigma_{\mathcal{A}}\rangle^{\sigma_{\mathcal{A}}}_{\text{BKM}} \qquad \forall x,z\in\mathscr{X}_{\mathcal{A}}, \quad (105)\\ &=\langle W(x),W(z)\rangle^{\sigma_{\mathcal{A}}}_{\text{BKM}}-1 \qquad \forall x,z\in\mathscr{X}_{\mathcal{A}}. \quad (106)\end{aligned}$$

For the case when $W(x)$, $W(z)$, and $\sigma_{\mathcal{A}}$ mutually commute the definition in (105) reduces to the one in (59) given in §V-A, as expected.

When $\mathscr{X}_{\mathcal{A}}$ is a finite set and $\max_{x,z}\Lambda_{\mathcal{A}}(x,z)$ is finite, then $\Lambda_{\mathcal{A}}$ is a positive semi-definite matrix because for all $v\in\mathbb{R}^n$ we have (63)

$$v^{T}\Lambda_{\mathcal{A}}v=\int_{0}^{\infty}\mathrm{Tr}\left[\left(\frac{\sigma_{v}-v^{T}\mathbf{1}\sigma_{\mathcal{A}}}{\sigma_{\mathcal{A}}+s\mathbf{I}}\right)^{2}\right]\mathrm{d}s \qquad \forall v\in\mathbb{R}^{n}. \quad (107)$$

Thus $v^{T}\Lambda_{\mathcal{A}}v\geq 0$ for all $v\in\mathbb{R}^n$ and consequently $\Lambda_{\mathcal{A}}$ defines a seminorm on $\mathbb{R}^n$ for classical-quantum channels, as well. Furthermore, as was the case in the classical channels, see (67), the resulting seminorm is related to the quantum χ^2 divergence; for any $p\in\mathbb{R}^n$ satisfying $p^{T}\mathbf{1}=1$ and $\overline{p}\in\mathbb{R}^n$ satisfying $\sigma_{\overline{p}}=\sigma_{\mathcal{A}}$, we have

$$\chi^{2}\left(\sigma_{p}\|\sigma_{\mathcal{A}}\right)=\|p-\bar{p}\|^{2}_{\Lambda_{\mathcal{A}}}, \quad (108)$$

where $\sigma_{\overline{p}}$ is defined in (99).

The operator absolute value q_v can be bounded from above for any v in terms $\|v\|$, whenever $\|v\|$ is finite, as follows.

$$\begin{aligned}(\sigma_{v})^{2}&=\left(\sum\nolimits_{x\in\mathscr{X}_{\mathcal{A}}}v(x)W(x)\right)^{2}\\ &=\sum\nolimits_{x,z\in\mathscr{X}_{\mathcal{A}}}v(x)v(z)W(x)W(z)\\ &=\|v\|^{2}\sum_{x\in\mathscr{X}_{\mathcal{A}}}W(x)^{2}-\sum_{x,z\in\mathscr{X}_{\mathcal{A}}}\tfrac{(v(x)W(z)-v(z)W(x))^{2}}{2}\\ &\leq\|v\|^{2}\left(\sum\nolimits_{x\in\mathscr{X}_{\mathcal{A}}}W(x)^{2}\right). \quad (109)\end{aligned}$$

where the inequality follows from the positive semi-definiteness of the operator $(v(x)W(z)-v(z)W(x))^2$ for all x and z in $\mathscr{X}_{\mathcal{A}}$. Since the square-root is operator monotone (see e.g., [35, §4]), we have

$$|\sigma_{v}|\leq\|v\|\cdot\sqrt{\sum\nolimits_{x\in\mathscr{X}_{\mathcal{A}}}W(x)^{2}}. \quad (110)$$

Lemma 5: For any classical-quantum channel $W:\mathscr{X}\to\mathcal{S}(\mathcal{H})$ with a finite input set $\mathscr{X}$ and finite-dimensional Hilbert space $\mathcal{H}$ and a closed convex constraint set $\mathcal{A}\subset\mathscr{P}(\mathscr{X})$ satisfying $\kappa_{\mathcal{A}}<\infty$, for all $p\in\mathcal{A}$ and $\overline{p}\in\Pi_{\mathcal{A}}$ we have

$$\left|D\left(\sigma_{p}\|\sigma_{\mathcal{A}}\right)-\tfrac{1}{2}\|p-\bar{p}\|^{2}_{\Lambda_{\mathcal{A}}}\right|\leq\tfrac{\kappa_{\mathcal{A}}}{2}\|p-\bar{p}\|^{3}, \quad (111)$$

where $\kappa_{\mathcal{A}}$ is defined as follows

$$\kappa_{\mathcal{A}}:=\int_{0}^{\infty}\mathrm{Tr}\left[\left(\frac{\sqrt{\sum_{x\in\mathscr{X}_{\mathcal{A}}}W(x)^{2}}}{\sigma_{\mathcal{A}}+s\mathbf{I}}\right)^{3}\right]\mathrm{d}s. \quad (112)$$

Lemma 5 is proved in Appendix F. When $\{W(x)\}_{x\in\mathscr{X}_{\mathcal{A}}}$ mutually commute, i.e., all the channel outputs $W(x)$'s share the same eigen-basis, $\kappa_{\mathcal{A}}$ defined in (112) reduces to the one in (70) and Lemma 5 recovers Lemma 4 in the classical setting for finite $\mathscr{Y}$ case.

