Augustin Information Measures on Fading Channels Under Certain Symmetry Hypothesis

Mücahit Furkan Yıldız
Department of Electrical and Electronics Engineering
Middle East Technical University
Ankara, Turkey
0000-0002-5414-5585

Barış Nakiboğlu
Department of Electrical and Electronics Engineering
Middle East Technical University
Ankara, Turkey
0000-0001-7737-5423

***Abstract*—On the fast-fading discrete memoryless channels (DMCs) with channel state information at the receiver a necessary and sufficient condition is determined for the tilted channel associated with an input distribution to be the product of a tilted channel state distribution and the tilted channel associated with the same input distribution on the non-fading DMC for the given channel state. On fading channels for which constituent channels associated with different channel states share a common Augustin capacity-achieving input distribution aforementioned necessary and sufficient condition is shown to hold, and a parametric form for the sphere packing exponent (SPE) is obtained in terms of the tilted channel state distribution and the tilted channel of the non-fading DMC for the given channel state. The SPE of fast-fading binary erasure and binary symmetric channels with channel state information at the receiver are analyzed as examples.**

I. Introduction

The ever-increasing number of users that need to be served and the challenges emerging from high mobility requirements are among the most pressing technical issues in contemporary IoT applications. Nevertheless, high reliability, low latency, and high transmission rate requirements still exist, and the resulting constraints play crucial roles in the applications of emerging technologies. Normal approximation-based approaches, [1]–[3], are used when the high transmission rate requirement is more dominant. As the low-latency and high-reliability requirements become dominant, the error exponent analysis [4]–[7] and its refinements [8]–[14], i.e., large deviation analysis-based methods, become more appropriate tools. Augustin's information measures [15], [16] provide a framework for both the error exponent analysis [17] and its refinements [12], [14].

Many IoT systems use wireless channels that are subject to rapid changes in the channel characteristics, which can, in principle, be tracked. These effects are often modeled with a channel state, and the resulting channel models are called fast-fading channels. If neither the transmitter nor the receiver knows the channel state, the resulting channel is called a non-coherent fast-fading channel. Alternatively, one may assume the channel state to be known at the receiver but not at the transmitter; such channels are called fast-fading channels with the channel state information at the receiver. We confine our discussion to the latter class of channels in the following.

Our ultimate aim is to investigate the effect of fading on the achievable performances for the channel coding schemes. To that end, we obtain a parametric form for the sphere packing exponent (SPE) —see (37), (38), (39)— which is valid for fading channels with certain symmetries. We make the following qualitative observation[1] for the binary symmetric channels: the SPE of a fast-fading binary symmetric channel is always larger than the SPE of the binary symmetric channel with the same channel capacity, at all rates less than the channel capacity, see Figure 2. Our primary technical result that leads to results above about SPE is Lemma 1. Lemma 1 provides a necessary and sufficient condition for the decomposition of the tilted channel associated with an input distribution on a fading channel as the product of a tilted fading distribution and the tilted channel associated with the same input distribution on the non-fading DMC for the given channel state.

The rest of the manuscript is organized as follows. In §II, we review Augustin information measures and the characterization of the SPE and the random coding exponent in terms of Augustin information measures. In §III, we formally describe the fast fading channels and then state and prove our main technical result, Lemma 1. In §IV, we derive a parametric form for the SPE for fading channels whose constituent channels associated with different channel states share a common Augustin capacity-achieving input distribution using Lemma 1. Then we obtain corresponding closed-form expressions for fast-fading binary erasure and symmetric channels, using this parametric form. In §V, we discuss extensions of our result to more general channel models.

II. Preliminaries

We use the word alphabet for finite sets and denote the set of all probability mass functions on an alphabet $\mathcal{Y}$ by $\mathcal{P}(\mathcal{Y})$. A $w \in \mathcal{P}(\mathcal{Y})$ is said to be absolutely continuous in a $q \in \mathcal{P}(\mathcal{Y})$, i.e., $w \prec q$, if $w(y) = 0$ for all y satisfying $q(y) = 0$. A w and q are said to be equivalent, i.e., $w \sim q$, if both $w \prec q$ and

This work is supported by the Science Academy, Turkey, under The Science Academy's Young Scientist Award Program (BAGEP), and by the Scientific and Technological Research Council of Turkey (TÜBİTAK) under Grant 119E053.

[1]This qualitative observation can be seen as a particular case of a more general result on the extremality of the binary symmetric channel in terms of the error exponent functions among all binary input symmetric channels that was reported in [18]–[20].

978-1-6654-2159-1/22/$31.00 ©2022 IEEE

$q \prec w$ hold. For any positive real number α and $w, q \in \mathcal{P}(\mathcal{Y})$, the order α Renyi divergence between w and q is defined as

$$D_{\alpha}(w\| q) := \begin{cases} \frac{1}{\alpha-1} \ln \sum_{y} [w(y)]^{\alpha} [q(y)]^{1-\alpha} & \text{if } \alpha \in \mathbb{R}_{+} \setminus \{1\} \\ \sum_{y} w(y) \ln \frac{w(y)}{q(y)} & \text{if } \alpha = 1 \end{cases}.$$

For any $W : \mathcal{X} \to \mathcal{P}(\mathcal{Y})$, $Q : \mathcal{X} \to \mathcal{P}(\mathcal{Y})$, and $p \in \mathcal{P}(\mathcal{X})$, the order α conditional Renyi divergence is defined as

$$D_{\alpha}(W\| Q| p) := \sum_{x} p(x) D_{\alpha}(W(x)\| Q(x)).$$

We use $D_{\alpha}(W\| q| p)$ instead of $D_{\alpha}(W\| Q| p)$, whenever $Q(x) = q$ for all x with a positive $p(x)$.

