

Ivan Dimov · István Faragó
Lubin Vulkov (Eds.)

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Contents

Invited Papers

From a Geometrical Interpretation of Bramble-Hilbert Lemma to a Probability Distribution for Finite Element Accuracy.	3
<i>Joel Chaskalovic and Franck Assous</i>	
A High Order Method on Graded Meshes for a Time-Fractional Diffusion Problem.	15
<i>Hu Chen and Martin Stynes</i>	
Numerical Solving a Boundary Value Problem for the Eikonal Equation	28
<i>Alexander G. Churbanov and Petr N. Vabishchevich</i>	
Iterative Semi-implicit Splitting Methods for Stochastic Chemical Kinetics. . .	35
<i>Jürgen Geiser</i>	
The Application of a Special Hermite Finite Element Coupled with Collocation to the Diffusion Equation.	48
<i>Lidiya Gileva, Evgeniya Karepova, and Vladimir Shaydurov</i>	
A 2nd-Order Numerical Scheme for Fractional Ordinary Differential Equation Systems.	60
<i>W. Li and S. Wang</i>	
Blow-Up of Fronts in Burgers Equation with Nonlinear Amplification: Asymptotics and Numerical Diagnostics.	72
<i>Dmitry Lukyanenko and Nikolay Nefedov</i>	
Improved Computer Scheme for a Singularly Perturbed Parabolic Convection–Diffusion Equation.	80
<i>Grigorii Shishkin and Lidia Shishkina</i>	
On Ignorance Regions and Spatial Aspects for South American sea lion (Otaria byronia) Operational Interaction with the Artisan Gillnet Fishery in Chile	92
<i>Milan Stehlik, Jean Paul Maidana, Claudia Navarro Villarroel, Maritza Sepúlveda, and Doris Oliva</i>	
Finite Difference Scheme for Stochastic Differential Games with Several Singular Control Variables and Its Environmental Application	102
<i>Hidekazu Yoshioka and Yuta Yaegashi</i>	

Stability Properties of Repeated Richardson Extrapolation Applied Together with Some Implicit Runge-Kutta Methods. 114
Zahari Zlatev, Ivan Dimov, István Faragó, Krassimir Georgiev, and Ágnes Havasi

Contributed Papers

Computer Modeling of Control the Thermal Imaging System Surveillance . . . 129
Tatiana Akimenko and Ekaterina Filippova

Generalized Multiscale Finite Element Method for Elasticity Problem in Fractured Media 137
V. Alekseev, A. Tyrylgina, and M. Vasilyeva

Automatic Time Step Selection for Numerical Solution of Neutron Diffusion Problems 145
A. V. Avvakumov, V. F. Strizhov, P. N. Vabishchevich, and A. O. Vasilev

Local Convergence Results for an Optimal Iterative Method for Multiple Roots. 153
Ramandeep Behl, Eulalia Martínez, Fabricio Cevallos, and Ali Saleh Alshomrani

Spline Iterative Method for Pantograph Type Functional Differential Equations 159
Alexandru Mihai Bica, Mircea Curila, and Sorin Curila

On Gas Dynamic Hierarchy 167
S. V. Bogomolov, N. B. Esikova, A. E. Kuvshinnikov, and P. N. Smirnov

The Error Analysis of Finite Difference Approximation for a System of Singularly Perturbed Semilinear Reaction-Diffusion Equations with Discontinuous Source Term. 175
S. Chandra Sekhara Rao and Sheetal Chawla

Efficiency and Stability of a Family of Iterative Schemes for Solving Nonlinear Equations 185
Alicia Cordero, Ivan Giménez, and Juan R. Torregrosa

Stability of a Family of Iterative Methods of Fourth-Order 193
Alicia Cordero, Lucía Guasp, and Juan R. Torregrosa

Multidimensional Real Dynamics for High-Order Processes 201
Alicia Cordero, Javier G. Maimó, Juan R. Torregrosa, and Maria P. Vassileva

Bi-parametric Family of Methods with Memory Based
of Ostrowski-Chun Method 208
*Alicia Cordero, Javier G. Maimó, Juan R. Torregrosa,
and Maria P. Vassileva*

Asymptotically Lyapunov-Stable Solutions with Boundary and Internal
Layers in the Stationary Reaction-Diffusion-Advection Problems
with a Small Transfer 216
M. A. Davydova, N. N. Nefedov, and S. A. Zakharova