Remark 3: Although it is not needed for proving Lemma 5, the following bound on $\chi^{3}\left(\sigma_{p}\|\sigma_{\mathcal{A}}\right)$ in terms of $\|p-\overline{p}\|^{3}$, holds

$$\chi^{3}\left(\sigma_{p}\|\sigma_{\mathcal{A}}\right)\leq\kappa_{\mathcal{A}}\cdot\|p-\bar{p}\|^{3} \quad (113)$$

for all $\bar{p} \in \mathbb{R}^n$ satisfying $\sigma_{\bar{p}} = \sigma_{\mathcal{A}}$ and $p \in \mathbb{R}^n$ provided the Hilbert space of the channel $\mathcal{H}$ is finite-dimensional. Evidently, (113) corresponds to (69) for finite $\mathscr{Y}$ case. The proof of (113) relies on certain majorization properties of eigenvalues of self-adjoint matrices [36].

In our analysis on classical-quantum channels, we will need an operator-norm bound analogous to (47), similar to (73) for classical channels, as well. To that end we bound $\|v\|_{\Lambda_{\mathcal{A}}}$ from above in terms of $\|v\|$ for an arbitrary $v \in \mathbb{R}^n$. First note that,

$$\begin{aligned}\left(\sum\nolimits_{x\in\mathscr{X}_{\mathcal{A}}} v(x)\tfrac{W(x)-\sigma_{\mathcal{A}}}{\sigma_{\mathcal{A}}+s\mathbf{I}}\right)^2 &= \|v\|^2\sum\nolimits_{x\in\mathscr{X}_{\mathcal{A}}}\left(\tfrac{W(x)-\sigma_{\mathcal{A}}}{\sigma_{\mathcal{A}}+s\mathbf{I}}\right)^2 \\ &\quad-\tfrac{1}{2}\sum\nolimits_{x,z\in\mathscr{X}_{\mathcal{A}}}\left(v(x)\tfrac{W(z)-\sigma_{\mathcal{A}}}{\sigma_{\mathcal{A}}+s\mathbf{I}}-v(z)\tfrac{W(x)-\sigma_{\mathcal{A}}}{\sigma_{\mathcal{A}}+s\mathbf{I}}\right)^2 \\ &\leq \|v\|^2\sum\nolimits_{x\in\mathscr{X}_{\mathcal{A}}}\left(\tfrac{W(x)-\sigma_{\mathcal{A}}}{\sigma_{\mathcal{A}}+s\mathbf{I}}\right)^2\end{aligned}$$

for all $s \geq 0$ because all of the operators in the sum with the coefficient $\frac{1}{2}$ are positive semi-definite. Thus using the monotonicity of the trace, we get

$$\begin{aligned}\|v\|_{\Lambda_{\mathcal{A}}}^2 &= \int_0^\infty \mathrm{Tr}\left[\left(\sum\nolimits_{x\in\mathscr{X}_{\mathcal{A}}} v(x)\frac{W(x)-\sigma_{\mathcal{A}}}{\sigma_{\mathcal{A}}+s\mathbf{I}}\right)^2\right]\mathrm{d}s \\ &\leq \int_0^\infty \mathrm{Tr}\left[\|v\|^2\sum\nolimits_{x\in\mathscr{X}_{\mathcal{A}}}\left(\frac{W(x)-\sigma_{\mathcal{A}}}{\sigma_{\mathcal{A}}+s\mathbf{I}}\right)^2\right]\mathrm{d}s \\ &= \|v\|^2\cdot\mathrm{Tr}\left[\Lambda_{\mathcal{A}}\right]. \end{aligned} \tag{114}$$

We apply the analysis of Theorem 2 given in §V by invoking Lemma 5, (95) and (114) in place of Lemma 4, (28), and (73) to obtain the following result of the exact characterization of the slowest decay for quantum mutual information on classical-quantum channels with finite-dimensional Hilbert space $\mathcal{H}$.

Theorem 4: For a classical-quantum channel $W : \mathscr{X} \to \mathcal{S}(\mathcal{H})$ with a finite input set $\mathscr{X}$ and a finite-dimensional Hilbert space $\mathcal{H}$, a closed convex polyhedral constraint set $\mathcal{A} \subset \mathscr{P}(\mathscr{X})$ satisfying both $\mathcal{A} \setminus \Pi_{\mathcal{A}} \neq \emptyset$ and $\kappa_{\mathcal{A}} < \infty$, where $\kappa_{\mathcal{A}}$ defined in (112),

$$I(p;W) \leq C_{\mathcal{A}} - \gamma_1 \left\|v_p\right\| \qquad \forall p \in \mathcal{A}, \tag{115}$$

for γ_1 defined in (77), where $v_p := p - P_{\Pi_{\mathcal{A}}}(p)$ and there exists a $p \in \mathcal{A} \setminus \Pi_{\mathcal{A}}$ satisfying

$$I(p(\tau);W) \geq C_{\mathcal{A}} - \gamma_1 \left\|v_p\right\|\tau - \mathrm{Tr}\left[\Lambda_{\mathcal{A}}\right]\cdot\left\|v_p\right\|^2\tau^2 \tag{116}$$