For a given $W : \mathcal{X} \to \mathcal{P}(\mathcal{Y})$ and $x \in \mathcal{X}$, each $W(x)$ is an element of $\mathcal{P}(\mathcal{Y})$. We denote the value of $W(x)$ for a given $y \in \mathcal{Y}$ by $W(y|x)$.

A. Augustin Information Measures

For any order $\alpha \in \mathbb{R}_{+}$, channel $W : \mathcal{X} \to \mathcal{P}(\mathcal{Y})$ and input distribution $p \in \mathcal{P}(\mathcal{X})$, the α Augustin information is defined, see [15], [16], [21], as

$$I_{\alpha}(p; W) := \inf_{q \in \mathcal{P}(\mathcal{Y})} D_{\alpha}(W\| q| p). \tag{1}$$

By [16, Lemma 13], there exists a unique $q_{\alpha,p} \in \mathcal{P}(\mathcal{Y})$, called the order α Augustin mean, satisfying

$$I_{\alpha}(p; W) = D_{\alpha}(W\| q_{\alpha,p}| p). \tag{2}$$

The order one Augustin information equals to the mutual information and the order one Augustin mean $q_{1,p}$ is the output distribution q_p for the input distribution p, i.e. $q_{1,p} = q_p$, where

$$q_p(y) := \sum_{x} p(x) W(y|x). \tag{3}$$

For other orders the Augustin mean does not have a closed form expression in general. But it has been characterized in terms of the Augustin operator defined in the following. The Augustin operator $\mathtt{T}_{\alpha,p}(\cdot) : \mathcal{Q}_{\alpha,p} \to \mathcal{P}(\mathcal{Y})$ is defined as

$$\mathtt{T}_{\alpha,p}(q) := \sum_{x} p(x) W_{\alpha}^{q}(x) \qquad \forall q \in \mathcal{Q}_{\alpha,p}, \tag{4}$$

where $\mathcal{Q}_{\alpha,p} := \{q \in \mathcal{P}(\mathcal{Y}) : D_{\alpha}(W\| q| p) < \infty\}$ and the tilted channel $W_{\alpha}^{q} : \mathcal{X} \to \mathcal{P}(\mathcal{Y})$ is defined as

$$W_{\alpha}^{q}(y|x) := [W(y|x)]^{\alpha} [q(y)]^{1-\alpha} e^{(1-\alpha) D_{\alpha}(W(x)\| q)}. \tag{5}$$

The Augustin mean $q_{\alpha,p}$ is a fixed point of the Augustin operator $\mathtt{T}_{\alpha,p}(\cdot)$ that is equivalent to q_p (i.e., $q_{\alpha,p} = \mathtt{T}_{\alpha,p}(q_{\alpha,p})$ and $q_{\alpha,p} \sim q_p$) by [16, Lemma 13]. Furthermore, any fixed point of the Augustin operator in which q_p is absolutely continuous in is equal to the Augustin mean (i.e. if $q_p \prec q$ and $q = \mathtt{T}_{\alpha,p}(q)$, then $q = q_{\alpha,p}$) by [16, Lemma 13], as well.

The order α, Augustin capacity of a $W : \mathcal{X} \to \mathcal{P}(\mathcal{Y})$ is defined as the supremum of the order α Augustin information

$$C_{\alpha,W} := \sup_{p \in \mathcal{P}(\mathcal{X})} I_{\alpha}(p; W). \tag{6}$$

$C_{\alpha,W}$ is finite for any discrete channel W. Thus there exists a unique Augustin center $q_{\alpha,W}$ satisfying

$$C_{\alpha,W} = \inf_{q \in \mathcal{P}(\mathcal{Y})} \max_{x \in \mathcal{X}} D_{\alpha}(W(x)\| q), \tag{7}$$

$$= \max_{x \in \mathcal{X}} D_{\alpha}(W(x)\| q_{\alpha,W}), \tag{8}$$

by [16, Theorem 1]. A necessary and sufficient condition for $I_{\alpha}(p; W) = C_{\alpha,W}$ is $I_{\alpha}(p; W) = \max_{x \in \mathcal{X}} D_{\alpha}(W(x)\| q_{\alpha,p})$ by [16, Theorem 1], as well.

