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Sujet :

MÉTHODES DE VOLUMES FINIS POUR LES LOIS DE CONSERVATION HYPERBOLIQUES NON-LINÉAIRES POSÉES SUR UNE VARIÉTÉ

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Introduction

Dans cette thèse, nous étudions plusieurs questions mathématiques concernant les équations hyperboliques non-linéaires. D'une part, nous nous intéressons aux lois de conservation sur des variétés suivant deux approches : la première étant basée sur une métrique et la seconde sur des champs de formes différentielles. D'autre part, nous étudions les estimations d'erreur pour les schémas de volumes finis et la mise en oeuvre d'un modèle de fluides relativistes.

La première partie de cette thèse est consacrée à l'étude de la méthode de volumes finis pour les lois de conservation hyperboliques sur une variété. Nous étudions tout d'abord une première approche qui nécessite l'existence d'une métrique lorentzienne. Notre résultat principal établit la convergence de schémas de volumes finis du premier ordre pour une large classe de maillages. Ensuite, nous proposons une nouvelle approche basée sur des champs de formes différentielles. Nous considérons alors les lois de conservation hyperboliques non-linéaires posées sur une variété différentielle avec bord, appelée *espace-temps*, dans laquelle le "flux" est défini comme un champ de flux de n-formes dépendant d'un paramètre. Dans ce travail, nous introduisons une nouvelle version de la méthode de volumes finis, qui requiert seulement la structure de n-forme sur la variété de dimension $(n + 1)$.

La seconde partie porte sur les estimations d'erreur pour la méthode des volumes finis et sur la mise en oeuvre d'un modèle de fluides. Nous considérons tout d'abord les lois de conservation hyperboliques posées sur une variété riemannienne. Nous établissons une estimation d'erreur en norme L^1 pour une classe de schémas de volumes finis pour l'approximation des solutions entropiques du problème de Cauchy. L'erreur en norme L^1 est d'ordre $h^{1/4}$, où h représente le diamètre maximal des éléments d'une famille de maillages géodésiques. Nous considérons ensuite les équations hyperboliques posées sur un espace-temps courbe. En imposant que le flux vérifie une propriété naturelle d'invariance de Lorentz, nous identifions une loi de conservation unique, à une normalisation près, qui peut être vue comme une version relativiste de l'équation classique de Burgers. Cette équation fournit un modèle simplifié de dynamique de fluides compressibles relativistes. Des tests numériques mettent en évidence la convergence et la pertinence du schéma de volumes finis proposé.

1 Méthode de volumes finis sur une variété : deux approches

1.1 Approche basée sur une métrique

Motivations et rappels

Les systèmes d'équations aux dérivées partielles de type hyperbolique non-linéaire décrivent de nombreux phénomènes de la dynamique des milieux continus et de la physique. Les modèles scalaires, malgré leur apparence très simple, ont permis la découverte de toutes les méthodes de calcul numérique dans ce domaine : schémas aux différences finies, schémas de volumes finis, flux de Godunov, flux de Lax-Friedrichs, etc. L'étude de ces équations est d'une grande importance car la plupart des pathologies présentes dans les systèmes d'équations hyperboliques non-linéaires apparaissent déjà dans le cas scalaire. De plus, de nombreux schémas numériques pour des systèmes sont basés sur des algorithmes développés pour les équations scalaires. Nous allons ici nous concentrer sur le problème de Cauchy pour les lois de conservation scalaires, dont l'inconnue u est une fonction à valeurs réelles

$$\begin{aligned} &\partial_t u + \sum_{i=1}^{n} \partial_i\Big(f(u,x)\Big) = 0, \quad u = u(t,x) \in \mathbb{R}, \quad t \geq 0,\ x = (x_1, \dots, x_n) \in \mathbb{R}^n, \\ &u(0,x) = u_0(x), \end{aligned} \tag{1}$$

où le flux $f : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ est donné et est régulier.

Le problème de l'unicité de la solution pour les lois de conservation scalaires est résolu avec l'introduction de la notion fondamentale d'entropie. Nous imposons que la solution faible de (1) vérifie les *inégalités d'entropie*

$$\partial_t U(u) + \sum_{i=1}^{n} \partial_i\Big(F(u,x)\Big) \leq \sum_{i=1}^{n} (\partial_i F)(u,x) - U'(u) \sum_{i=1}^{n} (\partial_i f)(u,x), \tag{2}$$

au sens des distributions, pour toute fonction convexe $U : \mathbb{R} \to \mathbb{R}$. Le champ de vecteurs $x \mapsto F(u,x)$ est défini par

$$\partial_u F(u,x) := \partial_u U(u) \partial_u f(u,x), \quad u \in \mathbb{R},\ x \in \mathbb{R}^n,$$

et il faut noter ici la différence entre la dérivée partielle de la fonction $x \mapsto F(u(t,x),x)$ notée $\partial_i\Big(F(u,x)\Big)$ et la fonction $\partial_i F$ prise au point $(u(t,x),x)$ notée $(\partial_i F)(u,x)$.

D'après la théorie fondamentale de Kruzkov [27], le problème (1)–(2) est bien posé pour des données initiales dans L^∞. De nombreux travaux portent sur les lois de conservation hyperboliques, on se contentera ici de renvoyer le lecteur aux livres de Dafermos [17], Hörmander [23] et LeFloch [31].

Rappels de géométrie différentielle

Par définition, une variété differentielle M de dimension n est localement C^∞- difféomorphe à l'espace euclidien $\mathbb{R}^n$. Une **variété riemannienne** (M, g) de dimension n, est une variété munie d'une métrique définie positive g. La métrique g est un tenseur de type $(0, 2)$ qui associe à chaque point $x \in M$ une forme bilinéaire $(X, Y) \to g(X, Y)$ sur $T_xM \times T_xM$, où T_xM est l'espace tangent à M en x. Cette forme bilinéaire est décrite en coordonnées locales par une matrice définie positive d'éléments g_{ij}. Autrement dit, g induit un produit scalaire sur T_xM représentant la géométrie de M autour de x. Nous noterons les coordonnées locales $x = (x_1, \dots, x_n)$, et nous utiliserons la notation d'Einstein, notamment dans

$$g_{ij}X^i := \sum_{i=1}^{d} g_{ij}X^i.$$

Pour tous vecteurs tangents $X, Y \in T_xM$, nous utilisons la notation

$$g_x(X, Y) = \langle X, Y\rangle_g, \text{ et } |X|_g := \langle X, X\rangle_g^{1/2}.$$

L'espace cotangent T_x^*M est le dual de T_xM pour $x \in M$. Ses éléments s'appellent les 1–formes. En coordonnées locales, la base duale de $(\partial_1, \dots, \partial_n)$ est notée $(dx^1, \dots, dx^n)$ et on a $dx^i(\partial_j) = \delta^i_j$, où δ^i_j est le delta de Kronecker. Etant donnée φ une fonction régulière, la différentielle $d\varphi$ de φ est la 1-forme définie par $d\varphi(X) = X(\phi)$ pour tout champ de vecteurs X ; en coordonnées locales, on a $d\varphi_i = \partial_i\varphi$. La dérivée covariante d'un champ de vecteurs X est un champ de tenseur $(1, 1)$ dont les coordonnées sont indiquées par $(\nabla_g X)^j_k$. Nous avons la formule suivante pour la divergence d'un champ de vecteurs régulier :

$$\begin{aligned}\operatorname{div}_g\Big(f(u, x)\Big) &= du(\partial_u f(u, x)) + \Big(\operatorname{div}_g f\Big)(u, x)\\ &= \partial_u f^i\,\frac{\partial u}{\partial x^i} + \frac{1}{\sqrt{|g|}}\partial_i(\sqrt{|g|}\, f^i).\end{aligned}$$

Nous notons par d_g la fonction de distance associée à la métrique, $dv_g = dv_M$ la mesure du volume et ∇_g la connexion de Levi-Civita associée à g. L'opérateur divergence d'un champ de vecteurs est défini intrinsèquement comme la trace de sa dérivée covariante. A la suite de la formule de Gauss-Green, pour chaque champ de vecteurs réguliers f et tout sous-ensemble $S \subset M$ ouvert et régulier, on a

$$\int_S \operatorname{div}_g f\, dv_M = \int_{\partial S} \langle f, n\rangle_g\, dv_{\partial S},$$

où ∂S est le bord de S, n la normale unitaire extérieur à ∂S et $dv_{\partial S}$ la mesure induite sur ∂S.

Nous allons utiliser la notation standard suivante pour les espaces de fonctions définies sur M. Pour $p \in [1, \infty]$, la norme usuelle d'une fonction h dans l'espace de Lebesgue $L^p(M;g)$ est notée par $\|h\|_{L^p(M;g)}$; lorsque $p = \infty$, on écrit aussi $\|h\|_\infty$. Etant donnés $f \in L^1_{\text{loc}}(M;g)$ et un sous-ensemble $N \subset M$, nous notons

$$\fint_N f(y)\, dv_g(y) := |N|_g^{-1} \int_N f(y)\, dv_g(y), \qquad |N|_g := \int_N dv_g.$$

On ne peut pas dire qu'un champ de vecteurs f est constant sur une variété, puisque pour deux points différents x et y, les vecteurs $f(x)$ et $f(y)$ appartiennent à deux espaces vectoriels distincts T_xM et T_yM, on ne dispose donc pas d'une façon canonique de les comparer sans faire intervenir des coordonnées particulières. Il faut donc tenir compte de la dépendance explicite du flux f en x. Le problème de Cauchy sur (M,g) s'écrit alors

$$\begin{aligned} &\partial_t u + \nabla_g \cdot f(u,x) = 0, \qquad u = u(t,x) \in \mathbb{R}, \quad t \geq 0,\ x \in M \\ &u(0,x) = u_0(x), \end{aligned} \tag{3}$$

où $u : \mathbb{R}_+ \times M \to \mathbb{R}$ est l'inconnue et le flux $f = f_x(u) = f(\overline{u}, x)$ est un champ de vecteur régulier qui est défini pour tout $x \in M$ et dépend du paramètre réel $\overline{u}$.

Nous remarquons que les inégalités d'entropie prennent la forme

$$\partial_t U(u) + \nabla_g \cdot F(u,x) \leq (\nabla_g \cdot F)(u,x) - U'(u)(\nabla_g \cdot f)(u,x),$$

où (U,F) est une paire d'entropie si $U : \mathbb{R} \to \mathbb{R}$ est une fonction continue lipschitzienne et $F = F(u,x)$ un champ de vecteurs tel que, pour presque tout $\bar{u} \in \mathbb{R}$ et tout $x \in M$,

$$\partial_u F(\bar{u},x) = \partial_u U(\bar{u}) \partial_u f(\bar{u},x).$$

Lois de conservation hyperboliques sur une variété lorentzienne

Une **variété lorentzienne** M de dimension $n+1$ est une variété régulière munie d'une métrique g ayant la signature $(-1, 1, \cdots, 1)$, autrement dit, le tenseur de la métrique g peut s'écrire en coordonnées locales comme la matrice diag $(-1, 1, \cdots, 1)$. Ainsi, puisque la forme bilinéaire associée à g n'est plus définie positive, la métrique g n'induit plus un produit scalaire sur l'espace tangent T_xM, comme c'était le cas pour les variétés riemanniennes. Nous faisons une distinction entre vecteurs tangents X de type temps ($g(X,X) < 0$), de type lumière ($g(X,X) = 0$) et de type espace ($g(X,X) > 0$). Ces appellations sont motivées par le fait que, en relativité générale, les trajectoires des rayons de lumière ont des tangentes de type lumière. Les objets physiques suivent des trajectoires dont les tangentes sont partout de type temps.

Le problème de Cauchy pour les lois de conservation hyperboliques sur une variété lorentzienne M s'écrit

$$\text{div}_g\Big(f(u,p)\Big) = 0, \qquad u : M \to \mathbb{R} \tag{4}$$

et les inégalités d'entropie prennent la forme

$$\operatorname{div}_g\Big(F(u)\Big) - (\operatorname{div}_g F)(u) + U'(u)(\operatorname{div}_g f)(u) \leq 0 \tag{5}$$

où (U, F) désigne le couple entropie/flux d'entropie. Le vecteur-flux f est *compatible avec la géométrie* si

$$\operatorname{div}_g f(\overline{u}, p) = 0, \quad \overline{u} \in \mathbb{R},\ p \in M,$$

et du type temps, si sa dérivée par rapport à u est un champ de vecteurs du type temps de sorte que

$$g\Big(\partial_u f(\overline{u}, p), \partial_u f(\overline{u}, p)\Big) < 0, \qquad p \in M,\ \overline{u} \in \mathbb{R}.$$

Pour donner un sens au problème de Cauchy pour (4)–(5), il est nécessaire de faire une hypothèse sur la variété assurant de bonnes propriétés de causalité. Nous supposons donc que M est globalement hyperbolique, c'est-à-dire que M admet un feuilletage par des hypersurfaces H_t compactes, du type espace et orientées, indexées par un paramètre t qui joue le rôle du temps de sorte que

$$M = \bigcup_{t \in \mathbb{R}} \mathcal{H}_t.$$

Chacune des hypersurfaces H_t est une surface de Cauchy. Nous pouvons alors définir un problème de Cauchy sur une certaine surface H_{t_0} , et ajouter aux équations (4)–(5) la condition initiale

$$u|_{\mathcal{H}_{t_0}} = u_0. \tag{6}$$

Convergence de la méthode de volumes finis

Nous cherchons à généraliser les résultats de convergence de la méthode des volumes finis de [15] (le cas plat) et [2] (le cas riemannien) au cas lorentzien. La principale difficulté vient du fait qu'une variété lorentzienne ne nous fournit pas de direction de temps privilégiée. Ce fait est d'une extrême importance pour l'élaboration de schémas de volumes finis, puisque la définition du schéma ainsi que les résultats de convergence doivent être suffisamment robustes pour en tenir compte. Cette difficulté apparaît dès que nous essayons de définir le schéma de volumes finis. En effet, cette définition doit prendre en compte précisément l'absence de toute coordonnée temporelle privilégiée.

Soient (M, g) une variété lorentzienne et $\mathcal{T}^h$ une triangulation en espace-temps de M qui est composée d'éléments K. Nous supposons que chaque élément K de la triangulation $\mathcal{T}^h$, qui sera un élément en espace-temps, a exactement deux faces du type espace, une face "supérieure" e_K^+ et une face

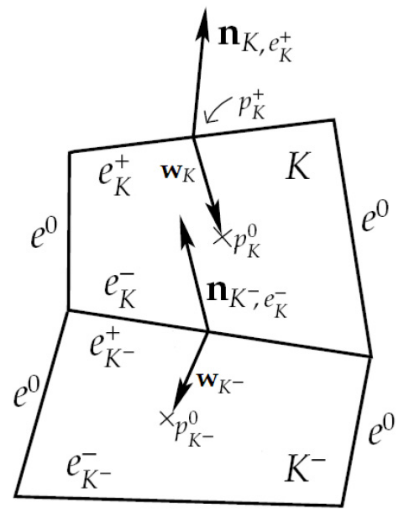

FIG. 1 – Les éléments du maillage

"inférieure" e_K^-, et le reste de son bord, noté par $\partial^0 K$, est du type temps. Le paramètre h est tel que pour tout K, $\operatorname{diam} e_K^\pm \leq h$. Nous notons le vecteur normal unitaire extérieur à K sur la face e par $n_{K,e}$, le centre de masse de e_K^+ par p_K^+, et le centre de masse de $\partial^0 K$ par p_K^0 . Nous notons par $\mathbf{w}_K$ le vecteur tangent à la géodésique reliant p_K^+ à p_K^0 et par K^- l'élément qui "précède" K, c'est-à-dire l'unique élément K^- partageant la face e_K^- avec K (voir la Figure 1).

