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Edited by Kamal Shah and Baver Okutmuştur

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Meet the editors

Dr. Kamal Shah is an associate professor in applied mathematics at the Department of Mathematics, University of Malakand, Khyber Pakhtankhwa, Pakistan. He was born on March 1, 1983 in Deolai, a large town in the beautiful valley of Swat in Khyber Pakhtankhwa, Pakistan. He has served at the University of Malakand since 2007. He received his PhD degree in Applied Mathematics from the University of Malakand in 2016. Before joining the University of Malakand he was a lecturer in mathematics at the Government Degree College, Mingora, Swat. In November 2006, he joined Government Degree College, Kabal, Swat, and taught there until 2007. Dr. Kamal Shah has published more than 100 articles related to the qualitative theory of fractional differential equations, numerical solutions, and stability analysis in world-reputed journals. He is also involved in fixed point theory and mathematical modeling of infectious diseases. Recently, he has been working on fuzzy integral, differential, and integro-differential equations and their existence theory, and numerical/approximate solutions of fractional as well as integer orders.

Dr. Baver Okutmuştur is an assistant professor at the Department of Mathematics, Middle East Technical University, Ankara, Turkey. He received his PhD degree in Mathematics from Université Pierre et Marie Curie-UPMC (Paris 6). He has published several articles based on finite volume techniques for hyperbolic conservation laws on manifolds. Moreover, he has studied various relativistic models and their numerical analysis on different space/time geometries.

Contents

Preface

The aim of this book is to present a broad overview of the theory and applications related to functional calculus. The book is based on two main subject areas: matrix calculus and applications of Hilbert spaces.

Functional analysis is the most important branch of mathematics, whose foundation was laid by the great Persian polymath Muhammad ibn Mūsā al-Khwārizmī, also known as Algorithmi, during 973–1048. He named this branch the "Theory of Functions." Later, Newton and Leibnitz enriched this branch by introducing the concept of derivatives and integrals during 1665–1742 and thus gave birth to another name: calculus. This branch of mathematics has been recently divided into several subbranches, including differential calculus, integral calculus, stochastic calculus, etc. In mathematics, a functional calculus is a theory that permits someone to apply mathematical functions to mathematical operators. Now, functional calculus is a branch that connects operator theory, classical calculus, algebra, and functional analysis. In daily life, functionals are increasingly used to model real-world situations, for example if $f: R \to R$ is real valued functional from real to real number system. If we apply f on some function $x \in R$, then $f(x)$ makes no sense but if we write it in equation form, then it makes sense, e.g. $f(x) = x$, which represents a physical process between two quantities such that there is direct proportionality. Similar problems occur daily in our surroundings. Therefore, it is necessary to understand what criteria should be satisfied by concerned functionals and operators used in modeling or in the description of daily life problems. It is functional calculus that guides and provides us with the path to how, when, and where particular functionals and operators may be used. Mostly, integral and differential equations are used when we wish to solve a technique or procedure that converts the mentioned equations into algebraic equations of known and unknown functions and functionals. Keeping these needs in mind, the editor of this book has been motivated to welcome international mathematicians and researchers to contribute various topics that address the areas of functional calculus and its applications in both pure and applied analysis. The editor has incorporated contributions from a diverse group of leading researchers in the field of functional calculus. This book aims to provide an overview of the present knowledge that addresses applications and results related to functional calculus. The main topics covered in this book are determinantal representations of the core inverse and its generalizations, which provides a foundation to solve matrix equations. Furthermore, new series formulae for matrix exponential series have been developed, which are used in solving algebraic equations. Also covered are results on fixed point theory, which is used for mapping the satisfying condition (D_A) in Banach space. Results that address folding on chaotic graph operations and their fundamental groups are also introduced. Such algebraic structures are largely used in biology and chemistry. Elsewhere in the book, a brief review is considered of Hilbert space with its fundamental features and features of reproducing kernels in corresponding spaces. Spectral theory is an important area that is most applicable in quantum mechanics. Therefore, a number of fundamental concepts have been investigated regarding analytical applications and observations of PM10 fluctuations. Optimal control is a very important procedure, which is increasingly used in the study of mathematical models of real-world problems. It is helpful in developing future predictions and control strategies of infectious diseases. Analytic and numerical results of the Euler–Bernoulli beam model with a two-parameter family of boundary conditions are also presented, where Chebyshev polynomial approximation has been used to approximate the solution. In recent times, fractional calculus has attracted great attention. Results on fractional integral inequalities are investigated. By using the principle of functional calculus, numerical analysis for boundary value problems of fractional differential equations are studied in the final chapter.

The theory of Hilbert spaces is the center around which functional analysis has developed. Hilbert spaces have a rich geometric nature as they are endowed with an inner product that permits the concept of orthogonality of vectors. Hilbert space methods are applied to several science and engineering areas such as optimization, variational and control problems, and to problems in approximation theory, nonlinear stability, and bifurcation as well as spectral theory and quantum mechanics. That is why a part of the book is devoted to a brief presentation and applications of Hilbert spaces. For the reader who has no previous experience in the theory of normed spaces with enough background for comprehending the theory of Hilbert spaces, there two chapters based on these topics in the book. An important application of the theory of Hilbert spaces to the reproducing kernels is also analyzed in this part. Spectral theory is an important area which is most applicable in quantum mechanics. In this content, a real-life application of Hilbert space where an investigation of the pollution and air quality in Caribbean region by the help of theoretical Hilbert frame aspect is also provided. Here some observations of PM10 fluctuations are analyzed by scaling and time-frequency properties of PM10 data in Hilbert frame and compared the functioning obtained in Hilbert space. Optimal control is also very important procedure which is increasingly used in study of mathematical models of real world problems. It is helpful in developing future predictions and control strategies of infectious disease. In this issue, analytic and numerical results of the Euler-Bernoulli beam model with a two-parameter family of boundary conditions have been presented where Chebyshev polynomial approximation has been used to approximate the solution.

We hope that this book will be of benefit to mathematicians, computational mathematicians, applied mathematicians, and researchers in the field of pure mathematics as well as in analysis. The book is written basically for those who have some knowledge of classical calculus and mathematical analysis. The authors of each section convey a strong emphasis on theoretical foundations.