B. Exponent Functions and Augustin Information Measures

The sphere packing exponent (SPE) and the random coding exponent (RCE) functions bound the exponential decay rate of the error probability with the block length from above and from below for all rates below the channel capacity. For rates above the critical rate, SPE and RCE are equal, characterizing the error exponent exactly [4], [5]. SPE and RCE were initially characterized in terms of Gallager's function, but they can also be expressed in terms of the Augustin capacity as follows, see [16], [17],

$$E_{sp}(R, W) := \sup_{\alpha \in (0,1)} \tfrac{1-\alpha}{\alpha} (C_{\alpha,W} - R), \tag{9}$$

$$E_{r}(R, W) := \sup_{\alpha \in (0.5,1)} \tfrac{1-\alpha}{\alpha} (C_{\alpha,W} - R). \tag{10}$$

The primary advantage of Augustin information measures over Gallager's functions (hence Renyi information [22] measures) is that when a cost constraint or a convex composition constraint is imposed on the codes the resulting exponent functions can be obtained by imposing the same constraint in the definition of the Augustin capacity in (6) and using the resulting constrained Augustin capacity in (9) and (10), see [17]. The SPE and the RCE expressions for the constant composition codes, [15], [23]–[26], can be seen as a particular incidence of this more general principle:

$$E_{sp}(R, W, p) := \sup_{\alpha \in (0,1)} \tfrac{1-\alpha}{\alpha} (I_{\alpha}(p; W) - R), \tag{11}$$

$$E_{r}(R, W, p) := \sup_{\alpha \in (0.5,1)} \tfrac{1-\alpha}{\alpha} (I_{\alpha}(p; W) - R). \tag{12}$$

For the constant composition codes, the SPE can be expressed in a parametric form in terms of the tilted channel. For any $R \in (\lim_{\alpha \downarrow 0} I_{\alpha}(p; W), I_1(p; W))$, there exists a unique order $\alpha \in (0,1)$ satisfying

$$E_{sp}(R, W, p) = D_1(W_{\alpha,p}\| W| p), \tag{13}$$

$$R = D_1(W_{\alpha,p}\| q_{\alpha,p}| p), \tag{14}$$

$$= I_1(p; W_{\alpha,p}), \tag{15}$$

where $W_{\alpha,p} = W_{\alpha}^{q_{\alpha,p}}$, see [17, Lemma 2]. This parametric characterization plays a critical role in refining the sphere packing bound, see [12]. Augustin information measures for orders greater than one play similar roles in characterizing the strong converse exponent, [27]–[29].

III. Fading Discrete Memoryless Channels

Let us first recall the discrete memoryless channel (DMC), which is characterized by the following equation:

$$\mathbf{P}[\mathsf{Y}_1^n = y_1^n | \mathsf{X}_1^n = x_1^n] = \prod_{t=1}^{n} W(y_t|x_t), \tag{16}$$

where W is a function of the form $W : \mathcal{X} \to \mathcal{P}(\mathcal{Y})$ and (16) holds for any positive integer n, length n string x_1^n of elements of the input alphabet $\mathcal{X}$, and length n string y_1^n of elements of the output alphabet $\mathcal{Y}$.

In certain cases, the conditional distribution of Y_1^n given X_1^n, can depend on a fading parameter H_1^n, i.e. the channel state, that can be observed at the receiver together with Y_1^n. The distribution of the fading parameter H_1^n is often assumed to be independent of X_1^n and to be independent and identically distributed over time. The resulting channel is called the fast-fading channel with channel state information (CSI) at the receiver. If H takes values from a finite state set $\mathcal{H}$, then resulting channel model is characterized by the following equation:

$$\mathbf{P}[\mathsf{H}_1^n = h_1^n, \mathsf{Y}_1^n = y_1^n | \mathsf{X}_1^n = x_1^n] = \prod\nolimits_{t=1}^{n} g(h_t)\, W(y_t|x_t, h_t), \quad (17)$$

where W is of the form $W : \mathcal{X} \times \mathcal{H} \to \mathcal{P}(\mathcal{Y})$, g is a probability mass function on $\mathcal{H}$, i.e. $g \in \mathcal{P}(\mathcal{H})$, and (17) holds for any positive integer n and length n strings x_1^n, h_1^n, y_1^n.

A fast-fading DMC with CSI at the receiver with input alphabet $\mathcal{X}$, output alphabet $\mathcal{Y}$, state alphabet $\mathcal{H}$ is also a DMC with input alphabet $\mathcal{X}$ and output alphabet $\mathcal{Z} = \mathcal{H} \times \mathcal{Y}$, where the transitions probabilities $V : \mathcal{X} \to \mathcal{P}(\mathcal{Z})$ satisfy[2]

$$V(h, y|x) = g(h)\, W(y|x, h), \qquad \forall x, h, y. \quad (18)$$

Thus as a result of [16, Lemma 13], we know that there exists a unique Augustin mean $s_{\alpha,p} \sim s_p$ satisfying

$$s_{\alpha,p}(z) = \sum\nolimits_{x} p(x)\, V_{\alpha}^{s_{\alpha,p}}(z|x), \quad (19)$$

where $s_p := \sum_x p(x)\, V(x)$ and $V_\alpha^s : \mathcal{X} \to \mathcal{P}(\mathcal{Z})$ is

$$V_\alpha^s(z|x) := [V(z|x)]^\alpha [s(z)]^{1-\alpha} e^{(1-\alpha) D_\alpha(V(x) \| s)}. \quad (20)$$

On the other hand for each $h \in \mathcal{H}$, $W(y|x, h)$ can be interpreted as a channel with the input alphabet $\mathcal{X}$ and output alphabet $\mathcal{Y}$. Thus as a result of [16, Lemma 13] for each h there exists a unique Augustin mean $q_{\alpha,p}(h) \sim q_p(h)$ satisfying

$$q_{\alpha,p}(y|h) = \sum\nolimits_{x} p(x)\, W_\alpha^{q_{\alpha,p}(h)}(y|x, h), \quad (21)$$

where $q_p(h) := \sum_x p(x)\, W(x, h)$ and $W_\alpha^{q(h)}(\cdot, h) : \mathcal{X} \to \mathcal{P}(\mathcal{Y})$ is the tilted channel defined (5) for each $h \in \mathcal{H}$.