Numerical Modeling of the Charge Transfer Along 1D Molecular
Chain “Donor-Bridge-Acceptor” at T = 300 K 225
Nadezhda Fialko and Victor D. Lakhno

Simulation of Influence of Special Regimes of Horizontal Flare Systems
on Permafrost 233
M. Yu. Filimonov and N. A. Vaganova

The Method of Fundamental Solutions Combined
with a Multi-level Technique 241
Csaba Gáspár

Generalized Multiscale Discontinuous Galerkin Method
for Helmholtz Problem in Fractured Media. 250
*U. Gavrileva, V. Alekseev, M. Vasilyeva, J. D. De Basabe, Y. Efendiev,
and R. L. Gibson Jr.*

Difference Schemes for the Nonlinear Equations in Partial Derivatives
with Heredity 258
T. V. Gorbova, V. G. Pimenov, and S. I. Solodushkin

Conservative Finite-Difference Algorithm for the Thermodiffusion
Stefan Problem 266
A. Gusev, O. Mazhorova, and O. Shcheritsa

On One Method for Solving of a Non-stationary Fluid Flows
with Free Surface 274
Valentin A. Gushchin and Vasilii G. Kondakov

The Third Order Iterative Method for Solving Nonlinear Parabolic
Equations and Its Application to the Biological Tissues Models 281
I. F. Iumanova and S. I. Solodushkin

Iterative Process for Numerical Recovering the Lowest
Order Space-Wise Coefficient in Parabolic Equations 289
D. Kh. Ivanov and P. N. Vabishchevich

P-Thinned Gamma Process and Corresponding Random Walk	297
<i>Pavlina Jordanova and Milan Stehlik</i>	
Analytic Theory of Singular Perturbations and Lomov's Regularization Method.	305
<i>Vasiliy I. Kachalov</i>	
On Inverse Problem of Determination of the Coefficient in the Black-Scholes Type Equation	313
<i>Vitaly L. Kamynin and Tatiana I. Bukharova</i>	
Finite Difference Schemes on Locally Refined Cartesian Grids for the Solution of Gas Dynamic Problems on the Basis of Quasigasdynamic Equations	321
<i>Yury N. Karamzin, Tatiana A. Kudryashova, Sergey V. Polyakov, and Viktoriia O. Podryga</i>	
Time to Start a Crowded Period in a Finite-Buffer Queue with Poisson Input Flow and General Processing Times.	329
<i>Wojciech M. Kempa</i>	
Two-Grid Newton Algorithms for a System of Heat Conducting Gas Equations	337
<i>Miglena N. Koleva and Lubin G. Vulkov</i>	
On Convergence of Difference Approximations of Extremum Problems Described by Elliptic Equations with Unbounded Nonlinearity	346
<i>Aigul Manapova</i>	
Spijker's Example and Its Extension	354
<i>Miklós E. Mincsovcis</i>	
Numerical Estimation of Seismic Wave Attenuation in Fractured Porous Fluid-Saturated Media.	362
<i>Mikhail Novikov, Vadim Lisitsa, and Tatiana Khachkova</i>	
Relativistic Burgers Models on Curved Background Geometries	370
<i>Baver Okutmustur</i>	
Consistent Discrete-Analytical Schemes for the Solution of the Inverse Source Problems for Atmospheric Chemistry Models with Image-Type Measurement Data	378
<i>Alexey Penenko, Vladimir Penenko, Elena Tsvetova, and Zhadyra Mukatova</i>	
Multiscale Mathematical Modeling of the Metal Nanoparticles Motion in a Gas Flow.	387
<i>Viktoriia Podryga and Sergey Polyakov</i>	

