Ivan Dimov · István Faragó
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Finite Difference Methods

Theory and Applications

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Relativistic Burgers Models on Curved Background Geometries

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Abstract. Relativistic Burgers model and its generalization to various spacetime geometries are recently studied both theoretically and numerically. The numeric implementation is based on finite volume and finite difference approximation techniques designed for the corresponding model on the related geometry. In this work, we provide a summary of several versions of these models on the Schwarzschild, de Sitter, Schwarzschild-de Sitter, FLRW and Reissner-Nordström spacetime geometries with their particular properties.

1 Introduction

The theory of derivation of relativistic type Burgers models on curved spacetimes was started and improved by LeFloch and collaborators [2–5]. The model was firstly derived on a flat spacetime both from the Euler system and from a hyperbolic balance law satisfying the Lorentz invariance property [5]. This analysis has recently been extended to the Schwarzschild, de Sitter (dS), Schwarzschild-de Sitter (SdS), FLRW and Reissner-Nordström (RN) spacetimes and examined numerically by means of finite volume and finite difference approximations. The current work provides a summary of [2–5,10] on different spacetime geometries. We are interested in compressible fluids developing on a curved background. The fluid under consideration may include shock/rarefaction waves and we study a class of weak solutions into the Euler system on the given geometry. The Levi-Civita connection is denoted by covariant derivative ∇. It follows that, the Euler equations for a compressible fluid on a curved spacetime is

$$\nabla_\alpha\big(T^{\alpha\beta}(\rho,u)\big) = \nabla_\alpha\big(\rho c^2 u^\alpha u^\beta + p(\rho)\big(u^\alpha u^\beta + g^{\alpha\beta}\big)\big) = 0, \tag{1}$$

where $T^{\alpha\beta}(\rho,u)$ is the energy-momentum, ρ is the mass-energy density, $u = (u^\alpha)$ is its unit velocity field, $c > 0$ is the light speed, p is the pressure. Further details on energy-momentum tensor and perfect fluids can be found in the articles [1,5,11]. The simplest form of these equations so called the Euler system of compressible fluids on the flat background reads

$$\partial_t\rho + \partial_x(\rho v) = 0, \tag{2}$$
$$\partial_t(\rho v) + \partial_x(\rho v^2 + p(\rho)) = 0. \tag{3}$$

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It can easily be checked that, imposing a vanishing pressure to this system and taking a combination of these equations, one can derive the inviscid Burgers equation We refer [2–5] for this derivation and for further details.

A relativistic generalization of the Burgers model on flat and Schwarzschild spacetimes are proposed in [5] and a numerical scheme was designed by finite volume approximation resulting weak solutions with shock waves. This analysis was extended to SdS spacetime in [2], dS spacetime in [3], FLRW spacetime in [4] and RN spacetime in [10]. The metric elements of each of these geometries describe a solution to the Einstein's field equations. We cite [1,6,8] for convergence and geometric formulation of finite volume methods on Lorentzian spacetimes. More details on the RN metric and its properties can be found in [7,9]. The reader can find general instruction for the general relativity theory and related topics in [11].

The outlook of this work is as follow. Firstly, we give some basic information about spacetime geometry. Then Schwarzschild, dS, SdS, FLRW and RN spacetime metrics and their particular properties are presented, respectively. Relativistic Burgers models for each background are also given in this part. We next describe a general approach for derivation of the model on any background. The final part is dedicated to description of a general geometric finite volume scheme on a curved background and it ends up with some concluding remarks.

2 Spacetime and Metric

In general relativity, different than Newton's theory, space and time are a single continuum as spacetime. By the relativistic point of view, there is no well-defined construct of two distinct events happening at the same time. For this reason, there is a light cone defined at any event that is the location of paths through spacetime. A spacetime is illustrated by an $(n+1)$ dimensional Lorentzian geometry, where n describes dimensions of space and 1 refers dimension of time. The sign of its spherically symmetric metric is denoted by $(-,+,\cdots,+)$. If we restrict the dimension $(n+1)$ to a particular dimension $(3+1)$, then a general spherically symmetric metric dimension will be of the form