Recall that the order one Augustin mean is equal to the output distribution, i.e. $s_{1,p} = s_p$ and $q_{1,p}(h) = q_p(h)$. Thus the following identity holds for all input distributions p

$$s_{1,p}(h, y) = g(h) q_{1,p}(y|h).$$

For Augustin means of other orders, Lemma 1 establishes a similar decomposition, see (26), assuming that an analogous decomposition holds for the tilted channels, see (24). More importantly Lemma 1 provides a necessary and sufficient condition for the decomposition in (24), see (25).

Let the tilted channel $V_{\alpha,p} : \mathcal{X} \to \mathcal{P}(\mathcal{H} \times \mathcal{Y})$ be

$$V_{\alpha,p}(h, y|x) := V_\alpha^{s_{\alpha,p}}(h, y|x), \quad (22)$$

where $s_{\alpha,p} \in \mathcal{P}(\mathcal{H} \times \mathcal{Y})$ is the Augustin mean satisfying (19).

Similarly, let the tilted channel $W_{\alpha,p} : \mathcal{X} \times \mathcal{H} \to \mathcal{P}(\mathcal{Y})$ be

$$W_{\alpha,p}(y|x, h) := W_\alpha^{q_{\alpha,p}(h)}(y|x, h) \quad (23)$$

where $q_{\alpha,p}(h) \in \mathcal{P}(\mathcal{Y})$ is the Augustin mean described in (21).

[2]More generally any fast-fading memoryless channel with CSI at the receiver can be interpreted as a memoryless channel, [17, §V].

Lemma 1. *For a given discrete channel $V : \mathcal{X} \to \mathcal{P}(\mathcal{H} \times \mathcal{Y})$ of the form* (18)*, input distribution $p \in \mathcal{P}(\mathcal{X})$, and order $\alpha \in \mathbb{R}_+$, the following two statements are equivalent*

i) There exist $g_\alpha \in \mathcal{P}(\mathcal{H})$ satisfying

$$V_{\alpha,p}(h, y|x) = g_\alpha(h)\, W_{\alpha,p}(y|x, h) \qquad \forall h, y, \quad (24)$$

and for all x s.t. $p(x) > 0$, for $V_{\alpha,p}$ defined in (22) *and $W_{\alpha,p}$ defined in* (23).

ii) There exist $a_\alpha : \mathcal{X} \to \mathbb{R}$ and $b_\alpha : \mathcal{H} \to \mathbb{R}$ satisfying

$$D_\alpha(W(x, h) \| q_{\alpha,p}(h)) = a_\alpha(x) + b_\alpha(h) \quad \forall h, \quad (25)$$

and for all x s.t. $p(x) > 0$.

Furthermore, if either statement holds then

$$s_{\alpha,p}(h, y) = g_\alpha(h) q_{\alpha,p}(y|h). \quad (26)$$

Proof. If (24) holds then $s_{\alpha,p}$ satisfies (26) as a result of (19) and (21). Furthermore, $\sum_y W_{\alpha,p}(y|x, h) = 1$ implies

$$\sum\nolimits_y V_\alpha^{s_{\alpha,p}}(h, y|x) = g_\alpha(h) \qquad \forall h,$$

and for all x s.t. $p(x) > 0$. Thus (18), (20), and (26) imply

$$D_\alpha(W(x, h) \| q_{\alpha,p}(h)) = \tfrac{\alpha}{\alpha-1} \ln \tfrac{g_\alpha(h)}{g(h)} + D_\alpha(V(x) \| s_{\alpha,p}).$$

Hence (25) holds for the following $a_\alpha : \mathcal{X} \to \mathbb{R}$ and $b_\alpha : \mathcal{H} \to \mathbb{R}$,

$$a_\alpha(x) = D_\alpha(V(x) \| g_\alpha q_{\alpha,p}), \quad (27)$$

$$b_\alpha(h) = \tfrac{\alpha}{\alpha-1} \ln \tfrac{g_\alpha(h)}{g(h)}. \quad (28)$$

Now assume that (25) holds and let $s(h, y)$ be

$$s(h, y) := \tfrac{g(h)}{\gamma} e^{\frac{\alpha-1}{\alpha} b_\alpha(h)} q_{\alpha,p}(y|h),$$

where $\gamma = \sum_h g(h) e^{\frac{\alpha-1}{\alpha} b_\alpha(h)}$. Then (20) and (25) imply

$$V_\alpha^s(h, y|x) = \tfrac{g(h)}{\gamma} e^{\frac{\alpha-1}{\alpha} b_\alpha(h)} W_\alpha^{q_{\alpha,p}(h)}(y|x, h).$$