for all $\tau \in [0,1]$, where $p(\tau) := P_{\Pi_{\mathcal{A}}}(p) + \tau v_p$ and $\Lambda_{\mathcal{A}}$ is defined in (105). Furthermore, if $\gamma_1 = 0$, then

$$I(p;W) \leq C_{\mathcal{A}} - \gamma_2 \left\|v_p\right\|^2 + \tfrac{\kappa_{\mathcal{A}}}{2}\left\|v_p\right\|^3 \qquad \forall p \in \Pi_{\mathcal{A}}^{\delta} \tag{117}$$

for positive constants γ_2 and δ, defined in (81a) and (81c) and there exists a $p \in \mathcal{A} \setminus \Pi_{\mathcal{A}}$ satisfying

$$I(p(\tau);W) \geq C_{\mathcal{A}} - \gamma_2 \left\|v_p\right\|^2\tau^2 - \tfrac{\kappa_{\mathcal{A}}\|v_p\|^3\tau^3}{2} \qquad \forall \tau \in [0,1]. \tag{118}$$

VII. Discussion

We have two main contributions. First, we have generalized Strassen's bound in (1) to channels with finite input sets and measurable output spaces for polyhedral constraint sets with explicit γ, and δ expressions, see Theorem 1. If we replace the Kullback-Leibler divergence, mutual information, and total variation norm, with the quantum relative entropy, quantum mutual information, and trace-norm, then the exact same proof applies to classical-quantum channels with separable output Hilbert spaces, see Theorem 3. Strassen's bound in (1) has not been proven either for channels with measurable output spaces or for classical-quantum channel before. Neither, has it been proven with explicit γ, and δ expressions even for channels with finite input and output sets. Our proof relied on Pinsker's inequality (i.e., (27)/(93)), Topsøe identity (i.e., (33)), polyhedral convexity (via Lemma 2), and the positivity of the angle between a pair of closed cones whose intersection is their common apex, see Lemma 1.

Second, we have determined the exact leading non-zero term in the Taylor series expansion of the slowest decay of the mutual information around the capacity-achieving input distributions for channels with finite input sets and measurable output spaces and for polyhedral constraint sets, under a finite moment constraint, i.e., under $\kappa_{\mathcal{A}} < \infty$ hypothesis for $\kappa_{\mathcal{A}}$ defined in (70), see Theorem 2. In particular, we have determined the largest γ_1 value satisfying

$$I(p;W) \leq C_{\mathcal{A}} - \gamma_1 \|p - \bar{p}\| \qquad \forall p \in \mathcal{A},$$

where $\bar{p}$ is the projection of p to $\Pi_{\mathcal{A}}$. Furthermore, for the cases when this largest γ_1 value is zero, we have determined the largest γ_2 value satisfying the following inequality for some $\delta > 0$

$$I(p;W) \leq C_{\mathcal{A}} - \gamma_2 \|p - \bar{p}\|^2 + \tfrac{\kappa_{\mathcal{A}}}{2}\|p - \bar{p}\|^3 \qquad \forall p \in \Pi_{\mathcal{A}}^{\delta},$$

showed that this largest γ_2 value is positive, and gave a closed form expression for the associated δ. We established the corresponding result for the classical-quantum channels under the additional hypothesis that Hilbert space at the output of the channel is finite-dimensional, see Theorem 4. Our proof relied on Moreau's decomposition theorem (i.e., Lemma 3) and Taylor's theorem with the remainder term.

We have also demonstrated that both the polyhedral constraint set assumption and the finite input set assumption are necessary. The channel in Example 1 has three input letters and two output letters. For a convex (but not polyhedral) constraint set $\mathcal{A}$, the only non-negative γ satisfying

$$I(p;W) \leq C_{\mathcal{A}} - \gamma \|p - \bar{p}\| \qquad \forall p \in \Pi_{\mathcal{A}}^{\delta},$$

for some $\delta > 0$ is zero, where $\bar{p}$ is the projection of p to $\Pi_{\mathcal{A}}$. The channel in Example 2 has a countably infinite input set and two output letters. For that channel only $f : [0,\delta] \to \mathbb{R}_{\geq 0}$ satisfying $I(p;W) \leq C - f(\|p - \bar{p}\|)$ for all input distributions satisfying $\|p - \bar{p}\| \leq \delta$ for a positive δ is $f = 0$, i.e., $f(z) = 0$ for all $z \in [0, 1 \wedge \delta]$.

The primary benefit of removing the finite output set assumption of [4], [17], and [18] is that it might be possible to generalize the proof techniques relying on (1) such as the ones in [4], [6], [7], [8], [9], and [10] to channels whose output set is not a finite set. There might be additional challenges in doing so because the finite output set assumption is often invoked implicitly elsewhere in those proofs. Nevertheless it might be possible to overcome those challenges. For example the exquisite net argument of Tomamichel and Tan in [8, §III-C], which is inspired by Hayashi's in [5, §X.A], constructs a net on the mass functions on the output set. However, one can construct a net around $\Pi_{\mathcal{A}}$ in $\mathcal{A}$ instead and it seems this new net might be used in place of the original one, with appropriate modifications to the argument and possibly with additional assumptions on the channel.