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Chapter 5

A Survey on Hilbert Spaces and Reproducing Kernels

Baver Okutmuştur

Abstract

The main purpose of this chapter is to provide a brief review of Hilbert space with its fundamental features and introduce reproducing kernels of the corresponding spaces. We separate our analysis into two parts. In the first part, the basic facts on the inner product spaces including the notion of norms, pre-Hilbert spaces, and finally Hilbert spaces are presented. The second part is devoted to the reproducing kernels and the related Hilbert spaces which is called the reproducing kernel Hilbert spaces (RKHS) in the complex plane. The operations on reproducing kernels with some important theorems on the Bergman kernel for different domains are analyzed in this part.

Keywords: Hilbert spaces, norm spaces, reproducing kernels, reproducing kernel Hilbert spaces (RKHS), operations on reproducing kernels, sesqui-analytic kernels, analytic functions, Bergman kernel

1. Framework

This chapter consists of introductory concept on the Hilbert space theory and reproducing kernels. We start by presenting basic definitions, propositions, and theorems from functional analysis related to Hilbert spaces. The notion of linear space, norm, inner product, and pre-Hilbert spaces are in the first part. The second part is devoted to the fundamental properties of the reproducing kernels and the related Hilbert spaces. The operations with reproducing kernels, inclusion property, Bergman kernel, and further properties with examples of the reproducing kernels are analyzed in the latter section.

2. Introduction to Hilbert spaces

We start by the definition of a vector space and related topics. Let $\mathbb{C}$ be the complex field. The following preliminaries can be considered as fundamental concepts of the Hilbert spaces.

2.1 Vector spaces and inner product spaces

Vector space. A vector space is a linear space that is closed under vector addition and scalar multiplication. More precisely, if we denote our linear space by $\mathcal{H}$ over the field $\mathbb{C}$, then it follows that

i. if $x, y, z \in \mathcal{H}$, then

$$x + y = y + x \in \mathcal{H}, \quad x + (y + z) = (x + y) + z \in \mathcal{H};$$

ii. if k is scalar, then $kx \in \mathcal{H}$.

Inner product. Let $\mathcal{H}$ be a linear space over the complex field $\mathbb{C}$. An *inner product* on $\mathcal{H}$ is a two variable function

$$\langle \cdot, \cdot \rangle : \mathcal{H} \times \mathcal{H} \to \mathbb{C}, \text{ satisfying}$$

i. $\langle f, g \rangle = \overline{\langle g, f \rangle}$ for $f, g \in \mathcal{H}$.

ii. $\langle \alpha f + \beta g, h \rangle = \alpha \langle f, h \rangle + \beta \langle g, h \rangle$ and $\langle f, \alpha g + \beta h \rangle = \overline{\alpha} \langle f, g \rangle + \overline{\beta} \langle f, h \rangle$ for $\alpha, \beta \in \mathbb{C}$ and $f, g, h \in \mathcal{H}$.

iii. $\langle f, f \rangle \geq 0$ for $f \in \mathcal{H}$ and $\langle f, f \rangle = 0 \Leftrightarrow f = 0$.

Pre-Hilbert space. A *pre-Hilbert space* $\mathcal{H}$ is a linear space over the complex field $\mathbb{C}$ with an inner product defined on it.

Norm space or inner product space. A norm on an inner product space $\mathcal{H}$ denoted by $\|\cdot\|$ is defined by

$$\|f\| = \langle f, f \rangle^{1/2} \text{ or } \|f\|_{\mathcal{H}} = \langle f, f \rangle_{\mathcal{H}}^{1/2}$$

where $f \in \mathcal{H}$ and $\langle \cdot, \cdot \rangle = \langle \cdot, \cdot \rangle_{\mathcal{H}}$ denote the inner product on $\mathcal{H}$. The corresponding space is called as the inner product space or the norm space.

Properties of norm. For all $f, g \in \mathcal{H}$, and $\lambda \in \mathbb{C}$, we have

- $\|f\| \geq 0$. (Observe that the equality occurs only if $f = 0$).
- $\|\lambda f\| = |\lambda| \|f\|$.

Schwarz inequality. For all $f, g \in \mathcal{H}$, it follows that

$$|\langle f, g \rangle| \leq \|f\| \|g\|. \tag{1}$$

In case if f and g are linearly dependent, then the inequality becomes equality.

Triangle inequality. For all $f, g \in \mathcal{H}$, it follows that

$$\|f + g\| \leq \|f\| + \|g\|. \tag{2}$$

In case if f and g are linearly dependent, then the inequality becomes equality.

Polarization identity. For all $f, g \in \mathcal{H}$, it follows that

$$\langle f, g \rangle = \frac{1}{4} \left(\|f + g\|^2 - \|f - g\|^2 + i\|f + ig\|^2 - \|f - ig\|^2 \right) \text{ for } f, g \in \mathcal{H}. \tag{3}$$

Parallelogram identity. For all $f, g \in \mathcal{H}$, it follows that

$$\|f + g\|^2 + \|f - g\|^2 = 2\|f\|^2 + 2\|g\|^2. \tag{4}$$

Metric. A metric on a set X is a function d: $X \times X \to \mathbb{R}$ satisfying the properties.

- $d(x, y) \geq 0$ and $d(x, y) = 0$ only if $x = y$;
- $d(x, y) = d(y, x)$;
- $d(x, y) \leq d(x, z) + d(z, y)$;

for all $x, y, z \in X$. Moreover the space (X, d) is the associated metric space. If we rearrange the metric with its properties for the inner product space $\mathcal{H}$, then it follows that for all $f, g, h \in \mathcal{H}$ and for all $\lambda \in \mathbb{C}$, where d satisfies all requirements to be a metric, we have

- $d(f, g) \geq 0$ and equality occurs only if $f = g$.
- $d(f, g) = d(g, f)$.
- $d(f, g) \leq d(f, h) + d(h, g)$.
- $d(f - h, g - h) = d(f, g)$.
- $d(\lambda f, \lambda g) = |\lambda| \cdot d(f, g)$.

Note. The binary function d given in the metric definition above represents the metric topology in $\mathcal{H}$ which is called *strong topology* or *norm topology*. As a result, a sequence $(f)_{n \geq 0}$ in the pre-Hilbert space $\mathcal{H}$ converges strongly to f if the condition

$$\|f_n - f\| \to 0 \text{ whenever } n \to \infty$$

is satisfied.

2.2 Introduction to linear operators

Linear operator. A map L from a linear space to another linear space is called *linear operator* if

$$L(\alpha f + \beta g) = \alpha L f + \beta L g$$

is satisfied for all $\alpha, \beta \in \mathbb{C}$ and for all $f, g \in \mathcal{H}$.