Consequently s is a fixed point of the Augustin operator by (21) and $s \sim s_p$. Thus $s = s_{\alpha,p}$ by [16, Lemma 13] and (24) holds for the following $g_\alpha \in \mathcal{P}(\mathcal{H})$

$$g_\alpha(h) = \left(\sum\nolimits_{\widetilde{h}} g(\widetilde{h}) e^{\frac{\alpha-1}{\alpha} b_\alpha(\widetilde{h})} \right)^{-1} g(h) e^{\frac{\alpha-1}{\alpha} b_\alpha(h)}. \quad (29)$$

∎

The decomposition in (24) expresses the tilted channel for the fading channel, i.e., $V_{\alpha,p}$, as the product of a tilted fading distribution, i.e., g_α, and the tilted channel for the particular realization of the fading parameter, i.e., $W_{\alpha,p}$. Lemma 1 establishes that when such a decomposition exists for the tilted channel $V_{\alpha,p}$ it is inherited by the Augustin mean $s_{\alpha,p}$, see (26). The decomposition in (24) also implies the following parametric form for the SPE of the constant composition codes on the fading channel via (13), (14), and (15)

$$E_{sp}(R, V, p) = D_1(g_\alpha \| g) + D_1(W_{\alpha,p} \| W | p g_\alpha), \quad (30)$$

$$R = D_1(W_{\alpha,p} \| q_{\alpha,p} | p g_\alpha), \quad (31)$$

$$= I_1(p g_\alpha; W_{\alpha,p}), \quad (32)$$

$$= I_1(p; V_{\alpha,p}). \quad (33)$$

Furthermore, Lemma 1 provides a necessary and sufficient condition for the existence of a decomposition of the form given in (24), in terms of the Augustin means corresponding to particular realizations of the fading parameter, see (25).

IV. Fading Channels with a Common Augustin Capacity-Achieving Input Distributions

The decomposition given (25) holds only for certain input distributions and channels. An important special case is when p is the Agustin capacity-achieving input distribution for the channel $W(\cdot,h):\mathcal{X}\to\mathcal{P}(\mathcal{Y})$ for all values of h with a positive $g(h)$. We will analyze this special case further in this section. With a slight abuse of notation, we denote $C_{\alpha,W(\cdot,h)}$ by $C_{\alpha,h}$. Thus the assumption we invoke in the rest of this section, can be expressed as the existence of a $p\in\mathcal{P}(\mathcal{X})$ satisfying

$$I_\alpha(p;W(\cdot,h))=C_{\alpha,h} \qquad \forall h: g(h)>0. \tag{34}$$

(34) and [16, Theorem 1] imply (25), because they imply

$$D_\alpha(W(x,h)\|\, q_{\alpha,p}(h))=C_{\alpha,h} \tag{35}$$

for all x s.t. $p(x)>0$ and h s.t. $g(h)>0$. Using (29), we get

$$g_\alpha(h)=\left(\sum\nolimits_{\widetilde{h}} g(\widetilde{h})e^{\frac{\alpha-1}{\alpha}C_{\alpha,\widetilde{h}}}\right)^{-1} g(h)e^{\frac{\alpha-1}{\alpha}C_{\alpha,h}}. \tag{36}$$

Note that $I_\alpha(p;V)=D_\alpha(V\|\, s_{\alpha,p}|\, p)$, (26), and (36) imply

$$I_\alpha(p;V)=\tfrac{\alpha}{\alpha-1}\ln\sum\nolimits_{\widetilde{h}} g(\widetilde{h})e^{\frac{\alpha-1}{\alpha}C_{\alpha,\widetilde{h}}}.$$

On the other hand, since $D_\alpha(W(x,h)\|\, q_{\alpha,p}(h))\leq C_{\alpha,h}$ for all x, as a result of (26) and (36) we have

$$D_\alpha(V(x)\|\, s_{\alpha,p})\leq\tfrac{\alpha}{\alpha-1}\ln\sum\nolimits_{\widetilde{h}} g(\widetilde{h})e^{\frac{\alpha-1}{\alpha}C_{\alpha,\widetilde{h}}}.$$

Thus (34) imply not only (25) but also $I_\alpha(p;V)=C_{\alpha,V}$.

If (34) holds for all orders $\alpha\in(0,1)$ on the fast-fading discrete channel V, then $E_{sp}(R,V)=E_{sp}(R,V,p)$ and as a result of (30) and (31), the SPE of the fast-fading discrete channel V is given by

$$E_{sp}(R,V)=D_1(g_\alpha\|\, g)+D_1(W_{\alpha,p}\|\, W|\, p g_\alpha)\,, \tag{37}$$
$$R=D_1(W_{\alpha,p}\|\, q_{\alpha,p}|\, p g_\alpha)\,, \tag{38}$$
$$=I_1(p g_\alpha;W_{\alpha,p}) \tag{39}$$

for g_α given in (36) and p satisfying (34).

Example 1 (Fading Binary Erasure Channel (FBEC))**.** A FBEC is a fading channel satisfying (18), for $\mathcal{X}=\{0,1\}$, $\mathcal{Y}=\{0,1,\mathrm{e}\}$, and $W(y|x,h)$ of the form

$$W(y|x,h)=\begin{cases}1-h & \text{if } y=x,\\ h & \text{if } y=e,\\ 0 & \text{else,}\end{cases} \tag{40}$$

for some finite subset $\mathcal{H}$ of the interval $[0,1]$.