Construction of Higher-Order Approximation Difference Scheme for Nonlinear Convection-Diffusion Equation Using Adaptive Artificial Viscosity in Case of Two-Phase Filtering Problems.	395
<i>I. V. Popov, Yu. A. Poveshchenko, and S. V. Polyakov</i>	
About Free-Volumetric Approximation of a Piezoconductive Medium with Gas Hydrate Inclusions.	403
<i>Yury Poveshchenko, Viktoriia Podryga, and Parvin Rahimly</i>	
Modeling of Fluidodynamic Processes in a Porous Medium with Gashydrate Deposits.	411
<i>Yu. A. Poveshchenko, P. I. Rahimly, I. V. Gasilova, G. I. Kazakevich, Yu. S. Sharova, V. O. Podryga, and S. B. Popov</i>	
Nonlinear Differential Equation of the Surface Section of Gas-Liquid	419
<i>L. A. Prokudina</i>	
Completely Conservative Difference Schemes for Simultaneous Calculations of Thawed Hydrated Zone and Piezoconductive Medium with Gas Hydrate Inclusions.	427
<i>Parvin Rahimly, Yury Poveshchenko, Viktoriia Podryga, and Orkhan Rahimly</i>	
Converging Shock Wave for Ignition of a Pre-compressed Target of Laser Thermonuclear Fusion.	435
<i>Orkhan Rahimly, Nikolai Zmitrenko, and Sergey Guskov</i>	
Monte Carlo Solution of Dirichlet Problem for Semi-linear Equation.	443
<i>Abdujabbor Rasulov, Gulnora Raimova, and Matyokub Bakoev</i>	
Parallel Numerical Method to Estimate the Effective Elastic Moduli of Rock Core Samples from 3D Tomographic Images	452
<i>Galina Reshetova and Tatiana Khachkova</i>	
Integral-Consistent Numerical Technique for Self-gravitating Medium Model.	461
<i>Yu. S. Sharova, Yu. A. Poveshchenko, V. A. Gasilov, N. S. Smirnova, and V. O. Podryga</i>	
Computation of Mean-Field Equilibria with Correlated Stochastic Processes	468
<i>V. Shaydurov, S. Zhang, and V. Kornienko</i>	
Development and Numerical Study of Robust Difference Schemes for a Singularly Perturbed Transport Equation.	476
<i>Lidia Shishkina and Grigorii Shishkin</i>	

A High Order Accurate Overlapping Domain Decomposition Method for Singularly Perturbed Reaction-Diffusion Systems	484
<i>Joginder Singh, Sunil Kumar, and Mukesh Kumar</i>	
Numerical Experiments for Some Markov Models for Solving Boundary Value Problems	493
<i>Alexander S. Sipin and Alexander I. Zeifman</i>	
Numerical Simulation of Deformations of Basalt Roving	501
<i>Anastasia V. Sivtseva and Petr V. Sivtsev</i>	
A Fractional Diffusion Model with Resetting	509
<i>Ercilia Sousa and Amal K. Das</i>	
Generalized Multiscale Finite Element Method for Unsaturated Filtration Problem in Heterogeneous Medium	517
<i>D. Spiridonov and M. Vasilyeva</i>	
A Continuous Model for the Ecological Collapse of Easter Island	525
<i>Bálint Takács</i>	
Solving a Singularly Perturbed Elliptic Problem by a Cascadic Multigrid Algorithm with Richardson Extrapolation.	533
<i>Svetlana Tikhovskaya</i>	
Iterative Algorithms for Coupled Physics Electrical Conductivity Imaging . . .	542
<i>Alex Timonov</i>	
Explicit and Conditionally Stable Combined Numerical Method for 1D and 2D Nonlinear Schrödinger Equation	550
<i>Vyacheslav A. Trofimov and Evgeny M. Trykin</i>	
On a Reliable Numerical Method for a Singularly Perturbed Parabolic Reaction-Diffusion Problem in a Doubly Connected Domain	558
<i>Irina Tselishcheva and Grigorii Shishkin</i>	
Generalized Multiscale Finite Element Method for Poroelasticity Problems in Heterogeneous Media	566
<i>A. Tyrylgina, M. Vasilyeva, and D. Brown</i>	
Past Surface Temperatures Reconstructed by Inversion of the Measured Borehole Temperature-Depth Profiles in Rock	574
<i>Sergey A. Tyufilin, Oleg V. Nagornov, and Tatiana I. Bukharova</i>	
Simulation of Cooling Devices and Effect for Thermal Stabilization of Soil in a Cryolithozone with Anthropogenic Impact.	580
<i>N. A. Vaganova and M. Yu. Filimonov</i>	

The Conjugate Gradient Method for the Dirichlet Problem and Its Modifications 588
V. I. Vasil'ev, A. M. Kardashevsky, and V. V. Popov

On a Digital Version of Pseudo-Differential Operators and Its Applications. 596
Vladimir B. Vasilyev

Upscaled Model for Mixed Dimensional Coupled Flow Problem in Fractured Porous Media Using Non-local Multicontinuum (NLMC) Method. 604
Maria Vasilyeva, Eric T. Chung, Yalchin Efendiev, Wing Tat Leung, and Yating Wang