Continuous operator. An operator L is said to be continuous if it is continuous at each point of its domain. Notice that the domain and range spaces must be convenient for appropriate topologies.

Lipschitz constant of a linear operator. If L is a linear operator from $\mathcal{H}$ to $\mathcal{G}$ where $\mathcal{H}$ and $\mathcal{G}$ are pre-Hilbert spaces, then the Lipschitz constant for L is its norm $\|L\|$ and it is defined by

$$\|L\| = \sup\left\{ \|Lf\|_{\mathcal{G}} / \|f\|_{\mathcal{H}} : 0 \neq f \in \mathcal{H} \right\}. \tag{5}$$

Theorem 1. Let L be a linear operator from the pre-Hilbert spaces $\mathcal{H}$ to $\mathcal{G}$. Then the followings are mutually equivalent:

i. L is continuous.

ii. L is bounded, that is,

$$\sup\{\|Lf\|_G : \|f\|_H \leq k\} < \infty$$

for $0 \leq k < \infty$.

iii. L is Lipschitz continuous, that is,

$$\|Lf - Lg\|_G \le \lambda \|f - g\|_{\mathcal{H}},$$

where $0 \le \lambda < \infty$ and $f, g \in \mathcal{H}$.

Some properties of linear operators. Let $B(\mathcal{H}, \mathcal{G})$ be the collection of all continuous linear operators from the pre-Hilbert spaces $\mathcal{H}$ to $\mathcal{G}$. Then

- $B(\mathcal{H}, \mathcal{G})$ is a linear space with respect to the natural addition and scalar multiplication satisfying

$$(\alpha L + \beta M)f = \alpha Lf + \beta Mf,$$

where L and M are linear operators, $f \in \mathcal{H}$ and $\alpha, \beta \in \mathbb{C}$.

- Whenever $\mathcal{H} = \mathcal{G}$, then $B(\mathcal{H}, \mathcal{G})$ is denoted by $B(\mathcal{H})$.

- If $\mathcal{K}$ is another pre-Hilbert space, $L \in B(\mathcal{H}, \mathcal{G})$ and $K \in B(\mathcal{G}, \mathcal{K})$. Then the product

$$(KL)f = K(Lf) \text{ for } f \in \mathcal{H} \in B(\mathcal{H}, \mathcal{K}).$$

In addition,

i. $K(\xi L + \zeta M) = \xi KL + \zeta KM$

ii. $\|\xi L\| = |\xi| \cdot \|L\|$

iii. $\|L + M\| \le \|L\| + \|M\|$ and

iv. $\|KL\| \le \|K\| \|L\|$.

are also satisfied.

2.3 Hilbert spaces and linear operators

Linear form (or linear functional). A linear operator from the pre-Hilbert space $\mathcal{H}$ to the scalar field $\mathbb{C}$ is called a *linear form* (or *linear functional*).

Hilbert spaces. A pre-Hilbert space $\mathcal{H}$ is said to be a *Hilbert space* if it is complete in metric. In other words if f_n is a Cauchy sequence in $\mathcal{H}$, that is, if

$$\|f_n - f_m\| \to 0 \text{ whenever } n, m \to \infty,$$

then there is $f \in \mathcal{H}$ such that

$$\|f_n - f\| \to 0 \text{ whenever } n \to \infty.$$

Note. Every subspace of a pre-Hilbert space is also a pre-Hilbert space with respect to the induced inner product. However, the reverse is not always true. For a subspace of a Hilbert space to be also a Hilbert space, it must be closed.

Completion. The canonical method for which a pre-Hilbert space $\mathcal{H}$ is embedded as a dense subspace of a Hilbert space $\tilde{\mathcal{H}}$ so that

$$\langle f, g \rangle_{\tilde{\mathcal{H}}} = \langle f, g \rangle_{\mathcal{H}} \text{ for } f, g \in \mathcal{H}$$

is called *completion*.

Note. If L is a continuous linear operator from a dense subspace $\mathcal{M}$ of a Hilbert space $\mathcal{H}$ to a Hilbert space $\mathcal{G}$, then it can be extended uniquely to a continuous linear operator from $\mathcal{H}$ to $\mathcal{G}$ with preserving norm.

Theorem 2. Let $\mathcal{M}$ and $\mathcal{N}$ be dense subspaces of the Hilbert spaces $\mathcal{H}$ and $\mathcal{G}$, respectively. For $f\in\mathcal{H}, g\in\mathcal{M}$ and $0\le\lambda<\infty$, if a linear operator L from $\mathcal{M}$ to $\mathcal{G}$ satisfies

$$|\langle Lf,g\rangle_{\mathcal{G}}|\le|\lambda|\|f\|_{\mathcal{H}}\|g\|_{\mathcal{G}}, \tag{6}$$

then L is uniquely extended to a continuous linear operator from $\mathcal{M}$ to $\mathcal{G}$ with norm $\le\lambda$ where the norm coincides with the minimum of such λ.

Theorem 3. Let (Ω,μ) denotes a measure space so that Ω is the union of subsets of finite positive measure and $L^2(\Omega,\mu)$ consists of all measurable functions $f(\omega)$ on Ω such that

$$\int_\Omega |f(\omega)|^2 d\mu(\omega)<\infty. \tag{7}$$

Then $L^2(\Omega,\mu)$ is a Hilbert space with respect to the inner product

$$\langle f,g\rangle := \int_\Omega f(\omega)\overline{g(\omega)}d\mu(\omega). \tag{8}$$

Theorem 4 (F. Riesz). For each continuous linear functional φ on a Hilbert space $\mathcal{H}$, there exists uniquely $g\in\mathcal{H}$ such that

$$\varphi(f)=\langle f,g\rangle \, for\, f\in\mathcal{H}. \tag{9}$$

Theorem 5. Let $\mathcal{M}$ be a closed subspace of a Hilbert space $\mathcal{H}$. Then the algebraic direct sum relation

$$\mathcal{H}=\mathcal{M}\oplus\mathcal{M}^\perp$$

is satisfied. In other words, $\forall f\in\mathcal{H}$ can be uniquely written by

$$f=f_{\mathcal{M}}+f_{\mathcal{M}^\perp} \text{ with } f_{\mathcal{M}}\in\mathcal{M},\ f_{\mathcal{M}^\perp}\in\mathcal{M}^\perp. \tag{10}$$

In addition, $\|f_{\mathcal{M}}\|$ coincides with the distance from f to $\mathcal{M}^\perp$

$$\|f_{\mathcal{M}}\|=\min\left\{\|f-g\|:g\in\mathcal{M}^\perp\right\}. \tag{11}$$

Remark. In a Hilbert space, the closed linear span of any subset A of a Hilbert space $\mathcal{H}$ coincides with $\left(A^\perp\right)^\perp$.