$$g = g_{00}\ dt^2 + g_{11}\ dr^2 + g_{22}\ d\theta^2 + g_{33}\ d\phi^2.$$

It follows that the line element for a $(3+1)$ dimensional form in terms of time t, the radial r and angular coordinates θ and φ can be written by the formula

$$g = -A(t,r)\ dt^2 + B(t,r)\ dr^2 + r^2(d\theta^2 + \sin^2\theta\ d\phi^2) \tag{4}$$

with nonzero covariant elements $g_{00} = -A(t,r),\ g_{11} = B(t,r),\ g_{22} = r^2,\ g_{33} = r^2\sin^2\theta$; and the corresponding contravariant elements $g^{00} = -\frac{1}{A(t,r)},\quad g^{11} = -\frac{1}{B(t,r)},\quad g^{22} = \frac{1}{r^2},\quad g^{33} = \frac{1}{r^2(\sin^2\theta)}$. Here $A(t,r)$ and $B(t,r)$ are functions depending on t and r variables. In the following, we introduce some of the well-known spacetime geometries having spherically symmetric metric elements and corresponding relativistic Burgers models on these geometries.

2.1 Minkowski Metric

This spacetime is also called as flat spacetime. We consider a $(3+1)$ dimensional coordinate system given by $(x^0, x^1, x^2, x^3) = (t, x, y, z)$ where $(x^0, x^1, x^2) = (x, y, z)$ and $(x^0) = (t)$ are spatial and time components, respectively. It follows that, the metric of a $(3+1)$ dimensional Minkowski spacetime is

$$g = -c^2 dt^2 + dx^2 + dy^2 + dz^2$$

where c is the light speed. In usual spherical coordinates r, θ, φ, it becomes

$$g = -c^2 dt^2 + dr^2 + r^2 d\theta^2 + r^2 \sin^2 \theta d\varphi^2. \tag{5}$$

Burgers Model on Flat Geometry. The relativistic Burgers equation on flat spacetime can be derived either by Lorentz invariance property or by the Euler system on a curved spacetime. The proofs of both derivation methods can be found in [5]. The relativistic Burgers equation on flat background is

$$\partial_t v + \partial_r \big(1/\epsilon^2 \big(-1 + \sqrt{1 + \epsilon^2 v^2}\big)\big) = 0, \quad \epsilon = 1/c. \tag{6}$$

2.2 Schwarzschild Metric

The Schwarzschild spacetime describes the gravitational field of the universe and defines a spherically symmetric black hole solution to the Einstein's field equations. Its metric is represented by

$$g = -\Big(1 - \frac{2M}{r}\Big) c^2 dt^2 + \Big(1 - \frac{2M}{r}\Big)^{-1} dr^2 + r^2 (d\theta^2 + \sin^2 d\varphi^2) \tag{7}$$

where $M > 0$ is the mass parameter, r is the radial spatial coordinate. The sign of the quantity M and r have significant effects on the geometry.

- If $M = 0$, the Schwarzschild metric reduces to the Minkowski metric.
- If $M \neq 0$, it is singular for $r = 0$ and has a coordinate singularity for $r = 2M$. That is, g_{00} vanishes and g_{11} becomes infinite, so the equation is not valid at that point. For $r < 2M$, $r \neq 0$, the Schwarzschild metric is a regular Lorentzian metric, but the timelike and spacelike behaviors of the coordinates t and r are interchanged, i.e., the Schwarzschild metric in standard coordinates is again a smooth Lorentzian metric, but t is a space coordinate while r is a time coordinate. If $r > 2M > 0$, the metric is a regular Lorentzian metric with t timelike and r spacelike. If $r = 2M$, the Schwarzschild metric with $M > 0$ is no more a smooth Lorentzian metric.

Burgers Model on Schwarzschild Geometry. According to the paper [5], relativistic Burgers model on Schwarzschild background is

$$\partial_t(v) + \Big(1 - \frac{2M}{r}\Big) \partial_r (\frac{v^2}{2}) - \frac{M}{r^2}(v^2 - c^2) = 0. \tag{8}$$

It can easily be observed that, if $M = 0$, we get the inviscid Burgers equation.