Under appropriate technical assumptions, similar results can be obtained for Augustin information [37], [38], [39], [40] using the same framework, as well, see [41].

Appendix

A. Gap in Strassen's Argument

The first two terms of the Taylor expansion characterizing the change of the mutual information around any capacity-achieving input distribution are determined in [4] to be

$$I(p;W) = C + f(p-\overline{p}) + o\left(\|p-\overline{p}\|^2\right)$$
$$f(v) = v^T \nabla I(p;W)\rvert_{\Pi} - \tfrac{1}{2} v^T H_W v \qquad \forall v \in \mathbb{R}^n$$

where $\overline{p}$ is the projection of p to Π and matrix H_W is defined in terms of the capacity achieving output distribution, i.e., the Shannon center, q_W as $H_W := W\mathrm{diag}\left(\frac{1}{q_W}\right)W^T$. [4, (4.41)] asserts that for small enough δ there exists a $\gamma > 0$ satisfying

$$f(p-\overline{p}) \le -\gamma \|p-\overline{p}\|^2 \qquad \forall p \in \Pi^{\delta}. \tag{4.41}$$

To establish (4.41) Strassen asserts that if (4.41) does not hold then there must exist a sequence $\{p_\jmath\}_{\jmath\in\mathbb{Z}_+} \subset \Pi^{\delta}$ satisfying

$$\liminf_{\jmath} f(p_\jmath - \overline{p_\jmath}) \ge 0. \tag{119}$$

Furthermore, Strassen asserts that since $(p-\bar{p})^T D(W\|q_W) \le 0$ for all $p \in \mathcal{P}(\mathcal{X})$, one can assume

$$\left\|p_\jmath - \overline{p_\jmath}\right\| = \delta \qquad \forall \jmath \in \mathbb{Z}_+. \tag{120}$$

We agree with Strassen's assertion because of the following reasoning: If Π is in the relative interior of the probability simplex, i.e., $\Pi \cap \partial\mathscr{P}(\mathscr{X}) = \emptyset$, then for small enough δ any point p on the boundary Π^{δ} will satisfy $\|p-\overline{p}\| = \delta$ and the identity $(p-\overline{p})^T \nabla I(p;W)\rvert_{\Pi} \le 0$ for all $p \in \mathscr{P}(\mathscr{X})$ implies

$$f(p-\overline{p}) \le \tfrac{\|p-\overline{p}\|^2}{\delta^2} f\left(\tfrac{p-\overline{p}}{\|p-\overline{p}\|}\delta\right) \qquad \forall p \in \Pi^{\delta}. \tag{121}$$

Thus if the sequence satisfying (119) does not satisfy (120), then we can replace each $p_\jmath$ with $b_\jmath = \overline{p_\jmath} + \frac{p_\jmath - \overline{p_\jmath}}{\|p_\jmath - \overline{p_\jmath}\|}\delta$ to get a sequence satisfying both (119) and (120). Note that $\overline{b_\jmath} = \overline{p_\jmath}$ and $\left\|b_\jmath - \overline{b_\jmath}\right\| = \delta$ for all $\jmath$ by construction.

However, for certain channels, Π might have points outside the relative interior of the probability simplex associated with the input set of the channel, i.e., $\Pi \cap \partial\mathscr{P}(\mathscr{X}) \neq \emptyset$ might hold. The unconstrained version of the channel considered in Example 1 is such a channel. The argument presented in the previous paragraph for $\Pi \cap \partial\mathscr{P}(\mathscr{X}) = \emptyset$ case will not work as is for this case because there might not be a positive δ for which infinitely many $b_\jmath$'s are guaranteed to be in the probability simplex $\mathscr{P}(\mathscr{X})$, and hence in Π^{δ}. Nevertheless, a sequence satisfying both (119) and (120) exists as claimed by Strassen. To see why first recall that the projection of a $p \in \mathscr{P}(\mathscr{X})$ to Π is $\overline{p}$ iff $p - \overline{p} \in \mathcal{N}_{\Pi}^{\mathscr{P}(\mathscr{X})}(\overline{p})$; see (10) and (11). Furthermore, both $\{\mathcal{N}_{\Pi}(\overline{p}) : \overline{p} \in \Pi\}$ and $\{\mathcal{T}_{\mathscr{P}(\mathscr{X})}(\overline{p}) : \overline{p} \in \Pi\}$ are finite sets as a result of the polyhedral convexity of Π and $\mathscr{P}(\mathscr{X})$. Thus the set $\mathcal{S} = \{\mathcal{N}_{\Pi}^{\mathscr{P}(\mathscr{X})}(\overline{p}) : \overline{p} \in \Pi\}$ is finite and for each $\varsigma \in \mathcal{S}$ there exists at least one (often uncountably many) $\overline{p} \in \Pi$ satisfying $\varsigma = \mathcal{N}_{\Pi}^{\mathscr{P}(\mathscr{X})}(\overline{p})$. For each $\varsigma \in \mathcal{S}$ we choose a $\widetilde{p}(\varsigma) \in \Pi$ satisfying $\varsigma = \mathcal{N}_{\Pi}^{\mathscr{P}(\mathscr{X})}(\widetilde{p}(\varsigma))$. Among $\{\mathcal{N}_{\Pi}^{\mathscr{P}(\mathscr{X})}(\overline{p_\jmath})\}_{\jmath\in\mathbb{Z}_+}$ at least one $\widehat{\varsigma} \in \mathcal{S}$ will be repeated infinitely often. Let $\{p_{\imath_\jmath}\}_{\jmath\in\mathbb{Z}_+}$ be a subsequence satisfying $\mathcal{N}_{\Pi}^{\mathscr{P}(\mathscr{X})}(\overline{p_{\imath_\jmath}}) = \widehat{\varsigma}$ for all $\jmath \in \mathbb{Z}_+$. Let us define $a_\jmath$ as $a_\jmath := \widetilde{p}(\widehat{\varsigma}) + \frac{p_{\imath_\jmath} - \overline{p_{\imath_\jmath}}}{\|p_{\imath_\jmath} - \overline{p_{\imath_\jmath}}\|}\delta$, for a constant δ that we will choose in the following. Then the projection of $a_\jmath$ onto Π is $\widetilde{p}(\widehat{\varsigma})$ for all $\jmath \in \mathbb{Z}_+$, because $a_\jmath - \widetilde{p}(\widehat{\varsigma}) \in \mathcal{N}_{\Pi}^{\mathscr{P}(\mathscr{X})}(\widetilde{p}(\widehat{\varsigma}))$. Furthermore, as a result of the polyhedral convexity of $\mathscr{P}(\mathscr{X})$ for each $\overline{p} \in \Pi$, there exists a $\delta(\overline{p}) > 0$ such that