Total subset of a Hilbert space. A subset $\mathcal{A}$ of a Hilbert space $\mathcal{H}$ is called *total* in $\mathcal{H}$ if 0 is the only element that is orthogonal to all elements of $\mathcal{A}$. In other words,

$$\mathcal{A}^\perp=\{0\}.$$

As a result, $\mathcal{A}$ is total if and only if every element of $\mathcal{H}$ can be approximated by linear combinations of elements of $\mathcal{A}$.

Orthogonal projection. If $\mathcal{M}$ is a closed subspace of $\mathcal{H}$, the map $f\mapsto f_{\mathcal{M}}$ gives a linear operator from $\mathcal{H}$ to $\mathcal{M}$ with norm ≤ 1. We call this operator as the *orthogonal projection* to $\mathcal{M}$ and denote it by $P_{\mathcal{M}}$.

Note. If I is the identity operator on $\mathcal{H}$, then $I - P_{\mathcal{M}}$ denotes the orthogonal projection to $\mathcal{M}^{\perp}$, and the relation

$$\|f\|^2 = \|P_{\mathcal{M}}f\|^2 + \|(I - P_{\mathcal{M}})f\|^2 \tag{12}$$

is satisfied for all $f \in \mathcal{H}$.

Weak topology. The weakest topology that makes continuous all linear functionals of the form $f \mapsto \langle f, g\rangle$ is called *the weak topology* of a Hilbert space $\mathcal{H}$.

Note. If $f \in \mathcal{H}$, then with respect to the weak topology, a fundamental system of neighborhoods of f is composed of subsets of the form

$$U(f; A, \epsilon) = \{h : |\langle f, g\rangle - \langle h, g\rangle| < \epsilon \text{ for } g \in \mathcal{A}\},$$

where $\mathcal{A}$ is a finite subset of $\mathcal{H}$ and $\epsilon > 0$. Then a directed net $\{f_\lambda\}$ converges weakly to f if and only if

$$\langle f_\lambda, g\rangle \xrightarrow{\lambda} \langle f, g\rangle \text{ for all } g \in \mathcal{H}.$$

Operator weak topology. The weakest topology that makes continuous all linear functionals of the form

$$L \mapsto \langle Lf, g\rangle \text{ for } f \in \mathcal{H}, g \in \mathcal{G}$$

is called *the operator weak topology* in the space $B(\mathcal{H}, \mathcal{G})$ of continuous linear operators from $\mathcal{H}$ to $\mathcal{G}$. In addition, a directed net $\{L_\lambda\}$ converges weakly to L if

$$\langle L_\lambda f, g\rangle \xrightarrow{\lambda} \langle Lf, g\rangle.$$

Operator strong topology. The weakest topology that makes continuous all linear operators of the form

$$L \mapsto Lf \text{ for } f \in \mathcal{H}$$

is called *the operator strong topology*. Moreover a directed net $\{L_\lambda\}$ converges strongly to L if

$$\|L_\lambda f - Lf\| \xrightarrow{\lambda} 0 \text{ for all } f \in \mathcal{H}.$$

Theorem 6. Let $\mathcal{H}$ and $\mathcal{G}$ be Hilbert spaces and $B(\mathcal{H}, \mathcal{G})$ be a continuous linear operator from $\mathcal{H}$ to $\mathcal{G}$. Then

- the closed unit ball $U := \{f : \|f\| \leq 1\}$ of $\mathcal{H}$ is weakly compact;
- the closed unit ball $\{L : \|L\| \leq 1\}$ of $B(\mathcal{H}, \mathcal{G})$ is weakly compact.

Theorem 7. Let $\mathcal{H}$ be a Hilbert space and $A \subseteq \mathcal{H}$. Then if A is weakly bounded in the sense

$$\sup_{f \in A} |\langle f, g\rangle| < \infty \text{ for } g \in \mathcal{H}, \tag{13}$$

then it is strongly bounded, that is, $\sup_{f \in A} \|f\| < \infty$.

Theorem 8. If $\mathcal{H}$ and $\mathcal{G}$ are Hilbert spaces and L is a linear operator from $\mathcal{H}$ to $\mathcal{G}$, then the strong continuity and weak continuity for L are equivalent.

Theorem 9. Let $\mathcal{H}$ and $\mathcal{G}$ be Hilbert spaces. Then the following statements for $\mathbf{L} \subseteq B(\mathcal{H}, \mathcal{G})$ are mutually equivalent:

(i) $\mathbf{L}$ is weakly bounded; that is, for $f \in \mathcal{H},\ g \in \mathcal{G}$, we have

$$\sup_{L \in \mathbf{L}} |\langle Lf, g \rangle| < \infty$$

(ii) $\mathbf{L}$ is strongly bounded; that is, for $f \in \mathcal{H},$ we have

$$\sup_{L \in \mathbf{L}} \|Lf\| < \infty.$$

(iii) $\mathbf{L}$ is norm bounded (or uniformly bounded); that is,

$$\sup_{L \in \mathbf{L}} \|L\| < \infty.$$

Theorem 10. A linear operator L from the Hilbert spaces $\mathcal{H}$ to $\mathcal{G}$ is said to be *closed* if its *graph*

$$G_L := \{ f \oplus Lf : f \in \mathcal{H} \} \tag{14}$$

is a closed subspace of the direct sum space $\mathcal{H} \oplus \mathcal{G}$, that is, whenever $n \to \infty$,

$$\|f_n - f\| \to 0 \ in\ \mathcal{H} \quad and \quad \|Lf_n - g\| \to 0 \quad in \quad \mathcal{G} \Rightarrow g = Lf.$$

Theorem 11. If L is a closed linear operator with a domain of a Hilbert space $\mathcal{H}$ to another Hilbert space $\mathcal{G}$, then it is continuous.