2.3 De Sitter Metric

The de Sitter (dS) spacetime is a particular background of the Lorentzian spacetime and its metric describes a cosmological solution to the Einstein's field equations. The metric element contains a cosmological constant Λ. If $\Lambda > 0$, the background geometry is called the de Sitter spacetime; if $\Lambda < 0$, the background geometry is called the Anti-de Sitter spacetime. Particularly, if $\Lambda = 0$, the metric turns to be a Minkowski metric and hence we get a flat geometry. In $(3+1)$ dimension, this metric is

$$g = -(1 - \Lambda r^2)dt^2 + \frac{1}{1 - \Lambda r^2}dr^2 + r^2(d\theta^2 + \sin^2\theta d\varphi^2). \tag{9}$$

It is easy to verify that for $\Lambda = 0$, it becomes the Minkowski metric.

Burgers Model on the dS Geometry. Following the paper [3], the relativistic Burgers equation on a dS background is

$$\partial_t v + (1 - \Lambda r^2)\, \partial_r(\frac{v^2}{2}) + \Lambda r(v - c^2 - 2v^2) = 0. \tag{10}$$

Substituting $\Lambda = 0$ in this equation, we recover the inviscid Burgers equation.

2.4 Schwarzschild-de Sitter (SdS) Metric

The SdS geometry is a spherically symmetric solution to the Einstein's field equations. Its metric is a composition of Schwarzschild and dS metrics. In a $(3+1)$ dimensional spherical coordinates, this metric is given by

$$g = -\Big(1 - \frac{2M}{r} - \frac{\Lambda r^2}{3}\Big)dt^2 + \Big(1 - \frac{2M}{r} - \frac{\Lambda r^2}{3}\Big)^{-1} dr^2 + r^2(d\theta^2 + \sin^2\theta d\varphi^2), \tag{11}$$

where $M > 0$ is the mass parameter, Λ is the cosmological constant, and c is the light speed. It can be observed that whenever $\Lambda = 0$, the SdS metric reduces to the Schwarzschild metric. If the mass parameter $M = 0$, the metric reduces to the dS metric. If both $\Lambda = M = 0$ then it turns to be the Minkowski metric.

Burgers Model on SdS Geometry. According to the article [2], the relativistic Burgers equation on SdS geometry is

$$\partial_t v + (1 - \frac{2M}{r} - \frac{\Lambda r^2}{3})\partial_r(\frac{v^2}{2}) = \frac{Mv^2}{r^2} - \frac{\Lambda r v^2}{3} - \frac{mc^2}{r^2} + \frac{\Lambda r c^2}{3}. \tag{12}$$

Substituting $\Lambda = 0$ in this model yields the Burgers model on the Schwarzschild background. Moreover, if $M = 0$ the model reduces to the model on the dS spacetime. Finally, if $\Lambda = M = 0$, then it gives the inviscid Burgers equation.

2.5 Friedmann–Lemaitre–Robertson–Walker (FLRW) Metric

The FLRW metric is a solution to the Einstein's field equations and is given by

$$g = -c^2 dt^2 + a(t)^2 \left(\frac{dr^2}{1-kr^2} + r^2 d\theta^2 + r^2 \sin^2\theta d\varphi^2 \right) \tag{13}$$

where t is the time, c is the light speed, $k = \{-1, 0, 1\}$ is the curvature, $a(t)$ is the cosmic expansion factor and r, θ, φ are the spherical coordinates.

Burgers Model on a FLRW Geometry. Following the paper [4], the relativistic Burgers equation on a FLRW background is

$$a\, v_t + \left(1-kr^2\right)^{1/2} \partial_r \left(\frac{v^2}{2}\right) + v\left(1 - \frac{v^2}{c^2}\right) a_t = 0. \tag{14}$$

For $k = -1, 0, 1$, we get three different models of interest on FLRW background.