$$\left\{\overline{p} + \tau v : v \in \mathcal{N}_{\Pi}^{\mathscr{P}(\mathscr{X})}(\overline{p}), \|v\| = 1, \text{and } \tau \in [0, \delta(\overline{p})]\right\} \subset \mathscr{P}(\mathscr{X}).$$

If we choose $\delta = \min_{\widehat{\varsigma}\in\mathcal{S}} \delta(\widetilde{p}(\widehat{\varsigma}))$ then all $a_\jmath$ are in $\mathscr{P}(\mathscr{X})$. Thus (120) holds for $a_\jmath$ by construction and (119) holds for $a_\jmath$ by (121). Hence, there exists a sequence satisfying both (119) and (120) when $\Pi \cap \partial\mathscr{P}(\mathscr{X}) \neq \emptyset$, as well.

B. A Counter-Example for [6, (500)]

Example 4: Let W be a channel with 9 input letters and 8 output letters given in the following

$$W = \begin{bmatrix} \varepsilon/3\,\mathbf{1}_{5\times 1} & \varepsilon/3\,\mathbf{1}_{5\times 1} & \varepsilon/3\,\mathbf{1}_{5\times 1} & (1-\varepsilon)\mathbf{I}_5 \\ 1/2 & 1/3 & 1/6 & \mathbf{0}_{1\times 5} \\ 1/6 & 1/2 & 1/3 & \mathbf{0}_{1\times 5} \\ 1/3 & 1/6 & 1/2 & \mathbf{0}_{1\times 5} \\ 1/3 & 1/2 & 1/6 & \mathbf{0}_{1\times 5} \end{bmatrix},$$

where $\mathbf{1}_{5\times 1}$ is a column vector of ones, $\mathbf{I}_5$ is 5-by-5 identity matrix, $\mathbf{0}_{1\times 5}$ is a row vector of zeros, and ε is the unique solution of the equation $\frac{\sqrt{3}\sqrt[3]{2}}{10} = \varepsilon 5^{-\varepsilon}$ on $\varepsilon \in \left(0, \frac{1}{\ln 5}\right)$.

With a slight abuse of notation when $\mathcal{A} = \mathscr{P}(\mathscr{X})$, we denote the Shannon capacity by C and the Shannon center by q_W. Let us assume $\mathcal{A} = \mathscr{P}(\mathscr{X})$. Then the capacity-achieving input distribution is unique and it is the uniform distribution on the first 5 input letters. Furthermore,

$$C = (1-\varepsilon)\ln 5 \quad \text{and} \quad q_W = \left[\tfrac{\varepsilon}{3}\ \tfrac{\varepsilon}{3}\ \tfrac{\varepsilon}{3}\ \tfrac{1-\varepsilon}{5}\mathbf{1}_{1\times 5}\right].$$

Note that $D(W(x)\|q_W) = C$ for all input letters x. Thus

$$\nabla I(p;W)\rvert_{\Pi} = D(W\|q_W) = C \cdot 1_{9\times 1}$$

On the other hand $\mathcal{K}_W = \{\tau u : \tau \in \mathbb{R}\}$ where the vector u is given by

$$u = \left[\mathbf{0}_{1\times 5}\ 2\ 2\ -1\ -3\right]^T.$$

Note that $u^T \nabla I(p;W)\rvert_{\Pi} = 0$. Thus $v_0^T \nabla I(p;W)\rvert_{\Pi} = 0$ for any p, where v_0 is the projection of $p - \overline{p}$ onto $\mathcal{K}_W$ considered in [6]. On the other hand if p puts non-zero probability only on one of the last four input letters then $\|v_0\| \neq 0$. Consequently,

$v_0^T \nabla I(p;W)|_\Pi \leq -\Gamma \|v_0\|$, i.e., [6, (500)], cannot be true for any positive Γ.