Sesqui-linear form. A function $\Phi : \mathcal{H} \times \mathcal{G} \to \mathbb{C}$ is a *sesqui-linear form* (or *sesqui-linear function*) if for $f, h \in \mathcal{H}, g, k \in \mathcal{G}$ and $\alpha, \beta \in \mathbb{C}$,

$$\text{(i) } \Phi(\alpha f + \beta h, g) = \alpha \Phi(f, g) + \beta \Phi(h, g) \tag{15}$$

$$\text{(ii) } \Phi(f, \alpha g + \beta k) = \overline{\alpha} \Phi(f, g) + \overline{\beta} \Phi(f, k) \tag{16}$$

are satisfied where $\mathcal{H}$ and $\mathcal{G}$ are Hilbert spaces.

Remark. If $L \in B(\mathcal{H}, \mathcal{G})$, then the sesqui-linear form Φ defined by

$$\Phi(f, g) = \langle Lf, g \rangle_G \tag{17}$$

is bounded in the sense that

$$|\Phi(f, g)| \leq \lambda \|f\|_{\mathcal{H}} \|g\|_{\mathcal{G}} \text{ for } f \in \mathcal{H},\ g \in \mathcal{G}, \tag{18}$$

where $\lambda \geq \|L\|$.

Remark. If a sesqui-linear form Φ satisfies the condition (18), then for $f \in \mathcal{H}$, the linear functional

$$g \mapsto \overline{\Phi(f, g)}$$

is continuous on $\mathcal{G}$. If we apply the Riesz theorem, then there exists uniquely $f' \in \mathcal{G}$ satisfying

$$\|f'\|_{\mathcal{G}} \le \lambda \|f\|_{\mathcal{H}} \text{ and } \Phi(f,g) = \langle f',g\rangle_{\mathcal{G}} \text{ for } g \in \mathcal{G}.$$

Hence $f \mapsto f'$ becomes linear, and as a result we obtain

$$\Phi(f,g) = \langle f',g\rangle_{\mathcal{G}} = \langle Lf,g\rangle_{\mathcal{G}}.$$

Adjoint operator. If $L \in B(\mathcal{H},\mathcal{G})$, then the unique operator $L^* \in B(\mathcal{G},\mathcal{H})$ satisfying

$$\Phi(f,g) = \langle f, L^*g\rangle_{\mathcal{H}} \text{ for } f \in \mathcal{H}, g \in \mathcal{G} \tag{19}$$

is called the *adjoint* of L.

Remark. By the definitions of L and L^*, it follows that

$$\langle Lf,g\rangle_{\mathcal{G}} = \langle f, L^*g\rangle_{\mathcal{H}} \text{ for } f \in \mathcal{H},\ g \in \mathcal{G}. \tag{20}$$

Isometric property. The adjoint operation is isometric if

$$\|L\| = \|L^*\| \text{ is satisfied.} \tag{21}$$

Remark. Let $\mathcal{H}, \mathcal{G}$, and $\mathcal{K}$ be Hilbert spaces and $K \in B(\mathcal{G},\mathcal{K})$ and $L \in B(\mathcal{H},\mathcal{G})$ be given. Then

$$KL \in B(\mathcal{H},\mathcal{K}) \text{ and } (KL)^* = L^*K^* \tag{22}$$

$$\text{Ker}(L) = (\text{Ran}(L^*))^{\perp} \text{ and } (\text{Ker}(L))^{\perp} = \text{Clos}\{\text{Ran}(L)^*\} \tag{23}$$

where Ker(L) is the kernel of L and Ran(L) is the range of L.

Theorem 12. If $L, M \in B(\mathcal{H},\mathcal{G})$, then the following statements are mutually equivalent.

i. $\text{Ran}(M) \subseteq \text{Ran}(L)$.

ii. There exists $K \in B(\mathcal{H})$ such that $M = LK$.

iii. There exists $0 \le \lambda < \infty$ such that

$$\|M^*g\| \le \lambda \|L^*g\| \text{ for } g \in \mathcal{G}.$$

Quadric form. Let $\mathcal{H}$ be a Hilbert space. A function

$$\varphi : \mathcal{H} \to \mathbb{C}$$

is a *quadratic form* if for all $f \in \mathcal{H}$ and $\zeta \in \mathbb{C}$,

$$\varphi(\zeta f) = |\zeta|^2 \varphi(f) \tag{24}$$

and

$$\varphi(f+g) + \varphi(f-g) = 2\{\varphi(f) + \varphi(g)\} \tag{25}$$

are satisfied.

Note. If $L \in B(\mathcal{H})$, the quadratic form φ on $\mathcal{H}$ is defined by

$$\varphi(f) = \langle Lf, f\rangle \text{ for } f \in \mathcal{H}, \tag{26}$$

and it is bounded

$$|\varphi(f)| \le \lambda \|f\|^2 \text{ for } f \in \mathcal{H}, \tag{27}$$

where $\lambda \ge \|L\|$.

Remark. The sesqui-linear form Φ associated with L can be recovered from the quadratic form φ by the equation

$$\Phi(f, g) = \frac{1}{4}\{\varphi(f+g) - \varphi(f-g)\} + \{\varphi(f+ig) - \varphi(f-ig)\} \tag{28}$$

for all $f, g \in \mathcal{H}$.

Self-adjoint operator. A continuous linear operator L on a Hilbert space $\mathcal{H}$ is said to be *self-adjoint* if $L = L^*$.

Remark. L is self-adjoint if and only if the associated sesqui-linear form Φ is Hermitian.

Remark. If L is self-adjoint, then the norm of L coincides with the minimum of λ given in (27) for the related quadratic form

Theorem 13. If L is a continuous self-adjoint operator, then

$$\|L\| = \sup\{|\langle Lf, f\rangle| : \|f\| \le 1\}. \tag{29}$$

Positive definite operator. A self-adjoint operator $L \in B(\mathcal{H})$ is said to be *positive* (or *positive definite*) if

$$\langle Lf, f\rangle \ge 0 \quad \text{for all } f \in \mathcal{H}.$$

If $\langle Lf, f\rangle = 0$ only when $f = 0$, then L is said to be *strictly positive* (or, *strictly positive definite*).