Some Features of the Asymptotic-Numerical Method for the Moving Fronts Description in Two-Dimensional Reaction-Diffusion Problems 612
Vladimir Volkov and Dmitry Lukyanenko

Conservative Semi-Lagrangian Numerical Algorithm with Decomposition of Integration Domain into Tetrahedrons for Three-Dimensional Advection Problem 621
Alexander Vyatkin and Elena Kuchunova

Numerical Method for Calculating the Pipe Spatial Vibrations 630
Natalia Yaparova

Finite Difference Computation of a Stochastic Aquaculture Problem Under Incomplete Information 638
Hidekazu Yoshioka, Kentaro Tsugihashi, and Yuta Yaegashi

Asymptotic and Numerical Analysis of Dynamics in a Generalized Free-Interfacial Combustion Model 646
Jun Yu, Kewang Chen, and Laura K. Gross

Analogue of Cubic Spline for Functions with Large Gradients in a Boundary Layer 654
Alexander Zadorin and Igor' Blatov

A Family of Optimal Eighth Order Multiple Root Finders with Multivariate Weight Function 663
Fiza Zafar, Alicia Cordero, and Juan Ramon Torregrosa

Multi-step Iterative Algorithm for Mathematical Modeling of Light Bullets in Anisotropic Media 670
Irina G. Zakharova, Aleksey A. Kalinovich, Maria V. Komissarova, and Sergey V. Sazonov

Absolute Stability and Implementation of the Two-Times Repeated Richardson Extrapolation Together with Explicit Runge-Kutta Methods 678
Zahari Zlatev, Ivan Dimov, István Faragó, Krassimir Georgiev, and Ágnes Havasi

Author Index 687



Relativistic Burgers Models on Curved Background Geometries

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Abstract. Relativistic Burgers model and its generalization to various spacetime geometries are recently studied both theoretically and numerically. The numeric implementation is based on finite volume and finite difference approximation techniques designed for the corresponding model on the related geometry. In this work, we provide a summary of several versions of these models on the Schwarzschild, de Sitter, Schwarzschild-de Sitter, FLRW and Reissner-Nordström spacetime geometries with their particular properties.

1 Introduction

The theory of derivation of relativistic type Burgers models on curved spacetimes was started and improved by LeFloch and collaborators [2–5]. The model was firstly derived on a flat spacetime both from the Euler system and from a hyperbolic balance law satisfying the Lorentz invariance property [5]. This analysis has recently been extended to the Schwarzschild, de Sitter (dS), Schwarzschild-de Sitter (SdS), FLRW and Reissner-Nordström (RN) spacetimes and examined numerically by means of finite volume and finite difference approximations. The current work provides a summary of [2–5, 10] on different spacetime geometries. We are interested in compressible fluids developing on a curved background. The fluid under consideration may include shock/rarefaction waves and we study a class of weak solutions into the Euler system on the given geometry. The Levi-Civita connection is denoted by covariant derivative ∇ . It follows that, the Euler equations for a compressible fluid on a curved spacetime is

$$\nabla_{\alpha}(T^{\alpha\beta}(\rho, u)) = \nabla_{\alpha}(\rho c^2 u^{\alpha} u^{\beta} + p(\rho)(u^{\alpha} u^{\beta} + g^{\alpha\beta})) = 0, \quad (1)$$

where $T^{\alpha\beta}(\rho, u)$ is the energy-momentum, ρ is the mass-energy density, $u = (u^{\alpha})$ is its unit velocity field, $c > 0$ is the light speed, p is the pressure. Further details on energy-momentum tensor and perfect fluids can be found in the articles [1, 5, 11]. The simplest form of these equations so called the Euler system of compressible fluids on the flat background reads

$$\partial_t \rho + \partial_x(\rho v) = 0, \quad (2)$$

$$\partial_t(\rho v) + \partial_x(\rho v^2 + p(\rho)) = 0. \quad (3)$$

It can easily be checked that, imposing a vanishing pressure to this system and taking a combination of these equations, one can derive the inviscid Burgers equation. We refer [2–5] for this derivation and for further details.

A relativistic generalization of the Burgers model on flat and Schwarzschild spacetimes are proposed in [5] and a numerical scheme was designed by finite volume approximation resulting weak solutions with shock waves. This analysis was extended to SdS spacetime in [2], dS spacetime in [3], FLRW spacetime in [4] and RN spacetime in [10]. The metric elements of each of these geometries describe a solution to the Einstein’s field equations. We cite [1,6,8] for convergence and geometric formulation of finite volume methods on Lorentzian spacetimes. More details on the RN metric and its properties can be found in [7,9]. The reader can find general instruction for the general relativity theory and related topics in [11].