Let p be the uniform distribution on $\mathcal{X}$ and let $q_\alpha(y|h)$ be

$$q_\alpha(y|h)=\begin{cases}\frac{1-h}{2-2h+2^{1/\alpha}h} & \text{if } y\in\{0,1\}\\ \frac{2^{1/\alpha}h}{2-2h+2^{1/\alpha}h} & \text{if } y=e\end{cases}.$$

Since both $\mathtt{T}_{\alpha,p}\left(q_\alpha(h)\right)=q_\alpha(h)$ and $q_p(h)\prec q_\alpha(h)$ hold for each $h\in[0,1]$, $q_\alpha(h)$ is the Augustin mean for p on the channel $W(\cdot,h):\mathcal{X}\to\mathcal{P}(\mathcal{Y})$, i.e. $q_\alpha(h)=q_{\alpha,p}(h)$, and thus

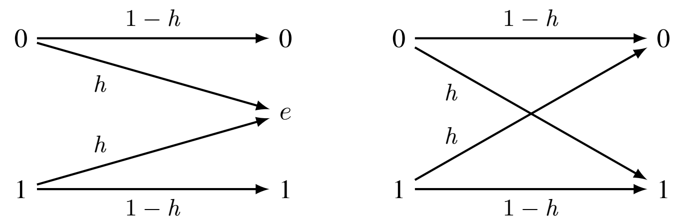

Fig. 1: Fading Binary Erasure and Symmetric Channels

$I_\alpha(p;W(\cdot,h))=D_\alpha(W(\cdot,h)\|\, q_\alpha(h)|\, p)$ for each $h\in[0,1]$ by [16, Lemma 13]. Consequently,

$$W_{\alpha,p}(y|x,h)=\begin{cases}\frac{2-2h}{2-2h+2^{1/\alpha}h} & \text{if } y=x,\\ \frac{2^{1/\alpha}h}{2-2h+2^{1/\alpha}h} & \text{if } y=e,\\ 0 & \text{else,}\end{cases} \tag{41}$$

Furthermore, $I_\alpha(p;W(\cdot,h))=D_\alpha(W(x,h)\|\, q_\alpha(h))$ for all $x\in\mathcal{X}$. Thus $I_\alpha(p;W(\cdot,h))=C_{\alpha,h}$ and (35) hold. Hence,

$$C_{\alpha,h}=\begin{cases}\frac{\alpha}{\alpha-1}\ln\left[h+2^{\frac{\alpha-1}{\alpha}}(1-h)\right] & \text{if } \alpha\neq1,\\ (1-h)\ln 2, & \text{if } \alpha=1,\end{cases} \tag{42}$$

$$g_\alpha(h)=\frac{g(h)}{\bar{h}+2^{\frac{\alpha-1}{\alpha}}(1-\bar{h})}\left[h+2^{\frac{\alpha-1}{\alpha}}(1-h)\right] \tag{43}$$

where $\bar{h}:=\sum_h g(h)h$. Plugging (41) and (43) in (37) and (39) we get

$$E_{sp}(R,V)=\ln\frac{1}{\bar{h}+2^{\frac{\alpha-1}{\alpha}}(1-\bar{h})}+\frac{\alpha-1}{\alpha}\frac{2^{\frac{\alpha-1}{\alpha}}(1-\bar{h})}{\bar{h}+2^{\frac{\alpha-1}{\alpha}}(1-\bar{h})}\ln 2, \tag{44}$$
$$R=\frac{2^{\frac{\alpha-1}{\alpha}}(1-\bar{h})}{\bar{h}+2^{\frac{\alpha-1}{\alpha}}(1-\bar{h})}\ln 2. \tag{45}$$

One can determine the value of α in terms of R and $\bar{h}$ from (45) and plug in the value in (44) to get

$$E_{sp}(R,V)=d_1\left(\tfrac{R}{\ln 2}\middle\|\, 1-\bar{h}\right), \tag{46}$$

where $d_\alpha(\cdot\|\cdot):[0,1]\times[0,1]\to[0,\infty]$ is defined as

$$d_\alpha(\varepsilon\|\,\tau):=\begin{cases}\frac{\ln\left(\varepsilon^\alpha\tau^{1-\alpha}+(1-\varepsilon)^\alpha(1-\tau)^{1-\alpha}\right)}{\alpha-1} & \text{if } \alpha\neq1\\ \varepsilon\ln\frac{\varepsilon}{\tau}+(1-\varepsilon)\ln\frac{1-\varepsilon}{1-\tau} & \text{if } \alpha=1\end{cases}. \tag{47}$$

Example 2 (Fading Binary Symmetric Channel (FBSC))**.** A FBSC is a fading channel satisfying (18), for $\mathcal{X}=\{0,1\}$, $\mathcal{Y}=\{0,1\}$, and $W(y|x,h)$ of the form

$$W(y|x,h)=\begin{cases}1-h & \text{if } y=x,\\ h & \text{if } y\neq x\end{cases} \tag{48}$$

for some finite subset $\mathcal{H}$ of the interval $[0,0.5]$.