Note. For any positive operator $L \in B(\mathcal{H})$, the Schwarz inequality holds in the following sense

$$|\langle Lf, g\rangle|^2 \le \langle Lf, f\rangle \cdot \langle Lg, g\rangle. \tag{30}$$

Theorem 14. Let L and M be continuous positive operators on $\mathcal{H}$ and $\mathcal{G}$, respectively. Then a continuous linear operator K from $\mathcal{H}$ to $\mathcal{G}$ satisfies the inequality

$$\left|\langle Kf, g\rangle_{\mathcal{G}}\right|^2 \le \langle Lf, f\rangle_{\mathcal{H}} \langle Mg, g\rangle_{\mathcal{G}} \text{ for } f \in \mathcal{H}, \in \mathcal{G} \tag{31}$$

if and only if the continuous linear operator

$$\begin{bmatrix} L & K^* \\ K & M \end{bmatrix}$$

on the direct sum Hilbert space $\mathcal{H} \oplus \mathcal{G}$ with

$$f \oplus g \mapsto (Lf + K^* g) \oplus (Kf + Mg)$$

is positive definite.

Theorem 15. Let L be a continuous positive definite operator. Then there exists a unique positive definite operator called the *square root* of L, denoted by $L^{1/2}$, such that $\left(L^{1/2}\right)^2 = L$.

Modulus operator. The square root of the positive definite operator L^*L is called the *modulus* (*operator*) of L if L is a continuous linear operator.

Isometry. A linear operator U between Hilbert spaces $\mathcal{H}$ and $\mathcal{G}$ is called *isometric* or an *isometry* if

$$\|Uf\|_{\mathcal{G}} = \|f\|_{\mathcal{H}} \text{ for } f \in \mathcal{H} \tag{32}$$

is satisfied, that is, it preserves the norm.

Note. Eq. (32) implies that a continuous linear operator U is isometric if and only if $U^*U = I_{\mathcal{H}}$; in other words,

$$\langle Uf, Ug\rangle_{\mathcal{G}} = \langle f, g\rangle_{\mathcal{H}} \text{ for } f, g \in \mathcal{H}, \tag{33}$$

that is, U preserves the inner product.

Unitary operator. A surjective isometry linear operator $U : \mathcal{H} \to \mathcal{H}$ is called a *unitary* (*operator*).

Note. Observe that if $U \in B(H)$ is a unitary operator, then $U^* = U^{-1}$.

Partial isometry. A continuous linear operator U between Hilbert spaces $\mathcal{H}$ and $\mathcal{G}$ is called a *partial isometry* if

$$f \in (\text{Ker}U)^{\perp} = \text{Ran}(U^*) \Rightarrow \|Uf\| = \|f\|.$$

The spaces $(\text{Ker}U)^{\perp}$ and $Ran(U)$ are called the *initial space* of U and the *final space* of U, respectively.

Note. If U is a partial isometry, then its adjoint U^* is also a partial isometry.

Theorem 17. Every continuous linear operator L on $\mathcal{H}$ admits a unique decomposition

$$L = U\tilde{L}, \tag{34}$$

where $\tilde{L}$ is a positive definite operator and U is a partial isometry with initial space the closure of $\text{Ran}(\tilde{L})$.

3. Reproducing kernels and RKHS

We continue our analysis on the abstract theory of reproducing kernels.

3.1 Definition and fundamental properties

Reproducing kernels. Let $\mathcal{H}$ be a Hilbert space of functions on a nonempty set X with the inner product $\langle f, g\rangle$ and norm $\|f\| = \langle f, f\rangle^{1/2}$ for f and $g \in \mathcal{H}$. Then the complex valued function $K(y, x)$ of y and x in X is called a *reproducing kernel of* $\mathcal{H}$ if

i. For all $x \in X$, it follows that $K_x(\cdot) = K(\cdot, x) \in \mathcal{H}$,

ii. For all $x \in X$ and all $f \in \mathcal{H}$,

$$f(x) = \langle f, K_x\rangle, \tag{35}$$

are satisfied.

Note. Let K be a reproducing kernel. Applying (35) to the function K_x at y, we get

$$K_x(y) = K(y, x) = \langle K_y, K_x \rangle, \text{ for } x, y \in X. \tag{36}$$

Then, for any $x \in X$, we obtain

$$\|K_x\| = \langle K_x, K_x \rangle^{1/2} = K(x, x)^{1/2}. \tag{37}$$

Note. Observe that the subset $\{K_x\}_{x \in X}$ is total in $\mathcal{H}$, that is, its closed linear span coincides with $\mathcal{H}$. This follows from the fact that, if $f \in \mathcal{H}$ and $f \perp K_x$ for all $x \in X$, then

$$f(x) = \langle f, K_x \rangle = 0 \text{ for all } x \in X,$$

and hence f is the 0 element in $\mathcal{H}$. As a result, $\{0\}^{\perp} = \mathcal{H}$.

RKHS. A Hilbert space $\mathcal{H}$ of functions on a set X is called a RKHS if there exists a reproducing kernel K of $\mathcal{H}$.

Theorem 18. If a Hilbert space $\mathcal{H}$ of functions on a set X admits a reproducing kernel K, then this reproducing kernel K is unique.

Theorem 19. There exists a reproducing kernel K for $\mathcal{H}$ for a Hilbert space $\mathcal{H}$ of functions on X, if and only if for all $x \in X$, the linear functional $\mathcal{H} \ni f \mapsto f(x)$ of evaluation at x is bounded on $\mathcal{H}$.

Hermitian and positive definite kernel. Let X be an arbitrary set and K be a kernel on X, that is, $K : X \times X \to \mathbb{C}$. The kernel K is called *Hermitian* if for any finite set of points $\{y_1, \dots, y_n\} \subseteq X$, we have

$$\sum_{i,j=1}^{n} \overline{\epsilon}_j \epsilon_i K\left(y_j, y_i\right) \in \mathbb{R}.$$

It is called *positive definite*, if for any complex numbers $\epsilon_1, \dots, \epsilon_n$, we have

$$\sum_{i,j=1}^{n} \overline{\epsilon}_j \epsilon_i K\left(y_j, y_i\right) \geq 0.$$

Note. From the previous inequality, it follows that for any finitely supported family of complex numbers $\{\epsilon_x\}_{x \in X}$, we have

$$\sum_{x,y \in X} \overline{\epsilon}_y \epsilon_x K(y, x) \geq 0. \tag{38}$$

Theorem 20. The reproducing kernel K of a reproducing kernel Hilbert space $\mathcal{H}$ is a positive definite matrix in the sense of E.H. Moore.