C. $\Lambda_{\mathcal{A}}$ Is a Fisher Information Matrix

Let ξ_p be

$$\xi_p := \tfrac{1}{p^T\mathbf{1}} \tfrac{\mathrm{d}q_p}{\mathrm{d}\nu} \qquad \forall p \in \mathcal{P}, \tag{122}$$

where ν is any σ-finite reference measure satisfying $q_{\mathcal{A}} \prec \nu$, q_p is defined in (32), and $\mathcal{P}$ is defined as

$$\mathcal{P} := \left\{ p \in \mathbb{R}^n : p^T\mathbf{1} > 0,\ q_p \prec q_{\mathcal{A}}, \tfrac{\mathrm{d}q_p}{\mathrm{d}\nu} \geq 0\ \nu\text{-a.e.} \right\}. \tag{123}$$

Then the Fisher information matrix for the parametric family of Radon-Nikodym derivatives $\{\xi_p : p \in \mathcal{P}\}$ at a $\overline{p}$ in the interior of $\mathcal{P}$ is defined as

$$J_\xi(\overline{p}) := \int \left(\tfrac{\partial}{\partial p} \ln \xi_p\right)^T \left(\tfrac{\partial}{\partial p} \ln \xi_p\right) \xi_p \mathrm{d}\nu \bigg|_{p=\overline{p}}. \tag{124}$$

On the other hand for all p in the interior of $\mathcal{P}$ we have,

$$\begin{aligned}
\tfrac{\partial}{\partial p} \tfrac{\mathrm{d}q_p}{\mathrm{d}\nu} &= \left(\tfrac{\mathrm{d}W}{\mathrm{d}\nu}\right)^T, \\
\ln \xi_p &= \ln\left(\tfrac{\mathrm{d}q_p}{\mathrm{d}\nu}\right) - \ln\left(p^T\mathbf{1}\right), \\
\tfrac{\partial}{\partial p} \ln \xi_p &= \tfrac{1}{\frac{\mathrm{d}q_p}{\mathrm{d}\nu}} \left(\tfrac{\mathrm{d}W}{\mathrm{d}\nu}\right)^T - \tfrac{\mathbf{1}^T}{p^T\mathbf{1}}, \\
&= \left(\tfrac{\mathrm{d}W}{\mathrm{d}q_p} - \tfrac{\mathbf{1}}{p^T\mathbf{1}}\right)^T.
\end{aligned}$$

For all $\overline{p}$ satisfying $q_{\overline{p}} = q_{\mathcal{A}}$, we have $p^T\mathbf{1} = 1$. Thus

$$\begin{aligned}
J_\xi(\overline{p}) &= \int \left(\tfrac{\mathrm{d}W}{\mathrm{d}q_{\mathcal{A}}} - \mathbf{1}\right)\left(\tfrac{\mathrm{d}W}{\mathrm{d}q_{\mathcal{A}}} - \mathbf{1}\right)^T \mathrm{d}q_{\mathcal{A}}, \\
&= \Lambda_{\mathcal{A}}.
\end{aligned} \tag{125}$$

Thus $\Lambda_{\mathcal{A}}$ is the Fisher information matrix for the parametric family of Radon-Nikodym derivatives $\{\xi_p : p \in \mathcal{P}\}$ defined in (122) at any $\overline{p}$ satisfying $q_{\overline{p}} = q_{\mathcal{A}}$.

D. Proof of (30)

Let us first recall Taylor's theorem with the remainder term, see [42, Appendix B]: Any function f that is n times continuously differentiable on an open interval including τ and x satisfies

$$f(\tau) = f(x) + \sum\nolimits_{\iota=1}^{n-1} \tfrac{f^{(\iota)}(x)}{\iota!}(\tau - x)^\iota + \Delta_n(x,\tau), \tag{126}$$

where $f^{(\iota)}(\cdot)$ is the ι^{th} derivative of $f(\cdot)$ and

$$\Delta_n(x,\tau) = \int_x^\tau \tfrac{f^{(n)}(z)}{(n-1)!}(\tau - z)^{n-1}\mathrm{d}z. \tag{127}$$

Let us consider the function $f(\tau) = \tau \ln \tau$:

$$\begin{aligned}
f^{(1)}(\tau) &= 1 + \ln\tau \\
f^{(2)}(\tau) &= \tfrac{1}{\tau} \\
f^{(3)}(\tau) &= -\tfrac{1}{\tau^2} \\
\Delta_3(x,\tau) &= -\tfrac{1}{2}\int_x^\tau \left(\tfrac{\tau}{z} - 1\right)^2 \mathrm{d}z
\end{aligned}$$

Then

$$\begin{aligned}
|\Delta_3(1,\tau)| &= \tfrac{1}{2}\left|\int_1^\tau \left(\tfrac{\tau}{z} - 1\right)^2 \mathrm{d}z\right| \\
&\leq \tfrac{1}{2}\left|\int_1^\tau (\tau - 1)^2 \mathrm{d}z\right| \\
&\leq \tfrac{|\tau - 1|^3}{2}
\end{aligned}$$