The outlook of this work is as follow. Firstly, we give some basic information about spacetime geometry. Then Schwarzschild, dS, SdS, FLRW and RN spacetime metrics and their particular properties are presented, respectively. Relativistic Burgers models for each background are also given in this part. We next describe a general approach for derivation of the model on any background. The final part is dedicated to description of a general geometric finite volume scheme on a curved background and it ends up with some concluding remarks.

2 Spacetime and Metric

In general relativity, different than Newton’s theory, space and time are a single continuum as spacetime. By the relativistic point of view, there is no well-defined construct of two distinct events happening at the same time. For this reason, there is a light cone defined at any event that is the location of paths through spacetime. A spacetime is illustrated by an $(n+1)$ dimensional Lorentzian geometry, where n describes dimensions of space and 1 refers dimension of time. The sign of its spherically symmetric metric is denoted by $(-, +, \dots, +)$. If we restrict the dimension $(n+1)$ to a particular dimension $(3+1)$, then a general spherically symmetric metric dimension will be of the form

$$g = g_{00} dt^2 + g_{11} dr^2 + g_{22} d\theta^2 + g_{33} d\phi^2.$$

It follows that the line element for a $(3+1)$ dimensional form in terms of time t , the radial r and angular coordinates θ and φ can be written by the formula

$$g = -A(t, r) dt^2 + B(t, r) dr^2 + r^2(d\theta^2 + \sin^2 \theta d\phi^2) \quad (4)$$

with nonzero covariant elements $g_{00} = -A(t, r)$, $g_{11} = B(t, r)$, $g_{22} = r^2$, $g_{33} = r^2 \sin^2 \theta$; and the corresponding contravariant elements $g^{00} = -\frac{1}{A(t, r)}$, $g^{11} = -\frac{1}{B(t, r)}$, $g^{22} = \frac{1}{r^2}$, $g^{33} = \frac{1}{r^2(\sin^2 \theta)}$. Here $A(t, r)$ and $B(t, r)$ are functions depending on t and r variables. In the following, we introduce some of the well-known spacetime geometries having spherically symmetric metric elements and corresponding relativistic Burgers models on these geometries.

2.1 Minkowski Metric

This spacetime is also called as flat spacetime. We consider a (3+1) dimensional coordinate system given by $(x^0, x^1, x^2, x^3) = (t, x, y, z)$ where $(x^0, x^1, x^2) = (x, y, z)$ and $(x^0) = (t)$ are spatial and time components, respectively. It follows that, the metric of a (3 + 1) dimensional Minkowski spacetime is

$$g = -c^2 dt^2 + dx^2 + dy^2 + dz^2$$

where c is the light speed. In usual spherical coordinates r, θ, φ , it becomes

$$g = -c^2 dt^2 + dr^2 + r^2 d\theta^2 + r^2 \sin^2 \theta d\varphi^2. \tag{5}$$

Burgers Model on Flat Geometry. The relativistic Burgers equation on flat spacetime can be derived either by Lorentz invariance property or by the Euler system on a curved spacetime. The proofs of both derivation methods can be found in [5]. The relativistic Burgers equation on flat background is

$$\partial_t v + \partial_r (1/\epsilon^2 (-1 + \sqrt{1 + \epsilon^2 v^2})) = 0, \quad \epsilon = 1/c. \tag{6}$$

2.2 Schwarzschild Metric

The Schwarzschild spacetime describes the gravitational field of the universe and defines a spherically symmetric black hole solution to the Einstein’s field equations. Its metric is represented by

$$g = -\left(1 - \frac{2M}{r}\right) c^2 dt^2 + \left(1 - \frac{2M}{r}\right)^{-1} dr^2 + r^2(d\theta^2 + \sin^2 d\varphi^2) \tag{7}$$

where $M > 0$ is the mass parameter, r is the radial spatial coordinate. The sign of the quantity M and r have significant effects on the geometry.