Properties of RKHS. Given a reproducing kernel Hilbert space $\mathcal{H}$ and its kernel $K(y, x)$ on X, then for all $x, y \in X$, we have

i. $K(y, y) \geq 0$.

ii. $K(y, x) = \overline{K(x, y)}$.

iii. $|K(y, x)|^2 \leq K(y, y) K(x, x)$ (Schwarz inequality).

iv. Let $x_0 \in X$. Then the following statements are equivalent:

a. $K(x_0, x_0) = 0$.

b. $K(y, x_0) = 0$ for all $y \in X$.

c. $f(x_0) = 0$ for all $f \in \mathcal{H}$.

Theorem 21. For any positive definite kernel K on X, there exists a unique Hilbert space $\mathcal{H}_K$ of functions on X with reproducing kernel K.

Theorem 22. Every sequence of functions $(f_n)_{n\geq 1}$ that converges strongly to a function f in $\mathcal{H}_K(X)$ converges also in the pointwise sense, i.e., for any point $x \in X$,

$$\lim_{n\to\infty} f_n(x) = f(x).$$

In addition, this convergence is uniform on every subset of X on which $x \mapsto K(x,x)$ is bounded.

Theorem 23. A complex valued function g on X belongs to the reproducing kernel Hilbert space $\mathcal{H}_K(X)$ if and only if there exists $0 \leq \lambda < \infty$ such that,

$$\left[g(y)\overline{g(x)}\right] \leq \lambda^2 \left[K(y,x)\right] \text{ on } X. \tag{39}$$

$\|g\|$ coincides with the minimum of all such λ.

Theorem 24. If $K^{(1)}(y,x)$ and $K^{(2)}(y,x)$ are two positive definite kernels on X, then the following statements are mutually equivalent:

i. $\mathcal{H}_{K^{(1)}}(X) \subseteq \mathcal{H}_{K^{(2)}}(X)$.

ii. There exists $0 \leq \lambda < \infty$ such that

$$\left[K^{(1)}(y,x)\right] \leq \lambda^2 \left[K^{(2)}(y,x)\right].$$

Note. For any map φ from a set X to a Hilbert space $\mathcal{H}$, with the notation $x \mapsto \varphi_x$, a kernel K can be defined by

$$K(y,x) = \left\langle \varphi_x, \varphi_y \right\rangle \text{ for } x, y \in X. \tag{40}$$

Theorem 25. Let $\varphi : X \mapsto \mathcal{H}$ be an arbitrary map and for $x, y \in X$ let K be defined as

$$K(y,x) = \left\langle \varphi_x, \varphi_y \right\rangle.$$

Then K is a positive definite kernel.

Theorem 26. Let T be the linear operator from $\mathcal{H}$ to the space of functions on X, defined by

$$(Tf)(x) = \langle f, \varphi_x \rangle \text{ for } x \in X, f \in \mathcal{H}.$$

Then $\operatorname{Ran}(T)$ coincides with $\mathcal{H}_K(X)$ and

$$\|Tf\|_K = \|P_{\mathcal{M}} f\| \text{ for } f \in \mathcal{H},$$

where $\mathcal{M}$ is the orthogonal complement of $\mathrm{Ker}(T)$, $P_{\mathcal{M}}$ is the orthogonal projection onto $\mathcal{M}$, and $\|\cdot\|_K$ denotes the norm in $\mathcal{H}_K(X)$.

Kolmogorov decomposition. Let $K(y,x)$ be a positive definite kernel on an abstract set X. Then there exists a Hilbert space $\mathcal{H}$ and a function $\varphi : X \to \mathcal{H}$ such that

$$K(y,x) = \left\langle \varphi_x, \varphi_y \right\rangle \text{ for } x, y \in X.$$

3.2 Operations with RKHSs

Theorem 27. Let $K^{(0)}$ be the restriction of the positive definite kernel K to a nonempty subset X_0 of X and let $\mathcal{H}_{K^{(0)}}(X)$ and $\mathcal{H}_K(X)$ be the RKHS corresponding to $K^{(0)}$ and K, respectively. Then

$$\mathcal{H}_{K^{(0)}}(X_0) = \{ f\big|_{X_0} : f \in \mathcal{H}_K(X)\} \tag{41}$$

and

$$\|h\|_{K^{(0)}} = \min\left\{ \|f\|_K : f|_{X_0} = h \right\} \text{ for all } h \in \mathcal{H}_{K^{(0)}}(X_0). \tag{42}$$

Remark. If $K^{(1)}(y,x)$ and $K^{(2)}(y,x)$ are two positive definite kernels, then

$$K(y,x) = K^{(1)}(y,x) + K^{(2)}(y,x)$$

is also a positive definite kernel.

Remark. Let $\mathcal{H}_{K^{(1)}}, \mathcal{H}_{K^{(2)}}$, and $\mathcal{H}_K$ be RKHSs with reproducing kernels $K^{(1)}(y,x)$, $K^{(2)}(y,x)$, and $K(y,x)$, respectively, and let $K = K^{(1)} + K^{(2)}$. Then

$$\mathcal{H}_K(X) = \mathcal{H}_{K^{(1)}}(X) + \mathcal{H}_{K^{(2)}}(X),$$

and for $f \in \mathcal{H}_{K^{(1)}}(X)$ and $g \in \mathcal{H}_{K^{(2)}}(X)$, it follows that

$$\|f+g\|_K^2 = \min\left\{ \|f+h\|_{K^{(1)}}^2 + \|g-h\|_{K^{(2)}}^2 : h \in \mathcal{H}_{K^{(1)}}(X) \cap \mathcal{H}_{K^{(2)}}(X) \right\}. \tag{43}$$

Theorem 28. The intersection $\mathcal{H}_{K^{(1)}}(X) \cap \mathcal{H}_{K^{(2)}}(X)$ of Hilbert spaces $\mathcal{H}_{K^{(1)}}(X)$ and $\mathcal{H}_{K^{(2)}}(X)$ is again a Hilbert space of functions on X with respect to the norm

$$\|f\|^2 := \|f\|_{K^{(1)}}^2 + \|f\|_{K^{(2)}}^2.$$

In addition the intersection Hilbert space is a RKHS.