Le schéma des volumes finis pour une variété lorentzienne peut s'écrire en intégrant l'équation (4) sur un élément K. Nous trouvons, après une intégration par parties,

$$
\begin{aligned}
&- \int_{e_K^+} g_p(f(u,p), \mathbf{n}_{K,e_K^+}(p))\, dV_g - \int_{e_K^-} g_p(f(u,p), \mathbf{n}_{K,e_K^-}(p))\, dV_g \\
&+ \sum_{e^0 \in \partial^0 K} \int_{e^0} g_p(f(u,p), \mathbf{n}_{K,e^0}(p)) dp = 0,
\end{aligned}
$$

ce qui suggère le schéma de volume finis

$$
|e_K^+| \mu^f_{K^+, e_K^+}(u_K^+) := |e_K^-| \mu^f_{K, e_K^-}(u_K^-) - \sum_{e^0 \in \partial^0 K} |e^0|\, \mathbf{q}_{K,e^0}(u_K^-, u_{K_{e^0}}^-), \tag{7}
$$

avec la notation

$$
\mu^f_{K,e}(u) := \frac{1}{|e|} \int_e g_p\Big(f(u,p), \mathbf{n}_{K,e}(p)\Big)\, dp = \fint_e g_p\Big(f(u,p), \mathbf{n}_{K,e}(p)\Big)\, dp,
$$

où

$$
\begin{aligned}
&\int_{e_K^-} g_p\Big(f(u,p), \mathbf{n}_{K,e_K^-}(p)\Big)\, dV_g \simeq |e_K^-| \mu^f_{K,e_K^-}(u_K^-), \\
&\int_{e^0} g_p(f(u,p), \mathbf{n}_{K,e^0}(p))\, dV_{e^0} \simeq |e^0|\, \mathbf{q}_{K,e^0}(u_K^-, u_{K_{e^0}}^-).
\end{aligned}
$$

En notant que la fonction $u \mapsto \mu^f_{K,e^-_K}(u)$ est monotone croissante, nous pouvons réécrire le schéma de volume finis (7) en prenant l'inverse de μ_{K^+,e^+_K}

$$u^+_K := (\mu^f_{K^+,e^+_K})^{-1}\Big(\frac{|e^-_K|}{|e^+_K|}\mu^f_{K,e^-_K}(u^-_K) - \sum_{e^0\in\partial^0 K}\frac{|e^0|}{|e^+_K|}\,\mathbf{q}_{K,e^0}(u^-_K, u^-_{K_{e^0}})\Big).$$

Les fonctions-flux numériques $\mathbf{q}_{K,e^0}(u,v)$ vérifient les hypothèses suivantes.

– *Propriété de consistance*

$$\mathbf{q}_{K,e^0}(u,u) = \fint_{e^0} f_{e^0}(u,p)\,dV_g = \mu^f_{K,e^0}(u). \tag{8}$$

– *Propriété de conservation*

$$\mathbf{q}_{K,e^0}(u,v) = -\mathbf{q}_{K_{e^0},e^0}(v,u), \qquad u,v\in\mathbb{R}. \tag{9}$$

– *Propriété de monotonie*

$$\partial_u \mathbf{q}_{K,e^0}(u,v) \geq 0, \qquad \partial_v \mathbf{q}_{K,e^0}(u,v) \leq 0. \tag{10}$$

On suppose la condition de stabilité CFL suivante : pour tout $K\in\mathcal{T}^h$, $e^0\in\partial_0 K$,

$$\frac{|\partial^0 K|}{|e^+_K|}\sup_{u\in\mathbb{R}}\left|\partial_u \mu^f_{K,e^0}(u)\right| \sup_{u\in\mathbb{R}}\partial_u(\mu^f_{K^+,e^+_K})^{-1}(u) \leq 1. \tag{11}$$

De plus, on définit le paramètre

$$\tau_K = \frac{|K|}{|e^+_K|},$$

où $|K|$ est la mesure $(n+1)$-dimensionnelle de K, et on suppose que

$$\tau := \max_K \tau_K \to 0, \qquad \frac{h^2}{\min\limits_K \tau_K} \to 0, \tag{12}$$

lorsque $h\to 0$.

La formulation (7) est suffisamment flexible pour pouvoir être appliquée à des triangulations assez générales, et se réduit à la formulation riemannienne si l'évolution temporelle de la variété est triviale, c'est-à-dire, si $M=\mathbb{R}^+\times N$, avec N une variété riemannienne. En effet, dans ce cas les fonctions $\mu^f_{K,e^\pm_K}$ se réduisent à l'identité, et le schéma (7) coïncide avec le schéma du cas riemannien. Dans le cas général du schéma (7), on peut à chaque pas de temps récupérer la solution approchée u^+_K en inversant la fonction $\mu^f_{K^+,e^+_K}(\cdot)$.

Hypothèses sur la triangulation

Nous trouvons une condition optimale pour la triangulation en espace-temps de la variété $\mathcal{T}^h$ assurant la convergence du schéma. C'est une condition globale qui porte sur une quantité associée à l'évolution temporelle de la triangulation : la déviation locale du maillage. Cette quantité mesure localement l'écart de l'évolution temporelle du maillage par rapport à une évolution cartésienne uniforme. Plus précisément, nous définissons la quantité

$$\mathcal{E}(K) := \frac{1}{\tau_K} \mathbf{w}_K \otimes \mathbf{n}_{K,e_K^+},$$

où $\mathbf{w}_K$ est le vecteur tangent en p_K^+ à la géodésique reliant p_K^+ à p_K^0, et τ_K est un paramètre associé à la taille de K dans la direction temporelle. Cette quantité mesure la déformation de l'élément K par rapport à un prisme de base e_K^-. Nous considérons alors le taux de variation locale de cette quantité,

$$|K|\mathcal{E}(K) - |K^-|\mathcal{E}(K^-),$$

que nous appelons *déviation locale* associée à K et K^- . Notre hypothèse consiste à demander que la somme sur $K \in \mathcal{T}^h$ de ces quantités appliquées à des champs de vecteurs test X, Y tendent vers zéro avec h

$$\Big| \sum_{K \in \mathcal{T}^h} \Big(|K|\mathcal{E}(K) - |K^-|\mathcal{E}(K^-) \Big)(X, Y) \Big| \to 0, \qquad h \to 0.$$

Résultat principal

Munis de ce critère d'admissibilité, nous démontrons le résultat de convergence suivant.

Théorème 1. *Soit u^h la suite de fonctions générée par la méthode des volumes finis* (7) *sur une triangulation admissible, avec la donnée initiale $u_0 \in L^\infty(H_0)$ et des fonction-flux numériques vérifiant les conditions* (8)–(12)*. Alors, pour tout $T > 0$, la suite u^h est uniformément bornée dans $L^\infty(\cap_{0 \le t \le T} H_t)$, et converge presque partout lorsque $h \to 0$ vers l'unique solution entropique $u \in L^\infty(M)$ du problème de Cauchy* (4),(6).

La démonstration de ce théorème suit une stratégie proposée par Cockburn, Coquel et LeFloch [15] où les auteurs démontrent un résultat analogue dans le cas euclidien, et généralisé par Amorim, Artzi et LeFloch [2] aux variétés riemanniennes. Ces preuves s'appuient sur des estimations de dissipation d'entropie permettant de contrôler les termes d'erreur d'approximation et d'en déduire une inégalité d'entropie discrète. Dans notre travail, l'obtention d'estimations analogues s'avère bien plus difficile, à cause de l'influence de la géométrie en espace-temps de la variété. Pour les détails de la démonstration, on se renvoie au Chapitre 1.

1.2 Approche basée sur des champs de formes différentielles

Le deuxième chapitre de la thèse est consacré à l'étude d'une nouvelle version de la méthode de volumes finis basée sur des champs de formes différentielles sur un espace-temps.

Loi de conservation sur un espace-temps

Soit M une variété compacte, orientée et différentiable de dimension $(n+1)$, que nous appelons un espace-temps. Nous considérons les lois de conservation non-linéaires

$$d(\omega(u)) = 0, \quad u = u(x),\ x \in M, \tag{13}$$

où, pour chaque $\overline{u} \in \mathbb{R}$, $\omega = \omega(\overline{u})$ est un champ des n-formes sur M. Nous appelons $\omega = \omega(\overline{u})$ les **champs des flux** de la loi de la conservation (13). Nous remarquons que d représente l'opérateur dérivée extérieure et donc $d(\omega(u))$ un champ de formes différentielles de degré $(n+1)$ sur M. Avec les notations ci-dessus, en introduisant des coordonnées locales $x = (x^\alpha)$, nous pouvons écrire pour tout $\overline{u} \in \mathbb{R}$

$$\omega(\overline{u}) = \omega^\alpha(\overline{u})\,(\widehat{dx})_\alpha,$$
$$(\widehat{dx})_\alpha := dx^0 \wedge \ldots \wedge dx^{\alpha-1} \wedge dx^{\alpha+1} \wedge \ldots \wedge dx^n.$$

Les coefficients $\omega^\alpha = \omega^\alpha(\overline{u})$ sont des fonctions régulières définies dans le diagramme local choisi. Nous rappelons que l'opérateur d agit sur les formes différentielles de degré arbitraire et si ρ est une p-forme et ρ' une p'-forme, alors

$$d(d\rho) = 0, \quad \text{et} \quad d(\rho \wedge \rho') = d\rho \wedge \rho' + (-1)^p \rho \wedge d\rho'.$$

Etant donnée une solution régulière u de (2.4), nous pouvons appliquer le théorème de Stokes sur un sous-ensemble ouvert $\mathcal{U}$ qui est compact inclus dans M. Nous obtenons

$$0 = \int_{\mathcal{U}} d(\omega(u)) = \int_{\partial\mathcal{U}} i^*(\omega(u)),$$

où $\partial\mathcal{U}$ est le bord régulier du U. De même, si $\psi : M \to \mathbb{R}$ est une fonction régulière, on peut écrire

$$d(\psi\,\omega(u)) = d\psi \wedge \omega(u) + \psi\, d(\omega(u)),$$

où le différentiel $d\psi$ est un champ de 1-forme. A condition que u satisfait (2.4), nous trouvons

$$\int_M d(\psi\,\omega(u)) = \int_M d\psi \wedge \omega(u)$$

et, par le théorème de Stokes, nous obtenons

$$\int_M d\psi \wedge \omega(u) = \int_{\partial M} i^*(\psi\omega(u)).$$

Remarquons qu'une orientation appropriée du bord ∂M est requis pour cette formule. Cette identité est satisfaite par toute solution régulière de (2.4).

Etant données deux hypersurfaces H et H', telles que H' se trouve dans le futur de H, nous avons également créé une propriété de contraction de deux solutions entropiques u, v telle que

$$\int_{H'} \mathbf{\Omega}(u_{H'}, v_{H'}) \le \int_{H} \mathbf{\Omega}(u_H, v_H).$$

Nous remarquons que pour tous réels u et v, le champ de n-formes $\mathbf{\Omega}(\overline{u}, \overline{v})$ est déterminé par des champs des flux ω et il peut être considéré comme une généralisation de la notion d'entropie Kruzkov $|\overline{u} - \overline{v}|$. Si u est une solution régulière de (13), alors les inégalités d'entropie prennent la forme

$$d\Big(\Omega(u)\Big) - (d\Omega)(u) + \partial_u U(u)(d\omega)(u) \le 0,$$

inégalités satisfaites au sens des distributions pour toute paire d'entropie (U, Ω).

Un champ de flux ω est appelé géométrie-compatible s'il est fermé pour chaque valeur du paramètre

$$(d\omega)(\overline{u}) = 0, \qquad \overline{u} \in \mathbb{R}.$$

Cette condition de compatibilité est naturelle car elle assure que les constantes sont des solutions triviales de la loi de conservation. Il s'agit d'une propriété partagée par de nombreux modèles de dynamique des fluides. Si ω est géometrie-compatible, les inégalités d'entropie prennent la forme suivante

$$d\Big(\Omega(u)\Big) \le 0. \tag{14}$$

Hypothèses sur la triangulation

En relativité générale, il est une hypothèse classique que l'espace-temps est globalement hyperbolique. Nous supposons que la variété M est feuilletée par des hypersurfaces

$$M = \bigcup_{0 \le t \le T} H_t,$$

où chaque tranche a la topologie d'une n-variété N régulière avec bord. Topologiquement, nous avons $M = [0, T] \times N$, et le bord de M peut être décomposé comme

$$\begin{aligned} \partial M &= H_0 \cup H_T \cup B, \\ B = (0, T) \times N &:= \bigcup_{0 < t < T} \partial H_t. \end{aligned} \tag{15}$$

Soit $\mathcal{T}^h = \bigcup_{K \in \mathcal{T}^h} K$ une triangulation de M composée d'éléments K satisfaisant les conditions suivantes.

– Le bord $\partial K = \bigcup_{e \in \partial K} e$ d'un élément K est une n-variété régulière par morceaux contenant exactement deux faces du type espace e_K^-, e_K^+ et des éléments "verticaux"

$$e^0 \in \partial^0 K := \partial K \setminus \left\{ e_K^+, e_K^- \right\}.$$

– L'intersection $K \cap K'$ de deux éléments distincts $K, K' \in \mathcal{T}^h$ est soit une face commune des K, K' ou bien une sous-variété de dimension au plus $(n-1)$.
– La triangulation est compatible avec le feuilletage (14)-(15) s'il existe une suite de temps $t_0 = 0 < t_1 < \ldots < t_N = T$ telle que toutes les faces du type espace sont des sous-variétés de $H_n := H_{t_n}$ pour tous $n = 0, \ldots, N$. Nous désignons par $\mathcal{T}_0^h$ l'ensemble de tous les éléments K qui admettent une face appartenant à l'hypersurface initiale H_0.

Méthode de volume fini

Nous introduisons la méthode de volumes finis, afin de prendre la moyenne de la loi de conservation (13) sur chaque élément $K \in \mathcal{T}^h$. En appliquant le théorème de Stoke, où u est une solution régulière de (13), nous obtenons

$$0 = \int_K d(\omega(u)) = \int_{\partial K} i^* \omega(u).$$

Puis nous décomposons le bord ∂K en e_K^+, e_K^- et $\partial^0 K$, il vient

$$\int_{e_K^+} i^* \omega(u) - \int_{e_K^-} i^* \omega(u) + \sum_{e^0 \in \partial^0 K} \int_{e^0} i^* \omega(u) = 0.$$

Etant donnée la moyenne des valeurs u_K^- sur e_K^- et $u_{K_{e^0}}^-$ sur $e^0 \in \partial^0 K$, nous avons besoin d'une approximation u_K^+ de la valeur moyenne de la solution u le long de e_K^+. A cet effet, le second terme peut être approché par

$$\int_{e_K^-} i^* \omega(u) \approx \int_{e_K^-} i^* \omega(u_K^-) = |e_K^-| \varphi_{e_K^-}(u_K^-),$$

et le dernier terme

$$\int_{e^0} i^* \omega(u) \approx \mathbf{q}_{K,e^0}(u_K^-, u_{K_{e^0}}^-),$$

où le flux total discret $\mathbf{q}_{K,e^0} : \mathbb{R}^2 \to \mathbb{R}$ doit être prescrit.

Enfin, on arrive à la version proposée de la *méthode des volumes finis* pour la loi de conservation (13)

$$\int_{e_K^+} i^* \omega(u_K^+) = \int_{e_K^-} i^* \omega(u_K^-) - \sum_{e^0 \in \partial^0 K} \mathbf{q}_{K,e^0}(u_K^-, u_{K_{e^0}}^-)$$

ou bien de manière équivalente,

$$|e_K^+|\varphi_{e_K^+}(u_K^+) = |e_K^-|\varphi_{e_K^-}(u_K^-) - \sum_{e^0 \in \partial^0 K} \mathbf{q}_{K,e^0}(u_K^-, u_{K_{e^0}}^-), \tag{16}$$

où l'on a utilisé la notation

$$\varphi_e(u) := \int_e i^* \omega(u).$$

Nous supposons que les fonctions-flux numériques $\mathbf{q}_{K,e^0}$ vérifient les conditions de conservation, consistance et monotonie analogues à (8)–(10) suivantes.