- If $M = 0$, the Schwarzschild metric reduces to the Minkowski metric.
- If $M \neq 0$, it is singular for $r = 0$ and has a coordinate singularity for $r = 2M$. That is, g_{00} vanishes and g_{11} becomes infinite, so the equation is not valid at that point. For $r < 2M, r \neq 0$, the Schwarzschild metric is a regular Lorentzian metric, but the timelike and spacelike behaviors of the coordinates t and r are interchanged, i.e., the Schwarzschild metric in standard coordinates is again a smooth Lorentzian metric, but t is a space coordinate while r is a time coordinate. If $r > 2M > 0$, the metric is a regular Lorentzian metric with t timelike and r spacelike. If $r = 2M$, the Schwarzschild metric with $M > 0$ is no more a smooth Lorentzian metric.

Burgers Model on Schwarzschild Geometry. According to the paper [5], relativistic Burgers model on Schwarzschild background is

$$\partial_t(v) + \left(1 - \frac{2M}{r}\right) \partial_r\left(\frac{v^2}{2}\right) - \frac{M}{r^2}(v^2 - c^2) = 0. \tag{8}$$

It can easily be observed that, if $M = 0$, we get the inviscid Burgers equation.

2.3 De Sitter Metric

The de Sitter (dS) spacetime is a particular background of the Lorentzian spacetime and its metric describes a cosmological solution to the Einstein’s field equations. The metric element contains a cosmological constant Λ . If $\Lambda > 0$, the background geometry is called the de Sitter spacetime; if $\Lambda < 0$, the background geometry is called the Anti-de Sitter spacetime. Particularly, if $\Lambda = 0$, the metric turns to be a Minkowski metric and hence we get a flat geometry. In $(3 + 1)$ dimension, this metric is

$$g = -(1 - Ar^2)dt^2 + \frac{1}{1 - Ar^2}dr^2 + r^2(d\theta^2 + \sin^2 \theta d\varphi^2). \tag{9}$$

It is easy to verify that for $\Lambda = 0$, it becomes the Minkowski metric.

Burgers Model on the dS Geometry. Following the paper [3], the relativistic Burgers equation on a dS background is

$$\partial_t v + (1 - Ar^2) \partial_r \left(\frac{v^2}{2}\right) + Ar(v - c^2 - 2v^2) = 0. \tag{10}$$

Substituting $\Lambda = 0$ in this equation, we recover the inviscid Burgers equation.

2.4 Schwarzschild-de Sitter (SdS) Metric

The SdS geometry is a spherically symmetric solution to the Einstein’s field equations. Its metric is a composition of Schwarzschild and dS metrics. In a $(3 + 1)$ dimensional spherical coordinates, this metric is given by

$$g = -\left(1 - \frac{2M}{r} - \frac{Ar^2}{3}\right)dt^2 + \left(1 - \frac{2M}{r} - \frac{Ar^2}{3}\right)^{-1}dr^2 + r^2(d\theta^2 + \sin^2 \theta d\varphi^2), \tag{11}$$

where $M > 0$ is the mass parameter, Λ is the cosmological constant, and c is the light speed. It can be observed that whenever $\Lambda = 0$, the SdS metric reduces to the Schwarzschild metric. If the mass parameter $M = 0$, the metric reduces to the dS metric. If both $\Lambda = M = 0$ then it turns to be the Minkowski metric.

Burgers Model on SdS Geometry. According to the article [2], the relativistic Burgers equation on SdS geometry is

$$\partial_t v + \left(1 - \frac{2M}{r} - \frac{Ar^2}{3}\right) \partial_r \left(\frac{v^2}{2}\right) = \frac{Mv^2}{r^2} - \frac{Arv^2}{3} - \frac{mc^2}{r^2} + \frac{Arc^2}{3}. \tag{12}$$

Substituting $\Lambda = 0$ in this model yields the Burgers model on the Schwarzschild background. Moreover, if $M = 0$ the model reduces to the model on the dS spacetime. Finally, if $\Lambda = M = 0$, then it gives the inviscid Burgers equation.

2.5 Friedmann–Lemaître–Robertson–Walker (FLRW) Metric

The FLRW metric is a solution to the Einstein’s field equations and is given by

$$g = -c^2 dt^2 + a(t)^2 \left(\frac{dr^2}{1 - kr^2} + r^2 d\theta^2 + r^2 \sin^2 \theta d\varphi^2 \right) \tag{13}$$

where t is the time, c is the light speed, $k = \{-1, 0, 1\}$ is the curvature, $a(t)$ is the cosmic expansion factor and r, θ, φ are the spherical coordinates.