Let p be the uniform distribution on $\mathcal{X}$ and $q_\alpha(\cdot|h)$ be the uniform distribution on $\mathcal{Y}$ for all $h\in[0,0.5]$. Then both $\mathtt{T}_{\alpha,p}\left(q_\alpha(h)\right)=q_\alpha(h)$ and $q_p(h)\prec q_\alpha(h)$ hold. Consequently, $q_\alpha(h)=q_{\alpha,p}(h)$ and $I_\alpha(p;W(\cdot,h))=D_\alpha(W(\cdot,h)\|\, q_\alpha(h)|\, p)$ for each $h\in[0,0.5]$, by [16, Lemma 13]. Thus,

$$W_{\alpha,p}(y|x,h)=\begin{cases}\frac{(1-h)^\alpha}{h^\alpha+(1-h)^\alpha} & \text{if } y=x,\\ \frac{h^\alpha}{h^\alpha+(1-h)^\alpha} & \text{if } y\neq x.\end{cases} \tag{49}$$

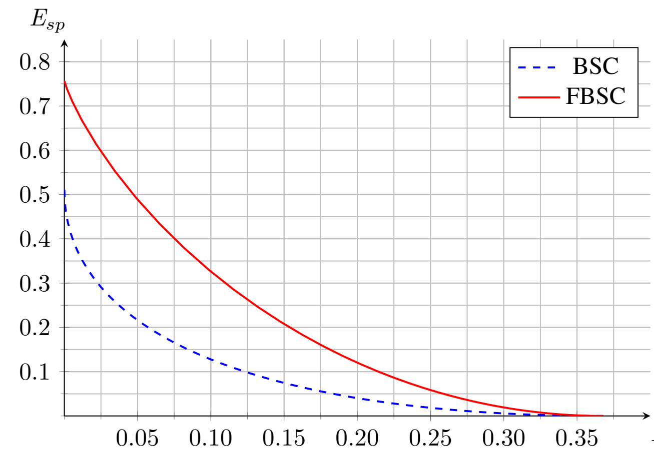

Fig. 2: The SPEs of BSC with crossover probability 0.1 and FBSC with $g(0) = 0.531$ and $g(0.5) = 0.469$.

Furthermore, $I_\alpha(p; W(\cdot, h)) = D_\alpha(W(x, h) \| q_\alpha(h))$ for all $x \in \mathcal{X}$. Thus $I_\alpha(p; W(\cdot, h)) = C_{\alpha,h}$ and (35) hold. Hence,

$$C_{\alpha,h} = d_\alpha(h \| {}^1\!/\!_2) \tag{50}$$

$$g_\alpha(h) = \frac{g(h)(h^\alpha + (1-h)^\alpha)^{\frac{1}{\alpha}}}{\sum_{\widetilde{h}} g(\widetilde{h})(\widetilde{h}^\alpha + (1-\widetilde{h})^\alpha)^{\frac{1}{\alpha}}} \tag{51}$$

Plugging (49) in (37) and (39) we get

$$E_{sp}(R, V) = D_1(g_\alpha \| g) + \sum_h g_\alpha(h) d_1\left(\frac{h^\alpha}{h^\alpha + (1-h)^\alpha} \middle\| h\right), \tag{52}$$

$$R = \sum_h g_\alpha(h) d_1\left(\frac{h^\alpha}{h^\alpha + (1-h)^\alpha} \middle\| \tfrac{1}{2}\right), \tag{53}$$

where g_α is given in (51).

To understand the effect of fading on reliable communication, we might compare the SPE of fading channel with a given channel capacity with a non-fading channel with the same channel capacity.

- For binary erasure channels as a result of (42) and (43) the channel capacity of a FBEC V is equal to the channel capacity of a non-fading BEC W, iff the average erasure probability $\bar{h}$ of V is equal to the erasure probability h of W. Thus as a result of (46), the SPE of V is equal to the SPE of W. Recall that BEC has the largest SPE among all binary input symmetric channels with the same channel capacity by [18]–[20].
- The channel capacity of BSC with crossover probability 0.1 is equal to the channel capacity of FBSC with $g(0) = 0.531$ and $g(0.5) = 0.469$. The SPEs of these channels are plotted in Figure 2, revealing that for all rates less than the channel capacity, the SPE of the fading BSC is larger than the SPE of the non-fading BSC with the same channel capacity. Qualitatively, this is expected because BSC has the smallest SPE among all binary input symmetric channels with the same channel capacity by [18]–[20].

V. Discussion

Lemma 1, establishing the necessary and sufficient conditions for the decomposition of the tilted channel is stated and proved for fast-fading DMCs to keep the presentation and the analysis brief and simple. However, the results reviewed in §II have already been generalized to more abstract channel models in [30]. Using the results of [30], one can generalize not only Lemma 1, but also the resulting parametric characterization of SPE given in (37), (38), (39) to more abstract channel models.

References

[1] Volker Strassen. Asymptotische abschätzungen in Shannons Informationstheorie. In *Trans. Third Prague Conf. Inf. Theory*, pages 689–723, 1962. (https://pi.math.cornell.edu/~pmlut/strassen.pdf).

[2] M. Hayashi. Information spectrum approach to second-order coding rate in channel coding. *IEEE Transactions on Information Theory*, 55(11):4947–4966, Nov 2009.

[3] Y. Polyanskiy, H. V. Poor, and S. Verdú. Channel coding rate in the finite blocklength regime. *IEEE Transactions on Information Theory*, 56(5):2307–2359, May 2010.