– *Propriété de consistance*

$$\mathbf{q}_{K,e^0}(\overline{u}, \overline{u}) = \int_{e^0} i^* \omega(\overline{u}). \tag{17}$$

– *Propriété de conservation*

$$\mathbf{q}_{K,e^0}(\overline{v}, \overline{u}) = -q_{K_{e^0},e^0}(\overline{u}, \overline{v}). \tag{18}$$

– *Propriété de monotonie*

$$\partial_{\overline{u}} \mathbf{q}_{K,e^0}(\overline{u}, \overline{v}) \geq 0, \qquad \partial_{\overline{v}} \mathbf{q}_{K,e^0}(\overline{u}, \overline{v}) \leq 0. \tag{19}$$

On suppose la condition de stabilité CFL suivante satisfaite. Pour tout $K \in \mathcal{T}^h$,

$$\frac{N_K}{|e_K^+|} \max_{e^0 \in \partial^0 K} \sup_u \left| \int_{e^0} \partial_u \omega(u) \right| < \inf_u \partial_u \varphi_{e_K^+}. \tag{20}$$

Ensuite, nous supposons les conditions suivantes sur la famille de triangulations

$$\lim_{h \to 0} \frac{\tau_{\max}^2 + h^2}{\tau_{\min}} = \lim_{h \to 0} \frac{\tau_{\max}^2}{h} = 0 \tag{21}$$

où $\tau_{\max} := \max_i (t_{i+1} - t_i)$ et $\tau_{\min} := \min_i (t_{i+1} - t_i)$. A titre d'exemple, ces conditions sont remplies si $\tau_{\max}$, $\tau_{\min}$ et h disparaissent au même ordre.

Résultat principal

Théorème 2. *Sous les hypothèses imposées ci-dessus sur les triangulations et à condition que le champ de flux soit compatible avec la géométrie, la famille des solutions approchées u^h générées par le schéma de volume fini* (16) *converge, lorsque $h \to 0$, vers une solution entropique de* (13).

Notre preuve de convergence de la méthode de volumes finis est une généralisation à un espace-temps de la technique présentée par Cockburn, Coquel et LeFloch pour le cas plat et déjà étendu aux variétés riemanniennes par Amorim, Ben-Artzi, et LeFloch [2] et aux variétés lorentziennes par Amorim, LeFloch et Okutmustur [3]. On se renvoie au Chapitre 2 pour les détails de la démonstration.

2 Estimation d'erreur et mise en oeuvre

2.1 Estimation d'erreur sur une variété

Le troisième chapitre de la thèse est consacré à l'étude des estimations d'erreur pour des schémas de volumes finis sur des variétés.

Le but est d'étendre l'estimation d'erreur pour les méthodes de volumes finis de Cockburn, Coquel, et LeFloch [15, 13] aux variétés. Pour parvenir à ce résultat, nous avons besoin de revoir la théorie de rapprochement de Kuznetzov [28, 29] et d'adapter la technique développée dans [15].

Soit (M, g) une variété connectée, compacte, n-dimensionelle avec la métrique g, c'est-à-dire, pour chaque $x \in M$, g_x est un produit scalaire sur l'espace tangent T_xM à x. Le problème de Cauchy sur (M, g) s'écrit

$$\begin{aligned} &\partial_t u + \nabla_g \cdot f(u, x) = 0, \qquad u = u(t, x) \in \mathbb{R}, \quad t \geq 0, \ x \in M \\ &u(0, x) = u_0(x), \end{aligned} \tag{22}$$

où $u : \mathbb{R}_+ \times M \to \mathbb{R}$ est l'inconnue et le flux $f = f_x(u) = f(\overline{u}, x)$ un champ de vecteur régulier défini pour tout $x \in M$ et dépendant du paramètre réel $\overline{u}$.

Nous rappelons les inégalités d'entropie

$$\partial_t U(u) + \nabla_g \cdot F(u, x) \leq (\nabla_g \cdot F)(u, x) - U'(u)(\nabla_g \cdot f)(u, x),$$

où (U, F) est une paire d'entropie si $U : \mathbb{R} \to \mathbb{R}$ est une fonction continue lipschitzienne et $F = F(u, x)$ un champ de vecteurs tels que pour presque tous $\bar{u} \in \mathbb{R}$ et $x \in M$,

$$\partial_u F(\bar{u}, x) = \partial_u U(\bar{u}) \partial_u f(\bar{u}, x).$$

Dans cette étude, nous nous intéressons à la discrétisation du problème (22) dans le cas où la donnée initiale est bornée et sa variation totale est finie,

$$u_0 \in L^\infty(M) \cap BV(M; g). \tag{23}$$

En particulier, il est établi en [6] que si les données initiales sont bornées, le principe du maximum suivant est établi :

$$\|u(t)\|_{L^\infty(M)} \leq C_0(T, g) + C_0'(T, g) \, \|u(s)\|_{L^\infty(M)}, \qquad 0 \leq s \leq t \leq T,$$

où les constantes $C_0, C_0' > 0$ dépendent de T et de la métrique g.

Nous rappelons la définition de la variation totale d'une fonction $w : M \to \mathbb{R}$ par

$$\mathrm{TV}_g(w) := \sup_{\|\phi\|_\infty \leq 1} \int_M w \ \mathrm{div}_g \phi \, dv_g,$$

où ϕ décrit tous les champs de vecteurs de la forme C^1 à support compact. Nous notons

$$BV(M; g) = \{u \in L^1(M; g) \ / \ \mathrm{TV}_g(u) < \infty\},$$

l'espace de toutes les fonctions à variation totale finie sur M. Il est bien connu que le plongement $BV(M;g) \subset L^1(M;g)$ est compact à condition que g est suffisamment régulier.

Une propriété importante des solutions d'entropie pour (22) est la suivante : u est à variation totale finie pour tout temps $t \geq 0$ si (23) est satisfaite, de plus

$$\mathrm{TV}_g(u(t)) \leq C_1(T,g) + C_1'(T,g)\ \mathrm{TV}_g(u(s)), \qquad 0 \leq s \leq t \leq T,$$

où les constantes $C_0, C_0' > 0$ dépendent de T et de la métrique g. (Voir [6] pour les détails). Cela implique un contrôle du flux de l'équation

$$\sup_{t\geq 0} \int_M \left| \mathrm{div}_g \left(f(u(t,\cdot),\cdot) \right) \right| dv_g \leq C\ \mathrm{TV}_g(u_0).$$

Cependant, comme indiqué dans [2], cette inégalité peut être dérivée plus directement de la loi de conservation et on vérifie que la constante C est indépendante de T et de g, mais dépend de la plus grande vague de vitesse qui se pose dans le problème.

La famille de triangulations

Soit $\tau > 0$. Nous considérons le maillage uniforme $t_n := n\,\tau$ ($n = 0, 1, 2, \ldots$) sur la demi-ligne $\mathbb{R}_+$. Soit $\mathcal{T}^h$ une triangulation sur la variété M composée d'éléments K dont les arêtes sont rejointes par des faces géodésiques. Nous supposons que, si deux éléments distincts $K_1, K_2 \in \mathcal{T}^h$ ont une intersection non vide notée $K_1 \cap K_2 = I$, alors soit I est une face géodésique de K_1 et de K_2, soit $\mathcal{H}^{n-1}(I) = 0$, avec $\mathcal{H}^{n-1}$ représentant la mesure de Hausdorff à n-dimension.

La bord ∂K de K est composé de l'ensemble de toutes les faces e de K. Nous notons K_e l'élément unique et distinct de K partageant la face e avec K. La normale unitaire extérieure à un élément K en un point $x \in e$ est notée $\mathbf{n}_{e,K}(x) \in T_xM$. De plus, $|K|$ est la mesure Hausdorff de n-dimension et $|e|$ la mesure Hausdorff de $(n-1)$-dimension. Posons

$$p_K := \sum_{e \in \partial K} |e|, \quad h := \sup\{h_K : K \in \mathcal{T}^h\},$$

et pour chaque $K \in \mathcal{T}^h$ le diamètre h_K de K est

$$h_K := \sup_{x,y \in K} d_g(x,y).$$

Nous posons

$$h := \sup\{h_K : K \in \mathcal{T}^h\},$$

qui tend vers zéro selon une séquence de triangulations géodésiques. Nous supposons aussi qu'il existe des constantes de $\gamma_1, \gamma_2 > 0$ telles que

$$\gamma_1^{-1} h \leq \tau \leq \gamma_1 h \tag{24}$$

et

$$\gamma_2^{-1}|K| \le h_K\, p_K \le \gamma_2|K| \tag{25}$$

pour tout $K \in \mathcal{T}^h$. Cette condition implique que, lorsque $h \to 0$,

$$\tau \to 0, \qquad h^2\tau^{-1} \to 0.$$

Enfin, nous posons $T = \tau\, n_T$ pour tout entier n_T.

Formulation du schéma

Comme dans le cas euclidien ([15]), nous introduisons la méthode de volumes finis afin de prendre la moyenne de la loi de la conservation (22) sur chaque élément $K \in \mathcal{T}^h$. Premièrement, nous définissons

$$u^h(t,x) = u_K^n \qquad (t,x) \in [t_n, t_{n+1}) \times M, \quad (n = 0, 1, \ldots) \tag{26}$$

où

$$u_K^n := \fint_K u(t_n, x)\, dv_g(x),$$

et

$$u_K^0 := \fint_K u_0(x) dv_g(x).$$

Puis, en vue de (22), nous écrivons

$$\begin{aligned} 0 &= \frac{d}{dt}\fint_K u(t,x)\, dv_g(x) + \fint_K \operatorname{div}_g f(u(t,x), x)\, dv_g(x) \\ &\approx \frac{u_K^{n+1} - u_K^n}{\tau} + \frac{1}{|K|}\sum_{e\in\partial K} \int_e \langle f(u(t,y)), n_{e,K}(y)\rangle_g\, d\Gamma_g(y). \end{aligned}$$

Enfin, nous formulons le schéma de volumes finis par

$$u_K^{n+1} := u_K^n - \frac{\tau}{|K|}\sum_{e\in\partial K} |e|\ f_{e,K}(u_K^n, u_{K_e}^n) \qquad (n = 0, 1, \ldots), \tag{27}$$

où les fonctions de flux $f_{e,K} : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ sont définies par

$$\fint_e \langle f(w(u_K^n, u_{K_e}^n), y), n_{e,K}(y)\rangle_g\, d\Gamma_g(y) \approx f_{e,K}(u_K^n, u_{K_e}^n),$$

satisfaisant les propriétés suivantes.

– *Propriété de consistence :* pour $u \in \mathbb{R}$,

$$f_{e,K}(u,u) = \fint_e \langle f(u,y), \mathbf{n}_{e,K}(y)\rangle_g\, d\Gamma_g(y). \tag{28}$$

– *Propriété de conservation :* pour $u, v \in \mathbb{R}$,

$$f_{e,K}(u,v) + f_{e,K_e}(v,u) = 0. \tag{29}$$

– *Propriété de monotonie*

$$\frac{\partial}{\partial u} f_{e,K} \geq 0, \qquad \frac{\partial}{\partial v} f_{e,K} \leq 0. \tag{30}$$

Pour des raisons de stabilité de la méthode numérique, nous imposons la condition de stabilité CFL suivante

$$\tau \sup_{K \in \mathcal{T}^h} \frac{p_K}{|K|} \operatorname{Lip}(f) \leq 1,$$

où $\operatorname{Lip}(f)$ est la constante de Lipschitz de f.

Résultat principal

Théorème 3. *Soit* $u : \mathbb{R}_+ \times M \to \mathbb{R}$ *la solution entropique associée au problème de Cauchy* (22) *pour une donnée initiale* $u_0 \in L^\infty(M) \cap BV(M; g)$*. Soit* u^h *la solution approchée définie par* (26) *et* (27)*. Alors, pour chaque* $T > 0$*, il existe des constantes*

$$C_0 = C_0(T, g, \|u_0\|_{L^\infty}), \quad C_1 = C_1(T, g, TV_g(u_0)),$$
$$C_2 = C_2(T, g, \|u_0\|_{L^2(M;g)})$$

telles que pour tout $t \in [0, T]$*, on ait*

$$\begin{aligned}
&\|u^h(t) - u(t)\|_{L^1(M;g)} \\
&\leq \Big(C_0\ |M|_g + C_1\Big)\, h + \Big(C_0\ |M|_g^{1/2} + \Big(C_0\ C_1\Big)^{1/2}\Big)\, |M|_g^{1/2}\ h^{1/2} \\
&\quad + \Big(\Big(C_0\ C_2\Big)^{1/2} |M|_g^{1/2} + \Big(C_1\ C_2\Big)^{1/2}\Big)\, |M|_g^{1/4}\ h^{1/4}.
\end{aligned}$$

La démonstration de ce résultat généralise une technique de Cockburn, Coquel et LeFloch [15] établie pour le cas plat. On se renvoie au Chapitre 3 pour les détails de la démonstration.

2.2 Version relativiste de l'équation de Burgers

Le quatrième chapitre de la thèse est consacré à l'étude de la version relativiste de l'équation de Burgers et la mise en oeuvre de ce modèle.

Nous considérons des lois d'équilibre hyperboliques posées sur un espace-temps courbe (M, ω) de dimension $(N+1)$ basé sur une forme volume

$$\operatorname{div}^\omega(T(v)) = S(v), \tag{31}$$

dont la fonction inconnue est le champ scalaire $v : M \to \mathbb{R}$ et div^ω l'opérateur divergence associé à ω. Le champ de vecteurs $T = T(v)$ est défini sur la variété M, et dépend de v en tant que paramètre. La variété M (avec bord) est supposée feuilletée par des hypersurfaces

$$M = \bigcup_{t \geq 0} H_t, \tag{32}$$

de sorte que chaque tranche H_t est une variété à N-dimensions basée sur un champ normal 1-forme N_t et a la même topologie que la tranche initiale de H_0. L' hyperbolicité globale de l'espace-temps et de l'équation (31) est assurée en supposant que la fonction

$$v \mapsto T^0(v) := \langle N_t, T(v) \rangle \quad \text{est strictement croissante.} \tag{33}$$

De plus, $S = S(v)$ est un champ scalaire donné, défini sur M en fonction de v en tant que paramètre.

Dérivation d'un modèle invariant de Lorentz

Nous recherchons les champs de flux $T(v)$ pour lesquels les solutions de l'équation (31) satisfont une propriété d'invariance de Lorentz. Afin de simplifier le calcul, nous supposons maintenant que $N = 1$, $S(v) \equiv 0$ et que la variété $M = [0, +\infty) \times \mathbb{R}$ est couverte par une coordonnée particulière (x^0, x^1) avec $\omega = dx^0 dx^1$. Avec ce choix, l'équation (31) prend la forme d'une loi de conservation

$$\partial_0 T^0(v) + \partial_1 T^1(v) = 0,$$

où $\partial_0 = \partial/\partial x^0$ et $\partial_1 = \partial/\partial x^1$. Nous supposons également que les fonctions $T^0 = T^0(v)$ et $T^1 = T^1(v)$ sont indépendantes de (x^0, x^1).

Nous rappelons que les transformations de Lorentz $(x^0, x^1) \mapsto (\overline{x}^0, \overline{x}^1)$ sont définies par

$$\begin{aligned} \overline{x}^0 &:= \gamma_\epsilon(V)\,(x^0 - \epsilon^2 V x^1), \\ \overline{x}^1 &:= \gamma_\epsilon(V)\,(-V\,x^0 + x^1), \qquad \gamma_\epsilon(V) = \left(1 - \epsilon^2\,V^2\right)^{-1/2}, \end{aligned} \tag{34}$$

où $\epsilon \in (-1, 1)$ représente l'inverse de la vitesse normalisée de la lumière, et $\gamma_\epsilon(V)$ est appelé facteur de Lorentz associé à une vitesse donnée $V \in (-1/\epsilon, 1/\epsilon)$.

Nous rappelons aussi que les équations d'Euler relativistes de fluides compressibles sont invariantes sous les transformations de Lorentz. Plus précisément, étant donnée une vitesse V et d'après les transformations de Lorentz (34), la composante de vitesse v du fluide dans le système de coordonnées (x^0, x^1) est liée à la composante $\overline{v}$ dans les coordonnées $(\overline{x}^0, \overline{x}^1)$ par

$$\overline{v} = \frac{v - V}{1 - \epsilon^2 V\,v}. \tag{35}$$

Dans la limite non relativiste correspondant à $\epsilon \to 0$, on retrouve, lorsque $\epsilon = 0$, les transformations galiléennes

$$\overline{x}^0 = x^0, \quad \overline{x}^1 = -V x^0 + x^1, \quad \overline{v} = v - V. \tag{36}$$

La proposition suivante montre la propriété d'invariance de la loi de conservation donnée, avec la forme exacte des fonctions de flux dans le cas non-relativiste.