Burgers Model on a FLRW Geometry. Following the paper [4], the relativistic Burgers equation on a FLRW background is

$$a v_t + (1 - kr^2)^{1/2} \partial_r \left(\frac{v^2}{2} \right) + v \left(1 - \frac{v^2}{c^2} \right) a_t = 0. \tag{14}$$

For $k = -1, 0, 1$, we get three different models of interest on FLRW background.

2.6 Reissner-Nordström (RN) Metric

The RN spacetime is a spherically symmetric solution to the Einstein’s field equations. The main difference between RN and Schwarzschild blackholes is that RN spacetime is electrically charged with an electrically charge term Q . The corresponding metric is described by

$$g = - \left(1 - \frac{2M}{r} + \frac{Q^2}{r^2} \right) dt^2 + \left(1 - \frac{2M}{r} + \frac{Q^2}{r^2} \right)^{-1} dr^2 + r^2 (d\theta^2 + \sin^2 \theta d\varphi^2). \tag{15}$$

RN metric becomes Minkowski metric in polar coordinates at very large radius r . On the other hand, if $Q = 0$, it turns to be a Schwarzschild metric. RN metric is smooth and Lorentzian under the condition that

$$\frac{2M}{r} - \frac{Q^2}{r^2} < 1.$$

Event horizon is located at where $g^{11} = 0$, that is, $1 - \frac{2M}{r} + \frac{Q^2}{r^2} = 0 \Rightarrow r^2 - 2Mr + Q^2 = 0$ with roots $r_{\pm} = M \pm \sqrt{M^2 - Q^2}$. Depending on the relation between M and Q , we get more information about the event horizon and the geometry.

- If $M^2 < Q^2$, r_{\pm} are not real and g^{11} is positive except at $r = 0$ where there is a singularity. As $g^{11} > 0$, r is spacelike coordinate. At $r = 0$ we have timelike line. There is no event horizon for this case and this solution is non-physical.
- If $M^2 > Q^2$, there are three regions

1st region ($r_+ < r < \infty$): In this region $g^{11} > 0$. Event horizon is the surface defined by $r = r_+$. The singularity at $r = 0$ is timelike line.

2nd region ($r_- < r < r_+$): If we set $r = r_+ - \delta$ then with the condition $r_+ > M$, we get $g^{11} = 1 - \frac{2M}{r_+ - \delta} + \frac{Q^2}{(r_+ - \delta)^2} \Rightarrow 2\delta(-Mr_+ + Q^2) < 0$.

3rd region ($0 < r < r_-$): If we set $r = r_- - \delta$ then with the condition $r_- < M$, we get $g^{11} = 1 - \frac{2M}{r_- - \delta} + \frac{Q^2}{(r_- - \delta)^2} \Rightarrow 2\delta(-Mr_- + Q^2) > 0$.

- If $M^2 = Q^2$: This case is known as the extreme Reissner-Nordström solution. Event horizon is $r_+ = r_- = M$ and $g^{11} = 0$ at $r = r_{\pm}$.

We address the reader to the articles [7, 9, 10] for further detail.

Burgers Model on RN Geometry. According to the paper [10], the derived model is

$$\partial_t(v) + \partial_r\left(\frac{v^2}{2}\left(1 - \frac{2M}{r} + \frac{Q^2}{r^2}\right)\right) = (2v^2 - 1)\left(\frac{M}{r^2} - \frac{Q^2}{r^3}\right). \tag{16}$$

Note that if $Q = 0$, it yields the Burgers model on the Schwarzschild geometry. Moreover, if both $M = Q = 0$, then the classical Burgers equation is recovered.

2.7 Derivation of the Burgers Models

A general approach to derive the Burgers model for a given metric element is briefly introduced in this part. We take a metric of the general form (4) and consider the Christoffel symbols given by

$$\Gamma_{\alpha\beta}^{\mu} = \frac{1}{2}g^{\mu\nu}(-\partial_{\nu}g_{\alpha\beta} + \partial_{\beta}g_{\alpha\nu} + \partial_{\alpha}g_{\beta\nu}). \tag{17}$$

We substitute $\alpha, \beta, \mu, \nu \in \{0, 1, 2, 3\}$ in (17) to obtain all the terms of $\Gamma_{\alpha\beta}^{\mu}$. Then by using the unit vector property of u^{α} , we find a relation between u^0, u^1 and a velocity component v depending on u^0 and u^1 . It follows to substitute all these values into the energy momentum tensor for perfect fluids relation given by

$$T^{\alpha\beta} = (\rho c^2 + p)u^{\alpha}u^{\beta} + pg^{\alpha\beta}. \tag{18}$$