Theorem 29. The reproducing kernel of the space

$$\mathcal{H}_K(X) = \mathcal{H}_{K^{(1)}}(X) \cap \mathcal{H}_{K^{(2)}}(X)$$

is determined, as a quadratic form, by

$$\begin{aligned}\sum_{x,y} \overline{\varepsilon_y} \varepsilon_x K(y,x) = \inf \{ &\sum_{x,y} \overline{\eta_y} \eta_x K^{(1)}(y,x) + \sum_{x,y} \overline{\zeta_y} \zeta_x K^{(2)}(y,x) : [\varepsilon_x] \\ &= [\eta_x] + [\zeta_x] \},\end{aligned}$$

where $[\epsilon_x]$, $[\eta_x]$, $[\zeta_x]$ are an arbitrary complex valued function on X with finite support.

Theorem 30. The tensor product Hilbert space

$$\mathcal{H}_{K^{(1)}}(X) \otimes \mathcal{H}_{K^{(2)}}(X)$$

is a RKHS on $X \times X$.

Theorem 31. The RKHS $\mathcal{H}_K(X)$ of the kernel $K(y,x) = K^{(1)}(y,x) \cdot K^{(2)}(y,x)$ consists of all functions f on X for which there are sequences $(g_n)_{n \geq 0}$ of functions in $\mathcal{H}_{K^{(1)}}(X)$ and $(h_n)_{n \geq 0}$ of functions in $\mathcal{H}_{K^{(2)}}(X)$ so that

$$\sum_1^\infty \|g_n\|^2_{K^{(1)}} \|h_n\|^2_{K^{(2)}} < \infty, \quad \sum_1^\infty g_n(x) h_n(x) = f(x), \ x \in X, \tag{44}$$

and the norm is given by

$$\|f\|_K^2 = \min \left\{ \sum_1^\infty \|g_n\|^2_{K^{(1)}} \|h_n\|^2_{K^{(2)}} \right\},$$

where the minimum is taken over the set of all sequences $(g_n)_{n \geq 0}$ and $(h_n)_{\geq 0}$ satisfying (44).

3.3 Examples of RKHS. Bergman and Hardy spaces

Bergman space. The space of all analytic functions f on Ω for which

$$\int\int_\Omega |f(z)|^2 dxdy < \infty, \qquad (z = x + iy)$$

is satisfied is called the *Bergman space* on Ω and denoted by $A^2(\Omega)$.

Remark. $A^2(\Omega)$ is a *RKHS* with respect to the inner product

$$\langle f, g \rangle \equiv \langle f, g \rangle_\Omega := \int\int_\Omega f(z) \overline{g(z)} dxdy,$$

and its kernel is called the *Bergman kernel* on Ω and denoted by $B^{(\Omega)}(w,z)$.

Bergman kernel for the unit disc. The Bergman kernel for the open unit disc $\mathbb{D}$ is given by

$$B^{(\mathbb{D})}(w,z) = \frac{1}{\pi} \frac{1}{(1 - w\bar{z})^2} \qquad \text{for } w, z \in \mathbb{D}. \tag{45}$$

Bergman kernel of a simply connected domain. The Bergman kernel of a simply connected domain $\Omega (\neq \mathbb{C})$ is given by

$$B^{(\Omega)}(w,z) = \frac{1}{\pi} \frac{\varphi'(w)\overline{\varphi'(z)}}{\left(1 - \varphi(w)\overline{\varphi(z)}\right)^2} \ \text{for } w, z \in \Omega, \tag{46}$$

where φ is any conformal mapping function from Ω onto $\mathbb{D}$.

Theorem 32. A conformal mapping from Ω to $\mathbb{D}$ can be recovered from the Bergman kernel of Ω.

Jordan curve. A *Jordan curve* is a continuous $1 - 1$ image of $\{|\xi| = 1\}$ in $\mathbb{C}$.

Green function. A *Green function* $G(w,z)$ of Ω is a function harmonic in Ω except at z, where it has logarithmic singularity, and continuous in the closure $\overline{\Omega}$, with boundary values $G(w,z) = 0$ for all $w \in \partial\Omega$, where Ω is a finitely connected domain of the complex plane.

Theorem 33. Let Ω be a finitely connected domain bounded by analytic Jordan curves, and let $G(w,z)$ be the Green's function of Ω. Then the Bergman kernel function is

$$B^{(\Omega)}(w,z) = -\frac{2}{\pi}\frac{\partial^2 G}{\partial w \partial \overline{z}}(w,z), \ w \neq z. \tag{47}$$

Hardy space. The closed linear span of $\{\varphi_n : n = 0, 1, \dots\}$ in $L^2((T)$ is called the (Hilbert type) *Hardy space* on $\mathbb{T}$ and is denoted by $H^2(\mathbb{T})$. Here $\varphi_n(\xi) = \xi^n$.

Remark. $f \in L^2(\mathbb{T})$ belongs to the Hardy space $H^2(\mathbb{T})$ if and only if it is orthonormal to all φ_n $(n < 0)$, that is, all Fourier coefficients of f with negative indices vanish. Then we have

$$\langle f, g \rangle_{L^2} = \sum_{n=0}^{\infty} a_n \overline{b}_n \text{ for } f, g \in H^2(\mathbb{T}), \tag{48}$$

where

$$a_n = \langle f, \varphi_n \rangle_{L^2} \text{ and } b_n = \langle g, \varphi_n \rangle_{L^2} \quad (n = 0, 1, \dots).$$

Szegö kernel. The kernel $S(\xi, z) \coloneqq \frac{1}{1-\xi\overline{z}}$ for $\xi \in \mathbb{T}, z \in \mathbb{D}$, or its analytic extension $\tilde{S}(w,z) \coloneqq \frac{1}{1-w\overline{z}}$ for $w, z \in \mathbb{D}$ is called the *Szegö kernel*.

Notes

This chapter intends to offer a sample survey for the fundamental concepts of Hilbert spaces and provide an introductory theory of reproducing kernels. We present the basic properties with important theorems and sometimes with punctual notes and remarks to support the subject. However, due to the limit of content and pages, we skipped the proofs of the theorems. The proofs of the first part can be found in [1, 2] and in most of the basic functional analysis books. Besides, the proofs of the second part (related with the reproducing kernels) can easily be found in [3]. The Hilbert space and functional analysis parts of this chapter are based on the books by J.B. Conway [1] and R.G. Douglas [2]. On the other hand, the reproducing kernel part is based on the lecture notes of T. Ando [4] and N. Aronszajn [5], the book of S. Saitoh and Y. Sawano [6], and the book of B. Okutmustur and A. Gheondea [3]. Moreover, the details of Bergman and Hardy spaces are widely explained in the books [7–9].

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