Proposition (Dérivation de l'équation de Burgers non-relativiste)**.** *La loi de conservation*

$$\partial_0 T^0(v) + \partial_1 T^1(v) = 0 \tag{37}$$

est invariante par transformation galiléenne si et seulement si le flux T^0 est linéaire et le flux T^1 est quadratique. Après normalisation, on obtient

$$\partial_0 v + \partial_1 (v^2/2) = 0.$$

Résultat principal

Théorème 4 (Version relativiste de l'équation de Burgers)**.** *La loi de conservation*

$$\partial_0 T^0(v) + \partial_1 T^1(v) = 0 \tag{38}$$

est invariante par transformation lorentzienne si et seulement si, après normalisation, on a

$$T^0(v) = \frac{v}{\sqrt{1-\epsilon^2 v^2}}, \qquad et \qquad T^1(v) = \frac{1}{\epsilon^2}\left(\frac{1}{\sqrt{1-\epsilon^2 v^2}} - 1\right), \tag{39}$$

où le champ scalaire v prend sa valeur en $(-1/\epsilon, 1/\epsilon)$.

On se réfère au Chapitre 4 pour la démonstration.

Propriétés de l'équation relativiste de Burgers

Nous proposons une version équivalente de la loi de conservation (38) satisfaisant certaines propriétés dans le cas relativiste et non relativiste en notant

$$w := T^0(v) = \frac{v}{\sqrt{1-\epsilon^2 v^2}}.$$

Nous avons les propriétés suivantes.

1.

$$w = \frac{v}{\sqrt{1-\epsilon^2 v^2}} \in \mathbb{R} \text{ est une carte croissante et injective de } (-1/\epsilon, 1/\epsilon) \text{ à } \mathbb{R}.$$

2. Avec la nouvelle inconnue $w \in \mathbb{R}$, l'équation (38) est équivalente à

$$\begin{aligned} &\partial_0 w + \partial_1 f_\epsilon(w) = 0, \\ &f_\epsilon(w) = \frac{1}{\epsilon^2}\Big(\pm \sqrt{1+\epsilon^2 w^2} - 1\Big), \end{aligned} \tag{40}$$

   où le flux f_ϵ est strictement convexe ou strictement concave et, par conséquent, la loi de conservation (38) est vraiment non-linéaire dans le sens que

$$\frac{\partial_v T^1(v)}{\partial_v T^0(v)} = f'_\epsilon(w)$$

   est strictement croissant ou strictement décroissant selon $T^0(v)$.
3. Dans la limite non relativiste $\epsilon \to 0$, on retrouve l'équation de type Burgers non visqueuse

$$\partial_0 u + \partial_1(u^2/2) = 0, \tag{41}$$

   où $u \in \mathbb{R}$.

Nous remarquons aussi que l'équation (40) proposée conserve plusieurs principales caractéristiques des équations d'Euler relativistes :

- l'équation (40) est de forme conservative, de manière analogue à la conservation de la masse-énergie dans le système d'Euler,
- notre inconnue v est contrainte de se situer dans l'intervalle $(-1/\epsilon, 1/\epsilon)$ limitée par l'inverse du paramètre vitesse de la lumière, de manière analogue à la restriction imposée à la composante de la vitesse dans le système d'Euler,
- en envoyant la vitesse de la lumière à l'infini, on retrouve le modèle non-relativiste.

Effet de la géométrie

Pour davantage de simplicité dans la présentation, nous supposons que l'espace-temps et la loi de conservation admettent des symétries qui permettent une réduction de dimension 1+1. Nous supposons que la variété est décrite par un diagramme particulier et, après identification, nous définissons $M = \mathbb{R}_+ \times \mathbb{R}$. En coordonnées (x^0, x^1) avec $\partial_\alpha := \partial/\partial x^\alpha$ pour $\alpha = 0, 1$, les lois d'équilibre hyperboliques prennent la forme suivante

$$\partial_0(\omega\, T^0(v)) + \partial_1(\omega\, T^1(v)) = \omega\, S(v), \tag{42}$$

où $v : M \to \mathbb{R}$ est la fonction inconnue, $T^\alpha = T^\alpha(v)$ est le champ de flux et $S = S(v)$ la source sur M, lorsque $\omega = \omega(x) > 0$ est une fonction poids. Cette équation est *hyperbolique dans le sens de* $\partial/\partial x^1$ à condition que

$$\partial_v T^0(v) > 0. \tag{43}$$

On constate que la loi l'équilibre ci-dessus peut être réécrite sous la forme

$$\partial_0 v + \partial_1 f(v) = \widetilde{S}(v) \tag{44}$$

avec

$$\begin{aligned} \partial_v f(v) &:= \frac{\partial_v T^1(v)}{\partial_v T^0(v)}, \qquad \Omega := \ln \omega, \\ \widetilde{S}(v) &:= \frac{1}{\partial_v T^0(v)} \Big(S(v) - \partial_0 \Omega\, T^0(v) - \partial_1 \Omega\, T^1(v) \Big). \end{aligned} \tag{45}$$

Equations d'Euler relativistes avec une pression nulle

Les équations d'Euler relativistes avec une pression nulle, sont données par

$$\begin{aligned} \partial_0 \Big(\frac{\rho}{c^2 - v^2} \Big) + \partial_1 \Big(\frac{\rho\, v}{c^2 - v^2} \Big) = 0, \\ \partial_0 \Big(\frac{\rho\, v}{c^2 - v^2} \Big) + \partial_1 \Big(\frac{\rho\, v^2}{c^2 - v^2} \Big) = 0, \end{aligned} \tag{46}$$

où ρ est la densité, v la vélocité et c la vitesse de la lumière. Soient $c = 1/\epsilon$ et ρ considérée comme une constante. On peut alors réécrire les équations (46) par

$$\begin{aligned} \partial_0 \Big(\frac{1}{1 - \epsilon^2 v^2} \Big) + \partial_1 \Big(\frac{v}{1 - \epsilon^2 v^2} \Big) = 0, \\ \partial_0 \Big(\frac{v}{1 - \epsilon^2 v^2} \Big) + \partial_1 \Big(\frac{v^2}{1 - \epsilon^2 v^2} \Big) = 0. \end{aligned} \tag{47}$$

En appliquant un changement de variable

$$z = \frac{v}{1 - \epsilon^2 v^2} \quad \text{telle que} \quad v = v_\pm = \frac{-1 \pm \sqrt{1 + 4\epsilon^2 z^2}}{2\epsilon^2 z},$$

et en reformulant la seconde équation de (47), nous obtenons

$$\partial_0 z + \partial_1 \left(\frac{-1 \pm \sqrt{1 + 4\epsilon^2 z^2}}{2\, \epsilon^2} \right) = 0.$$

Nous rappelons maintenant l'équation (40) de Burgers relativiste proposée

$$\partial_0 w + \partial_1 \Big(\frac{-1 \pm \sqrt{1 + \epsilon^2 w^2}}{\epsilon^2} \Big) = 0.$$

Enfin, on peut constater que les deux équations sont équivalentes.

Schéma de volumes finis bien équilibré

Nous supposons que l'espace-temps courbe (1 + 1)-dimensionnelle (M, ω) est globalement hyperbolique, c'est-à-dire qu'il existe un feuilletage de M par hypersurfaces H_t, $t \in \mathbb{R}$, compactes et orientées, du type espace telles que

$$M = \bigcup_{t \in \mathbb{R}} H_t,$$

où chaque tranche a la topologie de $\mathbb{R}$. Nous supposons également que M est feuilletée par ces tranches.

Soit $\mathcal{T}^h = \bigcup_{K \in \mathcal{T}^h} K$ une triangulation de la variété M, composée d'éléments espace-temps K. Le bord ∂K d'un élément K contient exactement deux faces "de type espace" désignées par e_K^+ et e_K^-, ainsi que les éléments "de type temps"

$$e^0 \in \partial^0 K := \partial K \setminus \left\{ e_K^+, e_K^- \right\}.$$

Nous notons que $|K|$ et $|e|$ représentent les mesures de K et e.

Nous introduisons la méthode des volumes finis en faisant la moyenne de la loi d'équilibre (42) sur chaque élément $K \in \mathcal{T}^h$ de la triangulation, en intégrant en espace et en temps. Il vient

$$\int_K (\omega S) dV_M = \int_K \mathrm{div}^{\omega}(T(v))\, dV_M.$$

Enfin, nous trouvons le schéma de volumes finis

$$\omega_{e_K^+} |e_K^+| \overline{T}^0_{e_K^+}(v_K^+) = \omega_{e_K^-} |e_K^-| \overline{T}^0_{e_K^-}(v_K^-) - \sum_{e^0 \in \partial^0 K} |e^0| \omega_{e^0} q_{K,e^0}(v_K^-, v_{K_{e^0}}^-) + \omega_{\partial^0 K} |\partial^0 K|\, \overline{S}_{\partial^0 K}(v_K^-), \quad (48)$$

où nous avons introduit les approximations suivantes

$$\int_{e_K^-} T^0 dV_e \simeq |e_K^-| \overline{T}_{e_K^-}(v_K^-), \qquad \int_{e^0} T^1 dV_{e^0} \simeq |e^0|\, q_{K,e^0}(v_K^-, v_{K_{e^0}}^-),$$

et

$$\int_{\partial^0 K} S\, dV_{\partial^0 K} \simeq |\partial^0 K|\, \overline{S}_{\partial^0 K},$$

où

$$\overline{T}^0_e(v) := \frac{1}{|e|} \int_e T^0(v) dV_e, \qquad \overline{S}_e(v) := \frac{1}{|e|} \int_e S(v) dV_e.$$

Nous avons associé à chaque élément K et $e^0 \in \partial^0 K$ une fonction flux numérique $q_{K,e^0} : \mathbb{R}^2 \to \mathbb{R}$ localement lipschitzienne satisfaisant certaines hypothèses, à savoir les propriétés de consistance, de conservation et de monotonie.

De plus, si nous choisissons un système de coordonnées locales sur M avec un maillage cartésien, nous trouverons en coordonnées locales la méthode des volumes finis

$$\omega_{i,j}\,\overline{T}_{i+1,j} = \omega_{i,j}\,\overline{T}_{i,j} - \lambda\Big(\omega_{i,j+1/2}\,q_{i,j+1/2} - \omega_{i,j-1/2}\,q_{i,j-1/2}\Big) + \omega_{K_{i,j}}|\Delta x^0|\,\overline{S}_{i,j},$$

où

$$K_{i,j} := I_i \times J_j = (x^0_i, x^0_{i+1}) \times (x^1_{j-1/2}, x^1_{j+1/2}).$$

Des tests numériques mettent en évidence la convergence et la pertinence de ce schéma. Pour ces résultats, on se renvoie au Chapitre 4 .

Partie I

Convergence de la méthode de volumes finis sur une variété: deux approches

Chapitre 1

Approche basée sur une métrique*

Approach based on a metric

1.1 Introduction

We are interested in discontinuous solutions to nonlinear hyperbolic conservation laws posed on a globally hyperbolic Lorentzian manifold, and we introduce a class of first-order, monotone finite volume schemes which enjoy geometrically natural stability properties. In turn, we conclude that the proposed finite volume schemes converge (in a strong topology) toward entropy solutions to hyperbolic conservation laws. Recall that the well-posedness theory for nonlinear hyperbolic equations posed on a manifold was recently established by Ben-Artzi and LeFloch [6] and LeFloch and Okutmustur [34, 35]. On the other hand, our proof of convergence of the finite volume method can be viewed as a generalization to Lorentzian manifolds of the technique introduced by Cockburn, Coquel and LeFloch [12, 15] for the (flat) Euclidean setting and already extended to Riemannian manifolds by Amorim, Ben-Artzi, and LeFloch [2].

Major conceptual and technical difficulties arise in the analysis of partial differential equations posed on a Lorentzian manifold. Several new difficulties also appear when trying to generalize the convergence results in [2, 12, 15] to Lorentzian manifolds. Most importantly, a space and time triangulation must be introduced and the geometry of the manifold must be taken into account in the discretization. We point out that, on a Lorentzian manifold, one cannot canonically choose a preferred foliation by spacelike hypersurfaces in general, so that it is important for the discretization to be robust enough to allow for a large class of foliations and of spacetime triangulations. From the numerical analysis standpoint, it is challenging to design and analyze discretization schemes that are consistent with the geometry of the given manifold. Our

*En collaboration avec P. Amorim et P. G. LeFloch [3].

guide in deriving the necessary estimates was to ensure that all of our arguments are intrinsic in nature, and thus do not explicitly rely on a choice of local coordinates.

The main assumption of global hyperbolicity made on the given Lorentzian background is natural, and ensures that the manifold enjoys reasonable causality properties. Furthermore, the class of schemes considered in the present paper is quite general and, essentially, requires that the numerical flux functions are monotone. It encompasses a large class of spacetime triangulations, in which the elements may become degenerate (in the limit) in the spatial direction.

More specifically, we show here that the proposed finite volume schemes can be expressed as a convex decomposition of essentially one-dimensional schemes, and we derive a discrete version of entropy inequalities as well as sharp estimates on the entropy dissipation. Strong convergence towards an entropy solution follows from DiPerna's uniqueness theorem [18].

For another approach to conservation laws on manifolds we refer to Panov [40] and for high-order numerical methods to Rossmanith, Bale, and LeVeque [41] and the references therein. DiPerna's measure-valued solutions were used to establish the convergence of schemes by Szepessy [43, 44], Coquel and LeFloch [9, 10, 11], and Cockburn, Coquel, and LeFloch [12, 15]. For many related results and a review about the convergence techniques for hyperbolic problems, we refer to Tadmor [46] and Tadmor, Rascle, and Bagneiri [47]. Further hyperbolic models, including also a coupling with elliptic equations, as well as many applications were successfully investigated by Kröner [25], and Eymard, Gallouet, and Herbin [20]. For higher-order schemes, see the paper by Kröner, Noelle, and Rokyta [26]. Also, an alternative approach to the convergence of finite volume schemes was proposed by Westdickenberg and Noelle [50]. Finally, note that Kuznetsov's error estimate, established in [12, 15] in the Euclidian setting, was recently extended to hyperbolic conservation laws on manifolds [33].

An outline of this paper follows. In section 3.2, we state some preliminary results from the theory of conservation laws on manifolds. In section 1.3, we introduce a class of finite volume schemes, and state the assumptions made on the discretization. Next, we state our main convergence result in Theorem 1.5. In section 1.4, we gather several important remarks and examples of particular interest. In section 1.5 we derive various stability estimates, which are of independent interest and are also later used, in section 1.6, to conclude with the convergence proof for the proposed schemes.

1.2 Preliminaries on conservation laws on a Lorentzian manifold

We will need some existence results, for which we refer to [6, 34, 35]. Let $(\mathbf{M}, g)$ be a time-oriented, $(d+1)$-dimensional Lorentzian manifold. Here, g is a metric with signature $(-,+,\ldots,+)$, and we recall that tangent vectors $X \in T_p\mathbf{M}$ at a point $p \in \mathbf{M}$ can be separated into timelike vectors ($g(X,X) < 0$), null vectors ($g(X,X) = 0$), and spacelike vectors ($g(X,X) > 0$). The manifold is assumed to be time-oriented, so that we can distinguish between past-oriented and future-oriented vectors. The Levi-Cevita connection associated to g is denoted by ∇ and, for instance, allows us to define the divergence operator div_g. Finally we denote by dV_g the volume element associated with the metric g.