2.6 Reissner-Nordström (RN) Metric

The RN spacetime is a spherically symmetric solution to the Einstein's field equations. The main difference between RN and Schwarzschild blackholes is that RN spacetime is electrically charged with an electrically charge term Q. The corresponding metric is described by

$$g = -\left(1 - \frac{2M}{r} + \frac{Q^2}{r^2}\right) dt^2 + \left(1 - \frac{2M}{r} + \frac{Q^2}{r^2}\right)^{-1} dr^2 + r^2 (d\theta^2 + \sin^2\theta d\varphi^2). \tag{15}$$

RN metric becomes Minkowski metric in polar coordinates at very large radius r. On the other hand, if $Q = 0$, it turns to be a Schwarzschild metric. RN metric is smooth and Lorentzian under the condition that

$$\frac{2M}{r} - \frac{Q^2}{r^2} < 1.$$

Event horizon is located at where $g^{11} = 0$, that is, $1 - \frac{2M}{r} + \frac{Q^2}{r^2} = 0 \Rightarrow r^2 - 2Mr + Q^2 = 0$ with roots $r_\pm = M \pm \sqrt{M^2 - Q^2}$. Depending on the relation between M and Q, we get more information about the event horizon and the geometry.

- If $M^2 < Q^2$, $r_\pm$ are not real and g^{11} is positive except at $r = 0$ where there is a singularity. As $g^{11} > 0$, r is spacelike coordinate. At $r = 0$ we have timelike line. There is no event horizon for this case and this solution is non-physical.
- If $M^2 > Q^2$, there are three regions
  1st region $(r_+ < r < \infty)$: In this region $g^{11} > 0$. Event horizon is the surface defined by $r = r_+$. The singularity at $r = 0$ is timelike line.
  2nd region $(r_- < r < r_+)$: If we set $r = r_+ - \delta$ then with the condition $r_+ > M$, we get $g^{11} = 1 - \frac{2M}{r_+ - \delta} + \frac{Q^2}{(r_+ - \delta)^2} \Rightarrow 2\delta(-Mr_+ + Q^2) < 0$.
  3rd region $(0 < r < r_-)$: If we set $r = r_- - \delta$ then with the condition $r_- < M$, we get $g^{11} = 1 - \frac{2M}{r_- - \delta} + \frac{Q^2}{(r_- - \delta)^2} \Rightarrow 2\delta(-Mr_- + Q^2) > 0$.

– If $M^2 = Q^2$: This case is known as the extreme Reissner-Nordström solution. Event horizon is $r_+ = r_- = M$ and $g^{11} = 0$ at $r = r_\pm$.

We address the reader to the articles [7,9,10] for further detail.

Burgers Model on RN Geometry. According to the paper [10], the derived model is

$$\partial_t(v) + \partial_r\Big(\frac{v^2}{2}(1 - \frac{2M}{r} + \frac{Q^2}{r^2})\Big) = \Big(2v^2 - 1\Big)\Big(\frac{M}{r^2} - \frac{Q^2}{r^3}\Big). \tag{16}$$

Note that if $Q = 0$, it yields the Burgers model on the Schwarzschild geometry. Moreover, if both $M = Q = 0$, then the classical Burgers equation is recovered.

2.7 Derivation of the Burgers Models

A general approach to derive the Burgers model for a given metric element is briefly introduced in this part. We take a metric of the general form (4) and consider the Christoffel symbols given by

$$\Gamma^{\mu}_{\alpha\beta} = \frac{1}{2} g^{\mu\nu}(-\partial_\nu g_{\alpha\beta} + \partial_\beta g_{\alpha\nu} + \partial_\alpha g_{\beta\nu}). \tag{17}$$

We substitute $\alpha, \beta, \mu, \nu \in \{0, 1, 2, 3\}$ in (17) to obtain all the terms of $\Gamma^{\mu}_{\alpha\beta}$. Then by using the unit vector property of u^α, we find a relation between u^0, u^1 and a velocity component v depending on u^0 and u^1. It follows to substitute all these values into the energy momentum tensor for perfect fluids relation given by

$$T^{\alpha\beta} = (\rho c^2 + p)\, u^\alpha\, u^\beta + p\, g^{\alpha\beta}. \tag{18}$$

We then obtain all the terms $T^{\alpha\beta}$ for $\alpha, \beta = 0, 1, 2, 3$. These values are plugged into the Euler system given by

$$\nabla_\alpha T^{\alpha\beta} = \partial_\alpha T^{\alpha\beta} + \Gamma^{\alpha}_{\alpha\gamma} T^{\gamma\beta} + \Gamma^{\beta}_{\alpha\gamma} T^{\alpha\gamma} = 0. \tag{19}$$

As a result of this calculation, we obtain a system of two equations. Next we impose vanishing pressure to this system and take a suitable combination of both equations in order to write it in one equation form. This final equation is the desired Burgers model depending on the given geometry [2–5,10].