Thus applying Taylor's theorem with the remainder term to the function $x\ln x$ around $x = 1$, we get

$$-\tfrac{|1-x|^3}{2} \leq x\ln x - (x-1) - \tfrac{(x-1)^2}{2} \leq \tfrac{|1-x|^3}{2}.$$

Thus for any w and q satisfying $\chi^3(w\|q) < \infty$ we have

$$-\tfrac{1}{2}\chi^3(w\|q) \leq D(w\|q) - \tfrac{1}{2}\chi^2(w\|q) \leq \tfrac{1}{2}\chi^3(w\|q).$$

E. Proof of (97)

For an invertible density operator σ and arbitrary density operator ρ, let $\rho(\tau)$ and $f(\tau)$ be

$$\begin{aligned}
\rho(\tau) &= \tau\rho + (1-\tau)\sigma & \forall \tau \in [0,1], \\
f(\tau) &= D(\rho(\tau)\|\sigma) & \forall \tau \in [0,1].
\end{aligned}$$

To obtain (97), we first apply Taylor's theorem with the remainder term, i.e., (126), at $x = \epsilon$ to calculate $f(\tau)$ for an $\epsilon \in (0,1)$ and an $\tau \in (\epsilon, 1)$ and then calculate the limits as $\tau \uparrow 1$ and $\epsilon \downarrow 0$.

By standard calculations (see e.g., [35, §3]), we have

$$\begin{aligned}
f^{(1)}(\tau) &= \mathrm{Tr}\left[(\rho - \sigma)\ln\rho(\tau) - (\rho - \sigma)\cdot\ln\sigma\right], \\
f^{(2)}(\tau) &= \int_0^\infty \mathrm{Tr}\left[\left(\frac{\rho - \sigma}{\rho(\tau) + s\mathbf{I}}\right)^2\right]\mathrm{d}s, \\
f^{(3)}(\tau) &= -2\cdot\int_0^\infty \mathrm{Tr}\left[\left(\frac{\rho - \sigma}{\rho(\tau) + s\mathbf{I}}\right)^3\right]\mathrm{d}s,
\end{aligned}$$

On the other hand for any self-adjoint operator δ and real numbers $s \geq 0$ and $\tau \in [0,1)$, we have

$$\begin{aligned}
\mathrm{Tr}\left[\left(\frac{\delta}{\rho(\tau)+s\mathbf{I}}\right)^3\right] &\overset{(a)}{\leq} \mathrm{Tr}\left[\left|\frac{\delta}{\rho(\tau)+s\mathbf{I}}\right|^3\right] \\
&\overset{(b)}{=} \mathrm{Tr}\left[\left(\frac{\delta\,(\rho(\tau)+s\mathbf{I})^{-1}\,\delta}{\rho(\tau)+s\mathbf{I}}\right)^{\frac{3}{2}}\right] \\
&\overset{(c)}{\leq} \mathrm{Tr}\left[\left(\frac{\delta\,((1-\tau)\sigma+s\mathbf{I})^{-1}\,\delta}{\rho(\tau)+s\mathbf{I}}\right)^{\frac{3}{2}}\right] \\
&\overset{(d)}{=} \mathrm{Tr}\left[\left(\frac{\delta\,(\rho(\tau)+s\mathbf{I})^{-1}\,\delta}{(1-\tau)\sigma+s\mathbf{I}}\right)^{\frac{3}{2}}\right] \\
&\overset{(e)}{\leq} \mathrm{Tr}\left[\left(\frac{\delta\,((1-\tau)\sigma+s\mathbf{I})^{-1}\,\delta}{(1-\tau)\sigma+s\mathbf{I}}\right)^{\frac{3}{2}}\right] \\
&= \tfrac{1}{(1-\tau)^3}\mathrm{Tr}\left[\left|\frac{\delta}{\sigma+\frac{s}{1-\tau}\mathbf{I}}\right|^3\right]
\end{aligned}$$

where (a) follows from the operator inequality $T \leq |T|$ and the monotonicity of the map $\mathrm{Tr}\left[(\cdot)^3\right]$ by [43, Theorem 2.10], (b) follows from the cyclic property of trace and (90) because the noncommutative quotient is self-adjoint, (c) follows from the operator inequality $\rho(\tau) \geq (1-\tau)\sigma$ because the inverse is operator monotone decreasing, the map $T(\cdot)T^*$ is a positive-preserving map by (91), and the map $\mathrm{Tr}\left[(\cdot)^{\frac{3}{2}}\right]$ is monotone increasing by [43, Theorem 2.10], (d) holds because T^*T and TT^* have the same eigenvalues, (e) follows from the operator inequality $\rho(\tau) \geq (1-\tau)\sigma$ with the reasoning of invoked for the inequality (c). Thus for all $\tau \in (\epsilon, 1)$ we have