Following [6], a *flux-vector* on a manifold is defined as a vector field $f = f(\overline{u}, p)$ depending on a real parameter $\overline{u}$, and the *conservation law* on $(\mathbf{M}, g)$ associated with f reads

$$\mathrm{div}_g \Big(f(u,p) \Big) = 0, \qquad u : \mathbf{M} \to \mathbb{R}. \tag{1.1}$$

Moreover, the flux-vector f is said to be *geometry compatible* if

$$\mathrm{div}_g\, f(\overline{u}, p) = 0, \quad \overline{u} \in \mathbb{R},\ p \in \mathbf{M}, \tag{1.2}$$

and to be *timelike* if its u-derivative is a timelike vector field

$$g\Big(\partial_u f(\overline{u}, p), \partial_u f(\overline{u}, p)\Big) < 0, \qquad p \in \mathbf{M},\ \overline{u} \in \mathbb{R}. \tag{1.3}$$

We are interested in the initial-value problem associated with (1.1). So, we fix a spacelike hypersurface $\mathcal{H}_0 \subset \mathbf{M}$ and a measurable and bounded function u_0 defined on $\mathcal{H}_0$. Then, we search for a function $u = u(p) \in L^\infty(\mathbf{M})$ satisfying (1.1) in the distributional sense and such that the (weak) trace of u on $\mathcal{H}_0$ coincides with u_0, that is,

$$u_{|\mathcal{H}_0} = u_0. \tag{1.4}$$

It is natural to require that the vectors $\partial_u f(\overline{u}, p)$, which determine the propagation of waves in solutions of (1.1), are timelike and future-oriented.

We assume that the manifold $\mathbf{M}$ is *globally hyperbolic,* in the sense that there exists a foliation of $\mathbf{M}$ by spacelike, compact, oriented hypersurfaces $\mathcal{H}_t$ ($t \in \mathbb{R}$):

$$\mathbf{M} = \bigcup_{t \in \mathbb{R}} \mathcal{H}_t.$$

Any hypersurface $\mathcal{H}_{t_0}$ is referred to as a *Cauchy surface* in $\mathbf{M}$, while the family of slices $\mathcal{H}_t$ ($t \in \mathbb{R}$) is called an *admissible foliation associated with* $\mathcal{H}_{t_0}$. The future of the given hypersurface will be denoted by

$$\mathbf{M}_+ := \bigcup_{t \geq 0} \mathcal{H}_t.$$

Finally we denote by n^t the future-oriented, normal vector field to each $\mathcal{H}_t$, and by g^t the induced metric. Finally, along $\mathcal{H}_t$, we denote the normal component of a vector field X by X^t, thus $X^t := g(X, n^t)$. In the following, when there is no risk of confusion, we write $F(u)$ instead of $F(u, p)$.

Definition 1.1. *A flux $F = F(\overline{u}, p)$ is called a* convex entropy flux *associated with the conservation law* (1.1) *if there exists a convex function $U : \mathbb{R} \to \mathbb{R}$ such that*

$$F(\overline{u}, p) = \int_0^{\overline{u}} \partial_u U(u')\, \partial_u f(u', p)\, du', \qquad p \in \mathbf{M},\ \overline{u} \in \mathbb{R}.$$

A measurable and bounded function $u = u(p)$ is called an entropy solution *of conservation law* (1.1)–(1.2) *if the following* entropy inequality

$$\begin{aligned}&\int_{\mathbf{M}_+} g(F(u), \nabla_g \phi)\, dV_g + \int_{\mathbf{M}_+} (\mathrm{div}_g F)(u)\, \phi\, dV_g \\ &+ \int_{\mathcal{H}_0} g_0(F(u_0), n_0)\, \phi_{\mathcal{H}_0}\, dV_{g_0} - \int_{\mathbf{M}_+} U'(u)(\mathrm{div}_g f)(u)\, \phi\, dV_g \geq 0\end{aligned}$$

holds for all convex entropy flux $F = F(\overline{u}, p)$ and all smooth functions $\phi \geq 0$ compactly supported in $\mathbf{M}_+$.

In particular, the requirements in the above definition imply the inequality

$$\mathrm{div}_g \Big(F(u)\Big) - (\mathrm{div}_g F)(u) + U'(u)(\mathrm{div}_g f)(u) \leq 0$$

in the distributional sense. Next, denoting the space of integrable functions defined on the Riemannian slice $(\mathcal{H}_t, g_t)$ by $L^1_{g_t}(\mathcal{H}_t)$, from [6, 34, 35] we recall the following result.

Theorem 1.2 (Well-posedness theory for conservation laws on a manifold)**.** *Consider a conservation law* (1.1) *posed on a globally hyperbolic Lorentzian manifold* $\mathbf{M}$ *with compact slices. Let $\mathcal{H}_0$ be a Cauchy surface in $\mathbf{M}$, and $u_0 : \mathcal{H}_0 \to \mathbb{R}$ be a function in $L^\infty(\mathcal{H}_0)$. Then, the initial-value problem* (1.1)–(1.4) *admits a unique entropy solution $u = u(p) \in L^\infty_{\mathrm{loc}}(\mathbf{M}_+)$. Moreover, for every admissible foliation $\mathcal{H}_t$ the trace $u_{|\mathcal{H}_t} \in L^1_{g_t}(\mathcal{H}_t)$ exists as a Lipschitz continuous function of t. When the flux is geometry compatible, the functions*

$$\|F^t(u_{|\mathcal{H}_t})\|_{L^1_{g_t}(\mathcal{H}_t)},$$

are non-increasing in time, for any convex entropy flux F. Moreover, given any two entropy solutions u, v, the function

$$\|f^t(u_{|\mathcal{H}_t}) - f^t(v_{|\mathcal{H}_t})\|_{L^1_{g_t}(\mathcal{H}_t)}$$

is non-increasing in time.

Throughout the rest of this paper, a globally hyperbolic Lorentzian manifold is given, and we tackle the problem of the discretization of the initial value problem associated with the conservation law (1.1) and a given initial condition where $u_0 \in L^\infty(\mathcal{H}_0)$. In the present paper, we *do not* assume that the flux f is geometry compatible, and we refer to [35] for the generalization of the above theory. Throughout the present paper, we require the following *growth condition*: there exist constants $C_1, C_2 > 0$ such that for all $(\overline{u}, p) \in \mathbb{R} \times M$

$$|(\mathrm{div}_g f)(\overline{u}, p)| \leq C_1 + C_2 |\overline{u}|. \tag{1.5}$$

Two important remarks are in order.

- First of all, the terminology here differs from the one in the Riemannian (and Euclidean) cases, where the conservative variable is singled out. The class of conservation laws on a Riemannian manifold is recovered by taking $\mathbf{M} = \mathbb{R} \times \widetilde{M}$, where $(\widetilde{M}, \bar{g})$ is a Riemannian manifold and $f(\overline{u}, p) = (\overline{u}, \bar{f}(\overline{u}, p)) \in \mathbb{R} \times T_p\widetilde{M}$. We can then write $\mathrm{div}_g \big(f(u,p)\big) = \partial_t u + \mathrm{div}_{\bar{g}} \big(\bar{f}(u,p)\big)$.

- Second, in the Lorentzian case no time-translation property is available in general, contrary to the Riemannian case. Hence, no time-regularity is implied by the $L^1_{g_t}$ contraction property.

1.3 Formulation and main result

1.3.1 Definition of the finite volume schemes

Before we can state our main result we must introduce some notation and motivate the formulation of the finite volume schemes under consideration. We consider a spacetime triangulation $\mathcal{T}^h = \bigcup_{K \in \mathcal{T}^h} K$ of the manifold $\mathbf{M}_+$, which is made of (compact) spacetime elements K and satisfies the following conditions:

- The boundary ∂K of an element K is a piecewise smooth d-dimensional manifold without boundary, $\partial K = \bigcup_{e \subset \partial K} e$, and each d-dimensional element e is a smooth manifold with piecewise smooth boundary and is either everywhere timelike or everywhere spacelike. The outward unit normal to $e \in \partial K$ is denoted by $\mathbf{n}_{K,e}$.

- Each element K contains exactly two spacelike elements, with disjoint interiors, denoted by e_K^+ and e_K^-, such that the outward unit normals to K, $\mathbf{n}_{K,e_K^+}$ and $\mathbf{n}_{K,e_K^-}$, are future- and past-oriented, respectively. They will be called the *outflow* and the *inflow elements*, respectively.

- For each element K, the set of the *lateral elements* $\partial^0 K := \partial K \setminus \{e_K^+, e_K^-\}$ is non-empty and timelike.

- For every pair of distinct elements $K, K' \in \mathcal{T}^h$, the set $K \cap K'$ is either a common element of K, K' or else a submanifold with dimension at most $(d-1)$.
- $\mathcal{H}_0 \subset \bigcup_{K \in \mathcal{T}^h} \partial K$, where $\mathcal{H}_0$ is the initial Cauchy hypersurface.
- For every $K \in \mathcal{T}^h$, $\operatorname{diam} e_K^\pm \leq h$.

For computing the diameter above, we assume that some reference Riemannian metric is fixed on the Lorentzian manifold; such a metric can be easily introduced from the expression of the Lorentzian metric (by replacing the signature $(-,+,\dots,+)$ by $(+,+,\dots,+)$ in the expression in local coordinates). Given an element K, we denote the unique element distinct from K sharing the element e_K^+ (resp. e_K^-) with K by K^+ (resp. K^-), and for each $e^0 \in \partial^0 K$, we denote the unique element sharing the element e^0 with K by K_{e^0}. In addition, it is convenient to assume that the boundary of the element does not widely "oscillate", in the sense that for all smooth vector fields X defined on e_K^+,

$$\|g_p(X(p), \mathbf{n}_{K,e_K^+}(p))\|_{C^2(e_K^+)} \lesssim \|X(p)\|_{C^2(e_K^+)}, \tag{1.6}$$

where the implied constant (in $\lesssim$) is fixed once for all. This condition is intended to rule out oscillations on the normal vector field due to the geometry of e_K^+. It restricts the variation of the normal on each element e_K^+, but *not* the variation from one element to the next.

The most natural way of introducing the finite volume method is to view the discrete solution as defined on the spacelike elements $e_K^\pm$ separating two elements. So, to a particular element K we may associate two values, u_K^+ and u_K^- associated to the unique outflow and inflow elements e_K^+, e_K^-. Then, one may determine that the value u_K of the discrete solution on the element K is the solution u_K^+ determined on the inflow element e_K^- (one could just as well say that u_K is the solution u_K^+ determined on the outflow element e_K^+, or some average of the two, as long as one does this coherently throughout the manifold).

Thus, for any element K, integrate equation (1.1), apply the divergence theorem and decompose the boundary ∂K into its parts e_K^+, e_K^-, and $\partial^0 K$:

$$\begin{aligned} &-\int_{e_K^+} g_p(f(u,p), \mathbf{n}_{K,e_K^+}(p))\, dV_g - \int_{e_K^-} g_p(f(u,p), \mathbf{n}_{K,e_K^-}(p))\, dV_g \\ &+ \sum_{e^0 \in \partial^0 K} \int_{e^0} g_p(f(u,p), \mathbf{n}_{K,e^0}(p)) dp = 0. \end{aligned} \tag{1.7}$$

For any hypersurface $e \subset M$, we will often denote simply by $dV_e = dV_{g_e}$ the volume element of the induced metric g_e associated with the Lorentzian metric

g. Note the minus sign in the first two terms which comes from the fact that, for a Lorentzian manifold, the divergence theorem reads

$$\int_{\Omega} \mathrm{div}_g\, f dV_{\Omega} = \int_{\partial\Omega} g(f, \tilde{\mathbf{n}}) dV_{\partial\Omega},$$

in which $\tilde{\mathbf{n}}$ is the outward normal if it is spacelike, and the inward normal if it is timelike. This formula is nothing but the standard divergence theorem, with the signs of the normals properly taken into account.

Given an element K, we want to compute an approximation u_K^+ of the average of $u(p)$ in the outflow element e_K^+, given the values of u_K^- on e_K^- and of $u_{K_{e^0}}^-$ for each $e^0 \in \partial^0 K$.

The following notation will be useful. Let f be a flux on the manifold $\mathbf{M}$, K an element of the triangulation, and $e \subset \partial K$, respectively. Define the function $\mu_{K,e}^f : \mathbb{R} \to \mathbb{R}$ by

$$\mu_{K,e}^f(u) := \frac{1}{|e|} \int_e g_p\Big(f(u,p), \mathbf{n}_{K,e}(p)\Big)\, dV_e = \fint_e g_p\Big(f(u,p), \mathbf{n}_{K,e}(p)\Big)\, dV_e, \tag{1.8}$$

where $|e|$ is the measure of e. Also, if $w : \mathbf{M} \to \mathbb{R}$ is a real-valued function, we write

$$\mu_e^w := \fint_e w(p)\, dV_e.$$

Using this notation, the second term in (1.7) is approximated by

$$\int_{e_K^-} g_p\Big(f(u,p), \mathbf{n}_{K,e_K^-}(p)\Big)\, dV_g \simeq |e_K^-| \mu_{K,e_K^-}^f(u_K^-),$$

and the last term is approximated by using

$$\int_{e^0} g_p(f(u,p), \mathbf{n}_{K,e^0}(p))\, dV_{e^0} \simeq |e^0|\, \mathbf{q}_{K,e^0}(u_K^-, u_{K_{e^0}}^-),$$

where to each element K, and each element $e^0 \in \partial^0 K$ we associate a locally Lipschitz *numerical flux function* $\mathbf{q}_{K,e^0}(u,v) : \mathbb{R}^2 \to \mathbb{R}$ satisfying certain assumptions listed below.

Therefore, in view of the above approximation formulas we may write, as a discrete approximation of (1.7),

$$|e_K^+| \mu_{K^+,e_K^+}^f(u_K^+) := |e_K^-| \mu_{K,e_K^-}^f(u_K^-) - \sum_{e^0 \in \partial^0 K} |e^0|\, \mathbf{q}_{K,e^0}(u_K^-, u_{K_{e^0}}^-), \tag{1.9}$$

which is the *finite volume method* of interest and, equivalently

$$u_K^+ := (\mu_{K^+,e_K^+}^f)^{-1}\Big(\frac{|e_K^-|}{|e_K^+|} \mu_{K,e_K^-}^f(u_K^-) - \sum_{e^0 \in \partial^0 K} \frac{|e^0|}{|e_K^+|}\, \mathbf{q}_{K,e^0}(u_K^-, u_{K_{e^0}}^-)\Big). \tag{1.10}$$

The second formula which may be carried out numerically (using for instance a Newton algorithm) is justified by the following observation:

Lemma 1.3. *For any $K \in \mathcal{T}^h$, the function $u \mapsto \mu^f_{K,e^-_K}(u)$ is monotone increasing.*

Proof. From (1.8) we deduce that

$$\partial_u \mu^f_{K,e^-_K}(u) = \fint_e g_p\Big(\partial_u f(u,p), \mathbf{n}_{K,e^-_K}(p)\Big)\, dV_e > 0,$$

since $\partial_u f(u,p)$ is future-oriented and $\mathbf{n}_{K,e^-_K}$ is past-oriented. □

Now, if $e^-_K \subset \mathcal{H}_0$, the initial condition (1.4) gives

$$u^-_K := \mu^{u_0}_{e^-_K} = \fint_{e^-_K} u_0(p)\, dV_{e^-_K}. \tag{1.11}$$

Finally, we define the function $u^h : \mathbf{M} \to \mathbb{R}$ by

$$u^h(p) := u^-_K, \qquad p \in K. \tag{1.12}$$

On the other hand, for all $e \in \partial K$ we introduce the notation

$$f_e(u,p) := g_p(f(u,p), \mathbf{n}_{K,e}(p)). \tag{1.13}$$

1.3.2 Assumptions on the numerical flux

For the numerical flux $\mathbf{q}_{K,e^0}(u,v) : \mathbb{R}^2 \to \mathbb{R}$ we impose the following properties.