3 Finite Volume Method (FVM) Formulation

This part is based on the papers [1–5]. We consider an $(n + 1)$–dimensional spacetime M and a hyperbolic balance law given by

$$div\big(T(v)\big) = S(v), \quad v : M \to \mathbb{R}, \tag{20}$$

where v is a scalar field, $div(\cdot)$ is the divergence operator, $T(v)$ is the flux vector field and $S(v)$ is the scalar field. We establish the FVM by averaging (20) over each element of the constructed triangulation. For convergence of the scheme on curved manifolds and assumptions on triangulation we refer the articles [1,5].

In local coordinates, we suppose that the spacetime is described in coordinates (t, r) and consider equally spaced cells $I_j = [r_{j-1/2}, r_{j+1/2}]$ of size Δr, centred at r_j, with $r_{j+1/2} = r_{j-1/2} + \Delta r$, and $r_{j-1/2} = j\Delta r, \quad r_j = (j + 1/2)\Delta r$. Next we rewrite (20) in $(1 + 1)$ dimension

$$\partial_t T^0(t, r) + \partial_r T^1(t, r) = S(t, r), \tag{21}$$

with T^0, T^1 are flux fields, and S is the source term. Integrating (21) over each grid cell $[t_n, t_{n+1}] \times [r_{j-1/2}, r_{j+1/2}]$ yields

$$\begin{aligned}\int_{r_{j-1/2}}^{r_{j+1/2}} T^0(t_{n+1}, r)\, dr = &\int_{r_{j-1/2}}^{r_{j+1/2}} T^0(t_n, r)\, dr \\ &- \int_{t_n}^{t_{n+1}} (T^1(t, r_{j+1/2}) - T^1(t, r_{j-1/2}))\, dt \\ &+ \int_{[t_n, t_{n+1}]\times[r_{j-1/2}, r_{j+1/2}]} S(t, r)\, dt\, dr.\end{aligned}$$

Approximate these terms by numerical fluxes

$$\widetilde{T}_j^n \approx \frac{1}{\Delta r}\int_{r_{j-1/2}}^{r_{j+1/2}} T^0(t_n, r)\, dr, \quad \widetilde{Q}_{j\pm 1/2}^n \approx \frac{1}{\Delta t}\int_{t_n}^{t_{n+1}} T^1(t, r_{j\pm 1/2})\, dt,$$

$$\widetilde{S}_j^n \approx \frac{1}{\Delta r\, \Delta t}\int_{[t_n, t_{n+1}]\times[r_{j-1/2}, r_{j+1/2}]} S(t, r)\, dt\, dr,$$

the scheme takes the form

$$\widetilde{T}_j^{n+1} = \widetilde{T}_j^n - \frac{\Delta t}{\Delta r}\big(\widetilde{Q}_{j+1/2}^n - \widetilde{Q}_{j-1/2}^n\big) + \Delta t \widetilde{S}_j^n. \tag{22}$$

The numerical implementation of the Burgers models via FVM are analyzed in the papers [2–5,10]. We address the reader to these works for further detail.

3.1 Concluding Remarks

- In [2–4,10], nonlinear Burgers models describing the propagation and interaction of shock waves on flat, Schwarzschild, dS, SdS, FLRW and RN spacetimes are studied and examined. Here we provide a review summary of these works.
- Depending on the geometry and the derived relativistic Burgers model equation, the finite volume/difference schemes are redesigned.
- The schemes are consistent with the conservative form of the Burgers models which results correct computations of weak solutions with shock/rarefaction waves.
- The convergence, efficiency and robustness of these schemes are numerically analyzed for each spacetime geometry of interest.

– One of the most obvious findings emerging from this study is that it allows us to make a comparison of the relativistic Burger models and numerical results on different spacetimes.

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