[4] R. G. Gallager. A simple derivation of the coding theorem and some applications. *IEEE Transactions on Information Theory*, 11(1):3–18, Jan. 1965.

[5] C. E. Shannon, R. G. Gallager, and E. R. Berlekamp. Lower bounds to error probability for coding on discrete memoryless channels. I. *Information and Control*, 10(1):65–103, 1967.

[6] C. E. Shannon, R. G. Gallager, and E. R. Berlekamp. Lower bounds to error probability for coding on discrete memoryless channels. II. *Information and Control*, 10(5):522–552, 1967.

[7] U. Augustin. Error estimates for low rate codes. *Zeitschrift für Wahrscheinlichkeitstheorie und Verwandte Gebiete*, 14(1):61–88, 1969.

[8] C. E. Shannon. Probability of error for optimal codes in a Gaussian channel. *The Bell System Technical Journal*, 38(3):611–656, May 1959.

[9] Y. Altuğ and A. B. Wagner. Refinement of the sphere-packing bound: Asymmetric channels. *IEEE Transactions on Information Theory*, 60(3):1592–1614, March 2014.

[10] Y. Altuğ and A. B. Wagner. Refinement of the random coding bound. *IEEE Transactions on Information Theory*, 60(10):6005–6023, Oct 2014.

[11] Y. Altuğ and A. B. Wagner. On exact asymptotics of the error probability in channel coding: Symmetric channels. *IEEE Transactions on Information Theory*, 67(2):844–868, 2021.

[12] B. Nakiboğlu. A simple derivation of the refined sphere packing bound under certain symmetry hypotheses. *Turkish Journal Of Mathematics*, 44(3):919–948, 2020. (arXiv:1904.12780 [cs.IT]).

[13] A. Lancho, J. Östman, G. Durisi, T. Koch, and G. Vazquez-Vilar. Saddlepoint approximations for short-packet wireless communications. *IEEE Transactions on Wireless Communications*, 19(7):4831–4846, 2020. (arXiv:1904.10442[cs.IT]).

[14] G. Vazquez-Vilar. Error probability bounds for gaussian channels under maximal and average power constraints. *IEEE Transactions on Information Theory*, 67(6):3965–3985, 2021. arXiv:1907.03163 [cs.IT].

[15] Udo Augustin. *Noisy Channels*. Habilitation thesis, Universität Erlangen-Nürnberg, 1978. (https://bit.ly/3bsWDgG).

[16] B. Nakiboğlu. The Augustin Capacity and Center. *Problems of Information Transmission*, 55(4):299–342, October 2019. (arXiv:1803.07937 [cs.IT]).

[17] B. Nakiboğlu. The Sphere Packing Bound for Memoryless Channels. *Problems of Information Transmission*, 56(3):201–244, July 2020. (arXiv:1804.06372 [cs.IT]).

[18] E. Arıkan and E. Telatar. BEC and BSC are E_0 extremal. unpublished technical note.

[19] A. Guillen i Fabregas, I. Land, and A. Martinez. Extremes of error exponents. *IEEE Transactions on Information Theory*, 59(4):2201–2207, 2013.

[20] M. Alsan. Extremality for Gallager's reliability function E_0. *IEEE Transactions on Information Theory*, 61(8):4277–4292, 2015.

[21] I. Csiszár. Generalized cutoff rates and Rényi's information measures. *IEEE Transactions on Information Theory*, 41(1):26–34, Jan 1995.

[22] B. Nakiboğlu. The Rényi Capacity and Center. *IEEE Transactions on Information Theory*, 65(2):841–860, Feb 2019. (arXiv:1608.02424 [cs.IT]).

[23] G. Sh. Poltyrev. Random coding bounds for discrete memoryless channels. *Problems of Information Transmission*, 18(1):9–21, 1982.

[24] M. Dalai. Some remarks on classical and classical-quantum sphere packing bounds: Rényi vs. Kullback-Leibler. *Entropy*, 19(7):355, 2017.

[25] M. Dalai and A. Winter. Constant compositions in the sphere packing bound for classical-quantum channels. *IEEE Transactions on Information Theory*, 63(9):5603–5617, Sept 2017.

[26] H.-C. Cheng, M. H. Hsieh, and M. Tomamichel. Quantum sphere-packing bounds with polynomial prefactors. *IEEE Transactions on Information Theory*, 65(5):2872–2898, May 2019. (arXiv:1704.05703 [quant-ph]).

[27] G. Dueck and J. Korner. Reliability function of a discrete memoryless channel at rates above capacity (corresp.). *IEEE Transactions on Information Theory*, 25(1):82–85, Jan 1979.

[28] M. Mosonyi and T. Ogawa. Divergence radii and the strong converse exponent of classical-quantum channel coding with constant compositions. *IEEE Transactions on Information Theory*, 67(3):1668–1698, 2021. arXiv:1811.10599v6 [cs.IT].

[29] H.-C. Cheng and B. Nakiboğlu. Refined strong converse for the constant composition codes. In *2020 IEEE International Symposium on Information Theory (ISIT)*, pages 2149–2154, June 2020. (arXiv:2002.11414 [cs.IT]).

[30] H.-C. Cheng and B. Nakiboğlu. On the existence of the augustin mean. In *2021 IEEE Information Theory Workshop (ITW)*, pages 1–6, 2021.