- *Consistency property :*

$$\mathbf{q}_{K,e^0}(u,u) = \fint_{e^0} f_{e^0}(u,p)\, dV_g = \mu^f_{K,e^0}(u). \tag{1.14}$$

- *Conservation property :*

$$\mathbf{q}_{K,e^0}(u,v) = -\mathbf{q}_{K_{e^0},e^0}(v,u), \qquad u,v \in \mathbb{R}. \tag{1.15}$$

- *Monotonicity property :*

$$\partial_u \mathbf{q}_{K,e^0}(u,v) \geq 0, \quad \partial_v \mathbf{q}_{K,e^0}(u,v) \leq 0. \tag{1.16}$$

For each element K, define the time-increment

$$\tau_K = \frac{|K|}{|e^+_K|},$$

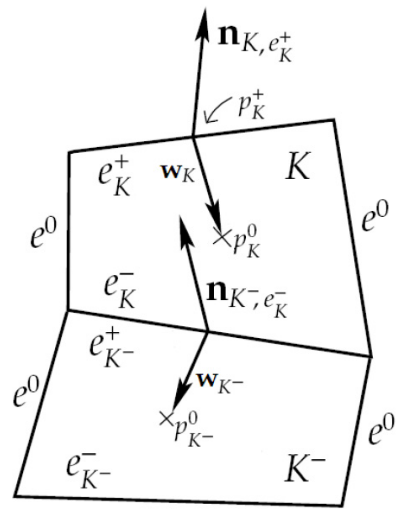

Figure 1.1: Elements of triangulation

where $|K|$ is the ($(d+1)$-dimensional) measure of K. We suppose that, as $h \to 0$,

$$\tau := \max_K \tau_K \to 0 \tag{1.17}$$

and

$$\frac{h^2}{\min_K \tau_K} \to 0. \tag{1.18}$$

For stability purposes, we also impose the following *CFL condition,* for all $K \in \mathcal{T}^h$, $e^0 \in \partial_0 K$,

$$\frac{|\partial^0 K|}{|e_K^+|} \sup_{u\in\mathbb{R}} \left|\partial_u \mu^f_{K,e^0}(u)\right| \sup_{u\in\mathbb{R}} \partial_u (\mu^f_{K^+,e_K^+})^{-1}(u) \leq 1. \tag{1.19}$$

1.3.3 Assumptions on the triangulation and main convergence result

For the triangulation $\mathcal{T}^h$, we will introduce an admissibility condition that is global and geometric in nature, and which is essentially optimal to ensure the convergence of the proposed schemes (see the following subsection for details). The condition only involves the time-evolution of the triangulation and is independent of the structure of the triangulation on spacelike elements. We stress that our method poses almost no restriction on the spacelike structure of the discretization.

The following notation will be used throughout this paper (see Figure 1.1). Let $K \in \mathcal{T}^h$. We denote by p_K^0 and p_K^+ the centers of mass of $\partial^0 K$ and e_K^+, respectively. Note that p_K^0 does not lie on $\partial^0 K$ (or even "close" to it), and that p_K^+ does not necessarily lie on e_K^+. Next, we define the vector $\mathbf{w}_K \in T_{p_K^+}\mathbf{M}$ as the vector at p_K^+ tangent to the geodesic line connecting p_K^+ to p_K^0 and with length $\mathrm{dist}(p_K^+, p_K^0)$ given by the reference Riemannian metric. This vector is well-defined if the discretization parameter h is small enough.

We will also make the following assumption on the triangulation, namely that for h sufficiently small, and for each element e_K^+, we may extend the normal vector field $\mathbf{n}_{K,e_K^+}$ by parallel transporting it (using the metric structure on the manifold) to a neighborhood of e_K^+ containing p_K^+. This is a natural assumption, which ensures that each element e_K^+ of the triangulation tends to become flat in the limit.

Consider the quantity

$$\mathcal{E}(K) := \frac{1}{\tau_K}\mathbf{w}_K \otimes \mathbf{n}_{K,e_K^+},$$

which can be viewed as a quadratic form on $T_{p_K^+}\mathbf{M}$. If X, Y are two vectors at the point p_K^+, we have

$$\mathcal{E}(K)(X,Y) = \frac{1}{\tau_K}\mathbf{w}_K \otimes \mathbf{n}_{K,e_K^+}(X,Y) = \frac{1}{\tau_K} g_{p_K^+}(X, \mathbf{w}_K)\, g_{p_K^+}(Y, \mathbf{n}_{K,e_K^+}).$$

We define the *local deviation* associated with K, K^- by

$$|K|\mathcal{E}(K) - |K^-|\mathcal{E}(K^-),$$

which measures the rate of change of the quantity $|K|\mathcal{E}(K)(X,Y)$ with respect to the timelike direction defined locally by the normals to the elements $e_K^\pm$. Our admissibility criterion below requires that this rate of change should tend to zero with h (after summation over all $K \in \mathcal{T}^h$).

Definition 1.4. *A triangulation $\mathcal{T}^h$ is called an admissible triangulation if for every vector field Φ with compact support and every family of smooth vector fields Ψ_K, $K \in \mathcal{T}^h$ (each of them being defined on the manifold and associated with a given K), the local deviation satisfies*

$$\Big| \sum_{K\in\mathcal{T}^h} |K|\mathcal{E}(K)(\Phi,\Psi_K) - |K^-|\mathcal{E}(K^-)(\Phi,\Psi_K)\Big| \lesssim \eta(h)\, \|\Phi\|_{L^\infty} \sup_K \|\Psi_K\|_{L^\infty} \tag{1.20}$$

for some fixed function $\eta(h)$ with $\eta(h) \to 0$.

Observe that, in (1.20), the vector $\mathbf{n}_{K,e_K^+}(p_K^+) \in T_{p_K^+}\mathbf{M}$ makes sense, since the normal vector field was extended to include a neighborhood of e_K^+. The assumption (1.20) is a global geometric condition on the local deviation of the triangulation. As further discussed in Section 1.4 below, this condition allows us to encompass a large class of implementable spacetime triangulations.

Finally, we are in a position to state the following theorem.

Theorem 1.5 (Convergence of the finite volume schemes)**.** *Let u^h be the sequence of functions generated by the finite volume method* (1.9)–(1.12) *on an admissible triangulation, with initial data $u_0 \in L^\infty(\mathcal{H}_0)$, with numerical flux satisfying the conditions* (1.14)–(1.16), *and the CFL condition* (1.19)*. Then, for every $T > 0$ the sequence u^h is uniformly bounded in $L^\infty\Big(\bigcup_{t\in[0,T]}\mathcal{H}_t\Big)$ in terms of the sup-norm of the initial data, and converges almost everywhere (when $h \to 0$) towards the unique entropy solution $u \in L^\infty_{loc}(\mathbf{M}_+)$ to the Cauchy problem* (1.1), (1.4).

In Section 1.5 below, we will derive the key estimates required for the proof of Theorem 1.5 which will be finally be given in Section 1.6. We follow here the strategy originally developed by Cockburn, Coquel and LeFloch [12, 15] for conservation laws posed on a fixed (time-independent) Euclidian background. New estimates are required here to take into account the geometric effects and, especially, during the time evolution in the scheme. We will start with local (both in time and in space) entropy estimates, and next deduce a global-in-space entropy inequality. We will also establish the L^∞ stability of the scheme and, finally, the global (spacetime) entropy inequality required for the convergence proof.

1.4 Examples and remarks on our assumptions

1.4.1 Admissible triangulations and lack of total variation estimate

Our assumption on the triangulation is essentially optimal. We argue by describing the setting in which the condition (1.20) will actually be used within our proof of Theorem 1.5. We also provide evidence that, in general, the finite volume method may not converge without this assumption.

In the proof of convergence (see Section 1.6), it is necessary to bound a term of the form

$$A^h(X) = \sum_{K\in\mathcal{T}^h}\Big(|e_K^+|g\Big(\mathbf{w}_K, X(p_K^+)\Big)g\Big(F(u_K^-,p_K^+),\mathbf{n}_{K,e_K^+}\Big) + |e_K^-|g\Big(\mathbf{w}_{K^-}, X(p_K^-)\Big)g\Big(F(u_K^-,p_K^-),\mathbf{n}_{K,e_K^-}\Big)\Big),$$

where X is a smooth vector field, $p_K^\pm$ is the center of mass of $e_K^\pm$, and the sum is taken over the whole spacetime triangulation. Recall that the vector $\mathbf{w}_K$ was defined earlier in this section. The above term must vanish in the limit for the finite volume schemes to converge and, furthermore, is an entirely new term that *does not arise* in the Euclidean nor Riemannian settings.

Note that both terms in the expression $A^h(X)$ involve $F(u_K^-,\cdot)$ and, consequently, the terms cannot be cancelled by re-ordering the expression. Therefore, if one were to integrate by parts the (discrete) sums, we would find ourselves in need of the uniform bound

$$\sum_{K \in \mathcal{T}^h} h \Big| |e_K^+| u_K^+ - |e_K^-| u_K^- \Big| = o(1). \tag{1.21}$$

However, it is well-known that this BV (bounded variation) time estimate is a very difficult open problem in the numerical analysis of finite volume schemes. Indeed, deriving (1.21) is open, even in the simplest Euclidean setting whenever the spatial discretization is not Cartesian.

On the other hand, one key observation made by Cockburn, Coquel, and LeFloch [12, 15] was that (1.21) is not necessary for the analysis of the convergence of the finite volume method, provided one considers L^∞ solutions rather than solutions with bounded variation.

The notion of admissible triangulation introduced in the present paper supplements the observation in [12, 15] and provides the precise condition ensuring the convergence of the schemes. In the Euclidian or Riemannian cases, our admissibility condition imposes no new constraint on triangulations. In view of our condition in Definition 1.4, it is easily checked that the term $A^h(X)$ converges to zero. Indeed, recalling the definition of u^h, we find

$$\begin{aligned}|A^h(X)| &= \Big| \sum_{K^n \in \mathcal{T}^h} \Big(|K| \mathcal{E}(K) - |K^-| \mathcal{E}(K^-) \Big)(X, F(u^h)) \Big| \\ &\lesssim \eta(h) \to 0.\end{aligned}$$

Thus, no control on the total variation of the discrete solution is required, and instead the proposed geometric condition on the triangulation suffices. See Section 1.4.4 for a further discussion.

1.4.2 Foliation by hypersurfaces and choice of triangulations

Our analysis is valid for any time-evolution that one may want to choose for the discretization, provided the assumptions on the triangulation in Definition 1.4 are met. These assumptions are independent of the actual foliation of the manifold appear to be essentially optimal, within the framework developed in the present paper. In fact, our method of proof is not tied to any particular time structure on the manifold – it only supposes that such a structure exists, which is a completely general assumption required for solving the initial value problem.

In particular, if a certain hypersurface $\mathcal{H}$ belongs to a given triangulation $\mathcal{T}^h$ (for some h), then this hypersurface need not be included in the triangulations $\mathcal{T}^{h'}$ with $h' < h$. That is to say, the discretization is *not* associated with any a priori fixed foliation, nor does the relation $\mathcal{T}^{h'} \subset \mathcal{T}^h$ hold for $h' < h$.

On the other hand, in Proposition 1.6 below, we are going to examine the special case where the triangulation is subordinate to a given foliation, and prove that it is admissible in the sense of Definition 1.4.

Furthermore, our formulas do coincide with the formulas already known in the Riemannian and Euclidean cases. In these cases, the function $\mu^f_{K,e^-_K}(u)$ coincides with the identity function $\mu^f_{K,e^-_K}(u) = u$ and, therefore, the finite volume scheme reduces to the scheme studied in [2, 12, 15]. Also, our expression for the time increment τ and the CFL condition (1.19) reduce to the usual formulas when specialized to the Euclidean or Riemannian setting.

1.4.3 Choice of flux-functions

Examples of scalar equations can be exhibited by taking any smooth, timelike vector field X and any smooth real function $\widetilde{f}(u)$ and setting $f(u,p) := X(p)\widetilde{f}(u)$. The conservation law then reads $\operatorname{div}_g\left(X(p)\bar{f}(u(p))\right) = 0$, and the flux is non-trivial and involves the geometry of the manifold.

In the interest of practical implementation, one may replace the right-hand side of the equations (1.8) and (1.11) with more realistic averages. For instance, one could take an average of $g(f(u,p), \mathbf{n}_{K,e})$ over N spatial points p_j given from some partition e^j of e,

$$\mu^f_{K,e}(u) = \frac{1}{|e|}\sum_{j=1}^{N} |e^j| g(f(u,p_j), \mathbf{n}_{K,e}(p_j)).$$

Hence, more generally, one could fix an averaging operator μ^f_{K,e^-_K}, and then use the equation (1.10) to iterate the method, with initial data given by

$$u^-_K := \mu^{u_0}_{e^-_K}.$$

However, any such average is just an approximation of the integral expression used in (1.8). This approximation can be chosen to be of arbitrary high-order in the parameter h, by choosing appropriate quadrature formulas. For the sake of clarity, we will present the proofs with the choice $\mu^f_{K,e}$ defined by (1.8) and we will omit the (straightforward) treatment of the error terms issuing from the above approximations.

As an example of numerical flux, one can consider the following generalization of the Lax–Friedrichs flux,

$$\mathbf{q}_{K,e^0}(u,v) = \frac{1}{2}\Big(\mu^f_{K,e^0}(u) + \mu^f_{K,e^0}(v)\Big) + \frac{D_{K,e^0}}{2}(u-v), \tag{1.22}$$

where the constants D_{K,e^0} satisfy $D_{K,e^0} = D_{K_{e^0},e^0}$ and

$$D_{K,e^0} \geq \frac{|e^+_K|}{|\partial^0 K|}\Big(\sup_{u\in\mathbb{R}} \partial_u(\mu^f_{K^+,e^+_K})^{-1}(u)\Big)^{-1}.$$

This numerical flux is conservative and consistent, and it is monotone, as may be checked using the CFL condition (1.19).

1.4.4 A class of examples based on a geometric condition

We provide here an explicit condition which is geometric in nature and suffices for a triangulation to be admissible in the sense of (1.20). Recall that $p_K^\pm$ denotes the center of mass of $e_K^\pm$ and that the vector $\mathbf{w}_K$ denotes the tangent at p_K^+ to the geodesic from p_K^+ to the center of $\partial^0 K$.

Proposition 1.6. *Let $\mathcal{T}^h$ be a triangulation and suppose that, for each element K, the rescaled exterior normals $|e_K^+|\mathbf{n}_{K,e_K^+}$ and $|e_K^-|\mathbf{n}_{K,e_K^-}$ and the vectors $\mathbf{w}_K$ and $\mathbf{w}_{K^-}$ satisfy the following conditions: for every smooth vector field X,*

$$\Big|g(|e_K^+|\mathbf{n}_{K,e_K^+}, X) - g(|e_K^-|\mathbf{n}_{K,e_K^-}, X)\Big| \lesssim \frac{\eta(h)}{h}\,|K|\,\|X\|_{L^\infty(K)}, \tag{1.23}$$

$$\Big|g(\mathbf{w}_K, X) - g(\mathbf{w}_{K^-}, X)\Big| \lesssim \eta(h)\,\tau_K\,\|X\|_{L^\infty(K)}, \tag{1.24}$$

where the expressions under consideration are evaluated at the centers of mass of e_K^+ and e_K^-, and $\eta(h)$ is such that $\eta(h) \to 0$. Then, $\mathcal{T}^h$ is an admissible triangulation in the sense of (1.20).

For instance, one can easily check that if a triangulation is subordinate to a given foliation (in the sense that the set of all outgoing elements $\{e_K^+ : K \in \mathcal{T}^h\}$ is contained in a certain Cauchy surface), and if, moreover, each lateral element e^0 is everywhere tangent to a given, fixed, smooth timelike vector field, then the hypotheses of Proposition 1.6 hold. However, our condition (1.20) or the ones in Proposition 1.6 allow for more general triangulations, which need not satisfy such regularity assumptions.