We then obtain all the terms $T^{\alpha\beta}$ for $\alpha, \beta = 0, 1, 2, 3$. These values are plugged into the Euler system given by

$$\nabla_{\alpha}T^{\alpha\beta} = \partial_{\alpha}T^{\alpha\beta} + \Gamma_{\alpha\gamma}^{\alpha}T^{\gamma\beta} + \Gamma_{\alpha\gamma}^{\beta}T^{\alpha\gamma} = 0. \tag{19}$$

As a result of this calculation, we obtain a system of two equations. Next we impose vanishing pressure to this system and take a suitable combination of both equations in order to write it in one equation form. This final equation is the desired Burgers model depending on the given geometry [2–5, 10].

3 Finite Volume Method (FVM) Formulation

This part is based on the papers [1–5]. We consider an $(n + 1)$ -dimensional spacetime M and a hyperbolic balance law given by

$$div(T(v)) = S(v), \quad v : M \rightarrow \mathbb{R}, \tag{20}$$

where v is a scalar field, $div(\cdot)$ is the divergence operator, $T(v)$ is the flux vector field and $S(v)$ is the scalar field. We establish the FVM by averaging (20) over

each element of the constructed triangulation. For convergence of the scheme on curved manifolds and assumptions on triangulation we refer the articles [1,5].

In local coordinates, we suppose that the spacetime is described in coordinates (t, r) and consider equally spaced cells $I_j = [r_{j-1/2}, r_{j+1/2}]$ of size Δr , centred at r_j , with $r_{j+1/2} = r_{j-1/2} + \Delta r$, and $r_{j-1/2} = j\Delta r$, $r_j = (j + 1/2)\Delta r$. Next we rewrite (20) in $(1 + 1)$ dimension

$$\partial_t T^0(t, r) + \partial_r T^1(t, r) = S(t, r), \tag{21}$$

with T^0, T^1 are flux fields, and S is the source term. Integrating (21) over each grid cell $[t_n, t_{n+1}] \times [r_{j-1/2}, r_{j+1/2}]$ yields

$$\begin{aligned} \int_{r_{j-1/2}}^{r_{j+1/2}} T^0(t_{n+1}, r) dr &= \int_{r_{j-1/2}}^{r_{j+1/2}} T^0(t_n, r) dr \\ &\quad - \int_{t_n}^{t_{n+1}} (T^1(t, r_{j+1/2}) - T^1(t, r_{j-1/2})) dt \\ &\quad + \int_{[t_n, t_{n+1}] \times [r_{j-1/2}, r_{j+1/2}]} S(t, r) dt dr. \end{aligned}$$

Approximate these terms by numerical fluxes

$$\begin{aligned} \tilde{T}_j^n &\approx \frac{1}{\Delta r} \int_{r_{j-1/2}}^{r_{j+1/2}} T^0(t_n, r) dr, \quad \tilde{Q}_{j\pm 1/2}^n \approx \frac{1}{\Delta t} \int_{t_n}^{t_{n+1}} T^1(t, r_{j\pm 1/2}) dt, \\ \tilde{S}_j^n &\approx \frac{1}{\Delta r \Delta t} \int_{[t_n, t_{n+1}] \times [r_{j-1/2}, r_{j+1/2}]} S(t, r) dt dr, \end{aligned}$$

the scheme takes the form

$$\tilde{T}_j^{n+1} = \tilde{T}_j^n - \frac{\Delta t}{\Delta r} (\tilde{Q}_{j+1/2}^n - \tilde{Q}_{j-1/2}^n) + \Delta t \tilde{S}_j^n. \tag{22}$$

The numerical implementation of the Burgers models via FVM are analyzed in the papers [2–5, 10]. We address the reader to these works for further detail.

3.1 Concluding Remarks

- In [2–4, 10], nonlinear Burgers models describing the propagation and interaction of shock waves on flat, Schwarzschild, dS, SdS, FLRW and RN spacetimes are studied and examined. Here we provide a review summary of these works.
- Depending on the geometry and the derived relativistic Burgers model equation, the finite volume/difference schemes are redesigned.
- The schemes are consistent with the conservative form of the Burgers models which results correct computations of weak solutions with shock/rarefaction waves.
- The convergence, efficiency and robustness of these schemes are numerically analyzed for each spacetime geometry of interest.

- One of the most obvious findings emerging from this study is that it allows us to make a comparison of the relativistic Burger models and numerical results on different spacetimes.

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