$$\begin{aligned}|\Delta_3(\epsilon,\tau)| &= \left|\int_\epsilon^\tau \tfrac{f^{(3)}(z)}{2}(\tau-z)^2 \mathrm{d}z\right| \\ &= \left|\int_\epsilon^\tau \int_0^\infty \mathrm{Tr}\left[\left(\tfrac{\rho-\sigma}{\rho(z)+s\mathbf{I}}\right)^3\right](\tau-z)^2 \mathrm{d}s\mathrm{d}z\right| \\ &\leq \left|\int_\epsilon^\tau \int_0^\infty \tfrac{1}{(1-z)^3}\mathrm{Tr}\left[\left|\tfrac{\rho-\sigma}{\sigma+\frac{s}{1-z}\mathbf{I}}\right|^3\right](\tau-z)^2 \mathrm{d}s\mathrm{d}z\right| \\ &= \tfrac{\chi^3(\rho\|\sigma)}{2}\left|\int_\epsilon^\tau \left(\tfrac{1-\tau}{1-z}-1\right)^2 \mathrm{d}z\right| \\ &\leq \tfrac{\chi^3(\rho\|\sigma)}{2}\left|\int_\epsilon^\tau \left(\tfrac{1-\tau}{1-\epsilon}-1\right)^2 \mathrm{d}z\right| \\ &= \tfrac{\chi^3(\rho\|\sigma)}{2}\tfrac{(\tau-\epsilon)^3}{(1-\epsilon)^2}\end{aligned}$$

Thus using the Taylor's theorem with a remainder term, i.e., (126), we get

$$\left|f(\tau)-f(\epsilon)-f^{(1)}(\epsilon)(\tau-\epsilon)-\tfrac{f^{(2)}(\epsilon)}{2}(\tau-\epsilon)^2\right| \leq \tfrac{\chi^3(\rho\|\sigma)}{2(1-\epsilon)^2}$$

for all $\epsilon \in (0,1)$ and $\tau \in (\epsilon, 1)$. Then (97) can be proved by taking the limits first as $\tau \uparrow 1$ and then as $\epsilon \downarrow 0$, provided that $\lim_{\tau\uparrow 0} f(\tau) = f(1)$, $\lim_{\tau\downarrow 1} f(\tau) = 0$, $\lim_{\tau\downarrow 0} f^{(1)}(\tau) = 0$, and $\lim_{\tau\downarrow 0} f^{(2)}(\tau) = \chi^2(\rho\|\sigma)$.

Note that $f(\tau) \leq \tau f(1)$ by the convexity of quantum relative entropy in its first argument, see [44], and Jensen's inequality because $f(0) = 0$. Thus $\lim_{\tau\downarrow 0} f(\epsilon) = 0$ by the non-negativity of the quantum relative entropy via (93). On the other hand $\liminf_{\tau\uparrow 1} f(\tau) \geq f(1)$ by the lower-semicontinuity of quantum relative entropy in its first argument, see [45, p.45] and [46, Theorem 4.1]. Thus $\lim_{\tau\uparrow 1} f(\tau) = f(1)$ because $f(\tau) \leq \tau f(1)$. The continuity of right derivative of proper closed convex functions, see [47, p.25] and $f_+(0) = 0$, imply $\lim_{\tau\downarrow 0} f^{(1)}(\tau) = 0$. The continuity of the matrix inversion, product, and the trace implies $\lim_{\tau\downarrow 0} f^{(2)}(\tau) = f^{(2)}(0)$, and hence $\lim_{\tau\downarrow 0} f^{(2)}(\tau) = \chi^2(\rho\|\sigma)$.

F. Proof of Lemma 5

We follow the proof of (97) presented in Appendix E, for the case when $\rho = \sigma_p$ and $\sigma = \sigma_{\mathcal{A}}$, but we will bound the third derivative of f in a slightly different way. First note that

$$\left|\sigma_p - \sigma_{\mathcal{A}}\right| \leq \|p-\overline{p}\| \cdot \sqrt{\sum\nolimits_{x\in\mathscr{X}_{\mathcal{A}}} W(x)^2}$$

by (110) because $\sigma_p - \sigma_{\mathcal{A}} = \sigma_{(p-\overline{p})}$. Then the monotonicity of the map $\mathrm{Tr}\left[(\cdot)^3\right]$ by [43, Theorem 2.10], implies

$$\begin{aligned}\mathrm{Tr}\left[\left(\tfrac{\sigma_p-\sigma_{\mathcal{A}}}{\rho(\tau)+s\mathbf{I}}\right)^3\right] &\leq \mathrm{Tr}\left[\left(\tfrac{|\sigma_p-\sigma_{\mathcal{A}}|}{\rho(\tau)+s\mathbf{I}}\right)^3\right] \\ &\leq \|p-\overline{p}\|^3\,\mathrm{Tr}\left[\left(\tfrac{\sqrt{\sum_{x\in\mathscr{X}_{\mathcal{A}}} W(x)^2}}{\rho(\tau)+s\mathbf{I}}\right)^3\right].\end{aligned}$$

Then following the analysis in Appendix E to bound the trace term on the right hand side, we get

$$|\Delta_3(\epsilon,\tau)| \leq \|p-\overline{p}\|^3 \cdot \kappa_{\mathcal{A}} \tfrac{(\tau-\epsilon)^3}{(1-\epsilon)^2}$$

for $\kappa_{\mathcal{A}}$ defined in (112). Then we apply the Taylor's theorem with a remainder term, i.e., (126), at ϵ for $f(\tau)$ for an $\epsilon \in (0,1)$ and a $\tau \in (\epsilon, 1)$; and calculate limiting values first as $\tau \uparrow 1$ and then as $\epsilon \downarrow 0$, as we did in Appendix E. Then the bound in (111) follows from (108).

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