Proof. Let Φ be a smooth vector field and, for each K, let Ψ_K be a family of smooth vector fields defined on $\mathbf{M}$. We have

$$\begin{aligned}
&\sum_{K\in\mathcal{T}^h} |K|\mathcal{E}(K)(\Phi,\Psi_K) - |K^-|\mathcal{E}(K^-)(\Phi,\Psi_K)\\
&= \sum_{K\in\mathcal{T}^h} \Big(|e_K^+|\mathbf{w}_K \otimes \mathbf{n}_{K,e_K^+} - |e_K^-|\mathbf{w}_{K^-} \otimes \mathbf{n}_{K,e_K^-}\Big)(\Phi,\Psi_K)\\
&= \sum_{K\in\mathcal{T}^h} g(\mathbf{w}_K,\Phi)\, g(|e_K^+|\mathbf{n}_{K,e_K^+},\Psi_K) - g(\mathbf{w}_{K^-},\Phi)\, g(|e_K^-|\mathbf{n}_{K,e_K^-},\Psi_K),
\end{aligned}$$

thus

$$\begin{aligned}
&\left|\sum_{K\in\mathcal{T}^h}|K|\mathcal{E}(K)(\Phi,\Psi_K)-|K^-|\mathcal{E}(K^-)(\Phi,\Psi_K)\right|\\
&=\Big|\sum_{K\in\mathcal{T}^h}g(\mathbf{w}_{K^-},\Phi)\Big(g(|e_K^+|\mathbf{n}_{K,e_K^+},\Psi_K)-g(|e_K^-|\mathbf{n}_{K,e_K^-},\Psi_K)\Big)\\
&\qquad+\Big(g(\mathbf{w}_K,\Phi)-g(\mathbf{w}_{K^-},\Phi)\Big)g(|e_K^+|\mathbf{n}_{K,e_K^+},\Psi_K)\Big|\\
&\lesssim\eta(h)\sum_{K\in\mathcal{T}^h}|K|\|\Phi\|_{L^\infty}\|\Psi_K\|_{L^\infty}\lesssim\eta(h)\|\Phi\|_{L^\infty}\sup_K\|\Psi_K\|_{L^\infty}.
\end{aligned}$$

In view of (1.20), this shows that $\mathcal{T}^h$ is an admissible triangulation. □

1.5 Discrete entropy estimates

1.5.1 Local entropy dissipation and entropy inequalities

We now introduce some notation which will simplify the statement of the results as well as the proofs. By defining

$$\mu_K^+(u):=\mu^f_{K^+,e_K^+}(u)=-\mu^f_{K,e_K^+}(u),$$

$$\mu_K^-(u):=\mu^f_{K,e_K^-},$$

the finite volume method (1.10) reads as

$$|e_K^+|\mu_K^+(u_K^+)=|e_K^-|\mu_K^-(u_K^-)-\sum_{e^0\in\partial^0K}|e^0|\mathbf{q}_{K,e^0}(u_K^-,u_{K_{e^0}}^-). \tag{1.25}$$

As in [2, 12, 15], we rely on a convex decomposition of $\mu_K^+(u_K^+)$, which allows us to control the entropy dissipation.

Define $\tilde{\mu}^+_{K,e^0}$ by the identity

$$\tilde{\mu}^+_{K,e^0}:=\mu_K^+(u_K^-)-\frac{|\partial^0K|}{|e_K^+|}\Big(\mathbf{q}_{K,e^0}(u_K^-,u_{K_{e^0}}^-)-\mathbf{q}_{K,e^0}(u_K^-,u_K^-)\Big),$$

and define

$$\overline{\mu}^+_{K,e^0}:=\tilde{\mu}^+_{K,e^0}-\frac{1}{|e_K^+|}\int_K\operatorname{div}_g f(u_K^-,p)\,dV_K. \tag{1.26}$$

Then, one has the following convex decomposition of $\mu_K^+(u_K^+)$, whose proof is immediate from (1.25).

$$\mu_K^+(u_K^+)=\frac{1}{|\partial^0K|}\sum_{e^0\in\partial^0K}|e^0|\overline{\mu}^+_{K,e^0}. \tag{1.27}$$

Lemma 1.7. *Let $(U(u), F(u,p))$ be a convex entropy pair (cf. Definition 1.1). For each K and for each $e = e_K^-, e_K^+$, let $V_{K,e} : \mathbb{R} \to \mathbb{R}$ be the convex function defined by*

$$V_{K,e}(a) := \mu_{K,e}^F\Big((\mu_{K,e}^f)^{-1}(a)\Big), \quad a \in \mathbb{R}. \tag{1.28}$$

Then there exists a family of numerical entropy fluxes $\mathbf{Q}_{K,e^0}(u,v) : \mathbb{R}^2 \to \mathbb{R}$ satisfying the following conditions.

- *$\mathbf{Q}_{K,e^0}$ is consistent with the entropy flux F:*

$$\mathbf{Q}_{K,e^0}(u,u) = \mu_{K,e^0}^F(u), \qquad K \in \mathcal{T}^h, e^0 \in \partial^0 K, u \in \mathbb{R}.$$

- *Conservation property:*

$$\mathbf{Q}_{K,e^0}(u,v) = -\mathbf{Q}_{K_{e^0},e^0}(v,u), \qquad u,v \in \mathbb{R}.$$

- *Discrete entropy inequality:*

$$\begin{aligned} &V_{K^+,e_K^+}\Big(\tilde{\mu}_{K,e^0}^+\Big) - V_{K^+,e_K^+}\Big(\mu_K^+(u_K^-)\Big) \\ &\quad + \frac{|\partial^0 K|}{|e_K^+|}\Big(\mathbf{Q}_{K,e^0}(u_K^-, u_{K_{e^0}}^-) - \mathbf{Q}_{K,e^0}(u_K^-, u_K^-)\Big) \le 0. \end{aligned} \tag{1.29}$$

From the inequality (1.29) we infer that

$$\begin{aligned} &V_{K^+,e_K^+}(\overline{\mu}_{K,e^0}^+) - V_{K^+,e_K^+}\Big(\mu_K^+(u_K^-)\Big) \\ &\quad + \frac{|\partial^0 K|}{|e_K^+|}\Big(\mathbf{Q}_{K,e^0}(u_K^-, u_{K_{e^0}}^-) - \mathbf{Q}_{K,e^0}(u_K^-, u_K^-)\Big) \le R_{K,e^0}^+, \end{aligned} \tag{1.30}$$

where R_{K,e^0}^+ is given by

$$R_{K,e^0}^+ := V_{K^+,e_K^+}(\overline{\mu}_{K,e^0}^+) - V_{K^+,e_K^+}(\tilde{\mu}_{K,e^0}^+). \tag{1.31}$$

Proof. To begin, we prove that the functions $V_{K,e}$ in (1.28) are indeed convex. First, note that it is sufficient to show that

$$V_{K,e}(\mu) = \int^{\mu} U'\Big((\mu_{K,e}^f)^{-1}(\sigma)\Big) d\sigma. \tag{1.32}$$

Indeed, using the convexity of U and the monotonicity of $(\mu_{K,e}^f)^{-1}$, for $e = e_K^+, e_K^-$ (cf. Lemma 1.3), the convexity of $V_{K,e}$ follows by differentiating this expression twice. To prove (1.32), note that setting $\alpha = (\mu^f_{K,e})^{-1}(\sigma)$, we find

$$\begin{aligned}\int^{\mu} U'\Big((\mu^f_{K,e})^{-1}(\sigma)\Big)d\sigma &= \int^{(\mu^f_{K,e})^{-1}(\mu)} U'(\alpha)\partial_\alpha \mu^f_{K,e}(\alpha)d\alpha \\ &= \fint_e g\Big(\int^{(\mu^f_{K,e})^{-1}(\mu)} U'(\alpha)\partial_\alpha f(\alpha,p)d\alpha, \mathbf{n}_{K,e}(p)\Big)\,dV_e \\ &= \fint_e g\Big(F((\mu^f_{K,e})^{-1}(\mu),p), \mathbf{n}_{K,e}(p)\Big)\,dV_e = \mu^F_{K,e}\Big((\mu^f_{K,e})^{-1}(\mu)\Big),\end{aligned}$$

which establishes (1.32).

We now proceed with the proof of the lemma. First of all, note that using (1.28) we may write the inequality (1.29) equivalently as

$$\begin{aligned}&\mu^F_{K^+,e^+_K}\Big((\mu^f_{K^+,e^+_K})^{-1}(\tilde\mu^+_{K,e^0})\Big) - \mu^F_{K^+,e^+_K}(u^-_K) \\ &\qquad + \frac{|\partial^0 K|}{|e^+_K|}\Big(\mathbf{Q}_{K,e^0}(u^-_K, u^-_{K_{e^0}}) - \mathbf{Q}_{K,e^0}(u^-_K,u^-_K)\Big) \le 0.\end{aligned} \tag{1.33}$$

Indeed, we have for instance

$$V_{K^+,e^+_K}\Big(\mu^+_K(u^-_K)\Big) = \mu^F_{K^+,e^+_K}\Big((\mu^f_{K^+,e^+_K})^{-1}(\mu^+_K(u^-_K))\Big) = \mu^F_{K^+,e^+_K}(u^-_K).$$

Next, introduce the following operator. For $u, v \in \mathbb{R}$, $e^0 \in \partial^0 K$, let

$$H_{K,e^0}(u,v) := \mu^+_K(u) - \frac{|\partial^0 K|}{|e^+_K|}\Big(\mathbf{q}_{K,e^0}(u,v) - \mathbf{q}_{K,e^0}(u,u)\Big).$$

We claim that H_{K,e^0} satisfies the following properties:

$$\frac{\partial}{\partial u}H_{K,e^0}(u,v) \ge 0, \qquad \frac{\partial}{\partial v}H_{K,e^0}(u,v) \ge 0, \tag{1.34}$$

$$H_{K,e^0}(u,u) = \mu^+_K(u). \tag{1.35}$$

The second and last properties are immediate. The first is a consequence of the CFL condition (1.19) and the monotonicity of the method. Indeed, from the definition of $H_{K,e^0}(u,v)$ we may perform exactly the same calculation as in the proof of Lemma 1.8 to prove that $H_{K,e^0}(u,v)$ is a convex combination of $\mu^+_K(u)$ and $\mu^+_K(v)$, which in turn are increasing functions. This establishes the first inequality in (1.34).

We now turn to the proof of the entropy inequality (1.33). Suppose first that (1.33) is already established for the Kruzkov family of entropies $\overline{U}(u,\lambda) = |u-\lambda|$, $\overline{F}(u,\lambda,p) = \operatorname{sgn}(u-\lambda)(f(u,p) - f(\lambda,p))$, $\lambda \in \mathbb{R}$. In this case, the Kruzkov numerical entropy flux are given by

$$\overline{\mathbf{Q}}_{K,e^0}(u,v,\lambda) := \mathbf{q}_{K,e^0}(u\vee\lambda, v\vee\lambda) - \mathbf{q}_{K,e^0}(u\wedge\lambda, v\wedge\lambda),$$

where $a \vee b = \max(a,b)$, and $a \wedge b = \min(a,b)$. It is easy to check that $\mathbf{Q}_{K,e^0}$ satisfies the first two conditions of the lemma.

We now show that it is enough to prove inequality (1.33) for Kruzkov's entropies only. Indeed, if U is a smooth function which is linear at infinity, we have (formally)

$$\begin{aligned}\frac{1}{2}\int_{\mathbb{R}} \overline{U}(u,\lambda)U''(\lambda)d\lambda &= \frac{1}{2}\int_{\mathbb{R}} \overline{U}''(u,\lambda)U(\lambda)d\lambda \\ &= \frac{1}{2}\langle \delta_{\lambda=u}, U(\lambda)\rangle = U(u),\end{aligned}$$

modulo an additive constant. Similarly, if (U,F) is a convex entropy pair, we obtain

$$\frac{1}{2}\int_{\mathbb{R}} \overline{F}(u,\lambda,p)U''(\lambda)d\lambda = F(u,p).$$

Since we shall prove an L^∞ bound for our approximate solutions, we may suppose that the u above varies in a bounded set $B \subset \mathbb{R}$. Thus, we may apply the same reasoning with any function which is not linear at infinity, by changing it into a linear function outside B. This shows that we can obtain the inequality (1.33) for any convex entropy pair (U,F) by first proving it in the special case of Kruzkov's entropies, multiplying by $U''(\lambda)/2$, and integrating. In that case, the numerical flux will be given by

$$\mathbf{Q}_{K,e^0}(u,v) = \frac{1}{2}\int_{\mathbb{R}} \overline{\mathbf{Q}}_{K,e^0}(u,v,\lambda)U''(\lambda)d\lambda.$$

Again, this numerical flux satisfies the first two assumptions of the lemma, since they are inherited from the corresponding properties for the Kruzkov numerical flux $\overline{\mathbf{Q}}_{K,e^0}(u,v,\lambda)$.

Therefore, we now proceed to prove the inequality (1.33) for Kruzkov's family of entropies. This is done in two steps. First, we will show that

$$\begin{aligned}\mu^{\overline{F}}_{K^+,e_K^+}(u_K^-,\lambda) - \frac{|\partial^0 K|}{|e_K^+|}\Big(\overline{\mathbf{Q}}_{K,e^0}(u_K^-,u_{K_{e^0}}^-,\lambda) - \overline{\mathbf{Q}}_{K,e^0}(u_K^-,u_K^-,\lambda)\Big) \\ = H(u_K^- \vee \lambda, u_{K_{e^0}}^- \vee \lambda) - H(u_K^- \wedge \lambda, u_{K_{e^0}}^- \wedge \lambda).\end{aligned} \tag{1.36}$$

Second, we will see that for any $u,v,\lambda \in \mathbb{R}$, we have

$$H(u \vee \lambda, v \vee \lambda) - H(u \wedge \lambda, v \wedge \lambda) \geq \mu^{\overline{F}}_{K^+,e_K^+}\Big((\mu_K^+)^{-1}(H(u,v)),\lambda\Big). \tag{1.37}$$

For ease of notation, we omit K, e^0 from the expression of H. The identity (1.36) and the inequality (1.37) (with $u = u_K^-, v = u_{K_{e^0}}^-$) combined give (1.33), for Kruzkov's entropies.

To prove (1.36), simply observe that

$$\begin{aligned}
\mu^{\overline{F}}_{K^+,e^+_K}(u^-_K,\lambda) &= \operatorname{sgn}(u^-_K-\lambda)\Big(\mu^f_{K^+,e^+_K}(u^-_K)-\mu^f_{K^+,e^+_K}(\lambda)\Big)\\
&= \operatorname{sgn}(\mu^+_K(u^-_K)-\mu^+_K(\lambda))\Big(\mu^+_K(u^-_K)-\mu^+_K(\lambda)\Big)\\
&= \Big(\mu^+_K(u^-_K)\vee\mu^+_K(\lambda)-\mu^+_K(u^-_K)\wedge\mu^+_K(\lambda)\Big)\\
&= \Big(\mu^+_K(u^-_K\vee\lambda)-\mu^+_K(u^-_K\wedge\lambda)\Big).
\end{aligned}$$

Here, we have repeatedly used that μ^+_K is a monotone increasing function. The identity (1.36) now follows from the expressions of the Kruzkov numerical entropy flux, $\overline{\mathbf{Q}}_{K,e^0}$, and of H.

Consider now the inequality (1.37). We have

$$\begin{aligned}
&H(u\vee\lambda,v\vee\lambda)-H(u\wedge\lambda,v\wedge\lambda)\\
&\qquad\qquad\geq\Big(H(u,v)\vee H(\lambda,\lambda)\Big)-\Big(H(u,v)\wedge H(\lambda,\lambda)\Big).
\end{aligned}$$

This is a consequence of the fact that if φ is an increasing function, then $\varphi(u\vee\lambda)=\varphi(u\vee\lambda)\vee\varphi(u\vee\lambda)\geq\varphi(u)\vee\varphi(\lambda)$, and (1.34). Thus, we have

$$\begin{aligned}
&H(u\vee\lambda,v\vee\lambda)-H(u\wedge\lambda,v\wedge\lambda)\\
&\geq\Big|H(u,v)-H(\lambda,\lambda)\Big|=\Big|H(u,v)-\mu^+_K(\lambda)\Big|\\
&=\operatorname{sgn}\Big(H(u,v)-\mu^+_K(\lambda)\Big)\Big(H(u,v)-\mu^+_K(\lambda)\Big)\\
&=\operatorname{sgn}\Big((\mu^+_K)^{-1}\Big(H(u,v)\Big)-\lambda\Big)\Big(\mu^+_K\Big((\mu^+_K)^{-1}\Big(H(u,v)\Big)\Big)-\mu^+_K(\lambda)\Big)\\
&=\mu^{\overline{F}}_{K^+,e^+_K}\Big((\mu^+_K)^{-1}\Big(H(u,v)\Big),\lambda\Big).
\end{aligned}$$

This establishes (1.37). We now choose $u=u^-_K, v=u^-_{K_{e0}}$ in (1.37), observe that $H_{K,e^0}(u^-_K,u^-_{K_{e0}})=\tilde{\mu}^+_{K,e^0}$, and combine this with (1.36) to obtain inequality (1.33) for Kruzkov entropies. As described above, (1.33) will hold for all convex entropy pairs (U,F). This completes the proof of Lemma 1.7. □

1.5.2 Entropy dissipation estimate and L^∞ estimate

We now discuss the time evolution of the triangulation. As we have said, the initial hypersurface $\mathcal{H}_0$ is composed of inflow elements e^-_K. We then define the hypersurfaces $\mathcal{H}_n$, for $n>0$, by

$$\mathcal{H}_n:=\bigcup_{e^-_K\subset\mathcal{H}_{n-1}}e^+_K,$$

and set

$$\mathcal{K}^n:=\Big\{K: e^-_K\subset\mathcal{H}_{n-1},\quad e^+_K\subset\mathcal{H}_n\Big\}.$$

It is important to note that the hypersurfaces $\mathcal{H}_n$ are not necessarily associated with a foliation $\{\mathcal{H}_t\}$ of the manifold; they are only restricted by our admissibility assumptions in Definition 1.4.

Next, we introduce the following notation, which we will use from now on. For $K \in \mathcal{K}^n$, we write

$$\mu_K^n := \mu_K^- = \mu_{K,e_K^-}^f, \quad u_K^n := u_K^-, \quad \bar{\mu}_{K,e^0}^{n+1} := \bar{\mu}_{K,e^0}^+,$$

so that, for instance, $\mu_K^+(u_K^+) = \mu_K^{n+1}(u_K^{n+1})$. Accordingly, we define

$$V_K^n(\mu) := V_{K,e_K^-}(\mu), \quad R_{K,e^0}^{n+1} := R_{K,e^0}^+, \tag{1.38}$$

where the timelike entropy flux $V_{K,e}$ and the error term R_{K,e^0}^+ are defined in Lemma 1.7.

Lemma 1.8. *The finite volume approximations satisfy the L^∞ bound*

$$\max_{K^n \in \mathcal{K}^n} |u_K^n| \leq \Big(\max_{K^0 \in \mathcal{K}^0} |u_K^0| + C_1 t_n \Big) e^{C_2 t_n} \tag{1.39}$$

for some constants $C_1, C_2 \geq 0$, where

$$t_n := \sum_{j=0}^{n} \tau_j = \sum_{j=0}^{n} \max_{K^j \in \mathcal{K}j} \frac{|K^j|}{|e_{K^j}^+|}. \tag{1.40}$$

Proof. First of all, observe that from the consistency condition (1.14), the definition of $\mu_{K,e}^f$ in (1.8) and the divergence theorem, we have for any $u \in \mathbb{R}$,

$$\begin{aligned}\int_{K^n} \operatorname{div}_g f(u,p)\, dV_K &= \int_{\partial K^n} g_p\Big(f(u,p), \tilde{\mathbf{n}}(p)\Big)\, dV_{\partial K} \\ &= |e_K^+|\mu_K^{n+1}(u) - |e_K^-|\mu_K^n(u) + \sum_{e^0 \in \partial^0 K} |e^0| \mathbf{q}_{K^n,e^0}(u,u)\end{aligned}$$

(recall that $\tilde{\mathbf{n}}$ is the interior unit normal if it is timelike, and the exterior unit normal if it is spacelike). Moreover, with our notation the finite volume scheme (1.9) reads as

$$|e_K^+|\mu_K^{n+1}(u_K^{n+1}) = |e_K^-|\mu_K^n(u_K^n) - \sum_{e^0 \in \partial^0 K} |e^0| \mathbf{q}_{K^n,e^0}(u_K^n, u_{K_{e^0}}^n).$$

Combining these two identities gives

$$\begin{aligned}\mu_K^{n+1}(u_K^{n+1}) = \mu_K^{n+1}(u_K^n) &- \frac{1}{|e_K^+|} \int_{K^n} \operatorname{div}_g f(u_K^n, p) dV_K \\ &- \sum_{e^0 \in \partial^0 K} \frac{|e^0|}{|e_K^+|} \Big(\mathbf{q}_{K^n,e^0}(u_K^n, u_{K_{e^0}}^n) - \mathbf{q}_{K^n,e^0}(u_K^n, u_K^n) \Big).\end{aligned} \tag{1.41}$$

Next, we rewrite the right-hand side as follows:

$$\begin{aligned}\mu_K^{n+1}(u_K^{n+1}) = (1 - \sum_{e^0\in\partial^0 K^n} \alpha_{K^n,e^0})\mu_K^{n+1}(u_K^n) + \sum_{e^0\in\partial^0 K^n} \alpha_{K^n,e^0}\mu_K^{n+1}(u_{K_{e^0}}^n)\\ - \frac{1}{|e_K^+|}\int_{K^n} \mathrm{div}_g\, f(u_K^n, p) dV_K,\end{aligned} \tag{1.42}$$

where

$$\alpha_{K^n,e^0} := \frac{|e^0|}{|e_K^+|}\frac{\mathbf{q}_{K^n,e^0}(u_K^n, u_{K_{e^0}}^n) - \mathbf{q}_{K^n,e^0}(u_K^n, u_K^n)}{\mu_K^{n+1}(u_K^n) - \mu_K^{n+1}(u_{K_{e^0}}^n)}.$$

This gives a convex combination of $\mu_K^{n+1}(u_K^n)$ and $\mu_K^{n+1}(u_{K_{e^0}}^n)$. Indeed, on one hand we have $\sum_{e^0\in\partial^0 K}\alpha_{K^n,e^0} \geq 0$, due to the monotonicity condition (1.16) and Lemma 1.3. On the other hand, the CFL condition (1.19) gives us

$$\begin{aligned}\sum_{e^0\in\partial^0 K}\alpha_{K^n,e^0} &< \left|\frac{u_K^n - u_{K_{e^0}}^n}{\mu_K^{n+1}(u_K^n) - \mu_K^{n+1}(u_{K_{e^0}}^n)}\right|(\mathrm{Lip}(\mu_K^{n+1})^{-1})^{-1}\\ &\leq \mathrm{Lip}(\mu_K^{n+1})^{-1}/\,\mathrm{Lip}(\mu_K^{n+1})^{-1} = 1.\end{aligned}$$

Thus, we find

$$\mu_K^{n+1}(u_K^{n+1}) \geq \min\Big(\mu_K^{n+1}(u_K^n), \min_{e^0\in\partial^0 K}\mu_K^{n+1}(u_{K_{e^0}}^n)\Big) - \frac{1}{|e_K^+|}\int_{K^n}\mathrm{div}_g\, f(u,p)\, dV_K,$$
$$\mu_K^{n+1}(u_K^{n+1}) \leq \max\Big(\mu_K^{n+1}(u_K^n), \max_{e^0\in\partial^0 K}\mu_K^{n+1}(u_{K_{e^0}}^n)\Big) - \frac{1}{|e_K^+|}\int_{K^n}\mathrm{div}_g\, f(u,p)\, dV_K.$$

Composing with the monotone increasing function $(\mu_K^{n+1})^{-1}$, we find

$$u_K^{n+1} \geq \min\Big(u_K^n, \min_{e^0\in\partial^0 K} u_{K_{e^0}}^n\Big) + \frac{\mathrm{Lip}(\mu_K^{n+1})^{-1}}{|e_K^+|}\int_{K^n}|\,\mathrm{div}_g\, f(u_K^n, p)|\, dV_K,$$
$$u_K^{n+1} \leq \max\Big(u_K^n, \max_{e^0\in\partial^0 K} u_{K_{e^0}}^n\Big) + \frac{\mathrm{Lip}(\mu_K^{n+1})^{-1}}{|e_K^+|}\int_{K^n}|\,\mathrm{div}_g\, f(u_K^n, p)|\, dV_K,$$

which in turn gives

$$|u_K^{n+1}| \leq \max_{K^n\in\mathcal{K}^n}|u_K^n| + \max_{K^n\in\mathcal{K}^n}\frac{\mathrm{Lip}(\mu_K^{n+1})^{-1}}{|e_K^+|}\int_{K^n}|\,\mathrm{div}_g\, f(u_K^n, p)|\, dV_K.$$

By induction we obtain

$$|u_K^{n+1}| \leq \max_{K^0\in\mathcal{K}^0}|u_K^0| + \sum_{j=0}^{n}\max_{K^j\in\mathcal{K}^j}\frac{\mathrm{Lip}(\mu_K^{j+1})^{-1}}{|e_{K^j}^+|}\int_{K^j}|\,\mathrm{div}_g\, f(u_K^j, p)|\, dV_K.$$

Now we use the growth condition (1.5) on the last term,

$$
\begin{aligned}
&\sum_{j=0}^{n} \max_{K^j\in\mathcal{K}^j} \frac{\mathrm{Lip}(\mu_K^{j+1})^{-1}}{|e^+_{K^j}|} \int_{K^j} |\operatorname{div}_g f(u_K^j, p)| \, dV_K \\
&\leq \sum_{j=0}^{n} \max_{K^j\in\mathcal{K}^j} \frac{\mathrm{Lip}(\mu_K^{j+1})^{-1}}{|e^+_{K^j}|} |K^j| \Big(C_1 + C_2 |u_K^j|\Big) \\
&\leq \Big(C_1 t_n + C_2 \sum_{j=0}^{n} \tau^j \max_j |u_K^j|\Big).
\end{aligned}
$$

Here, the constants $C_{1,2}$ may change at each occurrence, and we also used the fact that

$$
\max_{K^j\in\mathcal{K}^j} \mathrm{Lip}(\mu_K^{j+1})^{-1} \leq C,
$$

which is an easy consequence of our assumptions on the flux f. The result now follows from a discrete version of the Gronwall inequality (see [2, Lemma 6.1]). This completes the proof of Lemma 1.8. □

Recall that if V is a convex function, then its *modulus of convexity* on a set S is defined by $\beta := \inf\Big\{V''(w) : w \in S\Big\}$.

Proposition 1.9. *Let* V_K^n *be defined by* (1.28), (1.38), *and let* β_K^n *be the modulus of convexity of* V_K^n. *Then, one has*

$$
\begin{aligned}
&\sum_{K^n\in\mathcal{K}^n} |e_K^+| V_K^{n+1}(\mu_K^{n+1}(u_K^{n+1})) \\
&\qquad + \sum_{\substack{K^n\in\mathcal{K}^n \\ e^0\in\partial^0 K^n}} \frac{\beta_K^{n+1}}{2} \frac{|e^0||e_K^+|}{|\partial^0 K^n|} \Big|\overline{\mu}^{n+1}_{K,e^0} - \mu_K^{n+1}(u_K^{n+1})\Big|^2 \\
&\leq \sum_{K^n\in\mathcal{K}^n} |e_K^-| V_K^n(\mu_K^n(u_K^n)) \\
&\qquad + \sum_{K^n\in\mathcal{K}^n} \int_{K^n} \operatorname{div}_g F(u_K^n, p) \, dV_K + \sum_{\substack{K^n\in\mathcal{K}^n \\ e^0\in\partial^0 K^n}} \frac{|e^0||e_K^+|}{|\partial^0 K^n|} R^{n+1}_{K,e^0}
\end{aligned}
\tag{1.43}
$$

Proof. Consider the discrete entropy inequality (1.30). Multiplying by $\frac{|e^0||e_K^+|}{|\partial^0 K^n|}$ and summing in $K^n \in \mathcal{K}^n$, $e^0 \in \partial^0 K^n$ gives

$$\begin{aligned}\sum_{\substack{K^n \in \mathcal{K}^n \\ e^0 \in \partial^0 K^n}} \frac{|e^0||e_K^+|}{|\partial^0 K^n|} V_K^{n+1}(\overline{\mu}_{K,e^0}^{n+1}) - \sum_{K^n \in \mathcal{K}^n} |e_K^+| V_K^{n+1}\Big(\mu_K^{n+1}(u_K^n)\Big) \\ + \sum_{\substack{K^n \in \mathcal{K}^n \\ e^0 \in \partial^0 K^n}} |e^0| \Big(\mathbf{Q}_{K^n,e^0}(u_K^n, u_{K_{e^0}}^n) - \mathbf{Q}_{K^n,e^0}(u_K^n, u_K^n)\Big) \\ \leq \sum_{\substack{K^n \in \mathcal{K}^n \\ e^0 \in \partial^0 K^n}} \frac{|e^0||e_K^+|}{|\partial^0 K^n|} R_{K,e^0}^{n+1}.\end{aligned} \tag{1.44}$$

Next, observe that the conservation property (1.15) gives

$$\sum_{\substack{K^n \in \mathcal{K}^n \\ e^0 \in \partial^0 K^n}} |e^0| \mathbf{Q}_{K^n,e^0}(u_K^n, u_{K_{e^0}}^n) = 0. \tag{1.45}$$

Now, if V is a convex function, and if $v = \sum_j \alpha_j v_j$ is a convex combination of v_j, then an elementary result on convex functions gives

$$V(v) + \frac{\beta}{2} \sum_j \alpha_j |v_j - v|^2 \leq \sum_j \alpha_j V(v_j).$$

Now, apply this result with the convex combination (1.27) and with the convex function V_K^{n+1}, multiply by $|e_K^+|$, and sum up in $K^n \in \mathcal{K}^n$. Then, combining the resulting inequality with (1.44), (1.45), we obtain

$$\begin{aligned}\sum_{K^n \in \mathcal{K}^n} |e_K^+| V_K^{n+1}(\mu_K^{n+1}(u_K^{n+1})) - \sum_{K^n \in \mathcal{K}^n} |e_K^+| V_K^{n+1}\Big(\mu_K^{n+1}(u_K^n)\Big) \\ + \sum_{\substack{K^n \in \mathcal{K}^n \\ e^0 \in \partial^0 K^n}} \frac{\beta_K^{n+1}}{2} \frac{|e^0||e_K^+|}{|\partial^0 K^n|} \Big|\overline{\mu}_{K,e^0}^{n+1} - \mu_K^{n+1}(u_K^{n+1})\Big|^2 \\ - \sum_{\substack{K^n \in \mathcal{K}^n \\ e^0 \in \partial^0 K^n}} |e^0| \mathbf{Q}_{K^n,e^0}(u_K^n, u_K^n) \leq \sum_{\substack{K^n \in \mathcal{K}^n \\ e^0 \in \partial^0 K^n}} \frac{|e^0||e_K^+|}{|\partial^0 K^n|} R_{K,e^0}^{n+1}.\end{aligned} \tag{1.46}$$

Finally, using the identity

$$\begin{aligned}\int_{K^n} \mathrm{div}_g\, F(u,p)\, dV_K = \int_{\partial K^n} g_p\Big(F(u,p), \tilde{\mathbf{n}}(p)\Big)\, dV_{\partial K} \\ = |e_K^+| V_K^{n+1}(\mu_K^{n+1}(u)) - |e_K^-| V_K^n(\mu_K^n(u)) + \sum_{e^0 \in \partial^0 K^n} |e^0| \mathbf{Q}_{K^n,e^0}(u,u)\end{aligned} \tag{1.47}$$

(with $u = u_K^n$) yields the desired result. This completes the proof of Proposition 1.9. □

Corollary 1.10. *Suppose that for each $K \in \mathcal{T}^h$, $e = e_K^{\pm}$, the function $V_{K,e}$ is strictly convex, and that, moreover, one has*

$$\beta_K^n \geq \beta > 0, \tag{1.48}$$

uniformly in K and n. Then one has the following global estimate for the entropy dissipation,

$$\sum_{n=0}^{N} \sum_{\substack{K^n \in \mathcal{K}^n \\ e^0 \in \partial^0 K^n}} \frac{|e^0||e_K^+|}{|\partial^0 K^n|} \left| \overline{\mu}_{K,e^0}^{n+1} - \mu_K^{n+1}(u_K^{n+1}) \right|^2 = \mathcal{O}(t_N), \tag{1.49}$$

where t_N is defined in (1.40).

Proof. Summing the inequality (1.43) for $n = 0, \ldots, N$, we observe that the first terms on each side of the inequality cancel, leaving only the terms with $n = 0$ and $n = N$. Moreover, using the growth condition (1.5) on the divergence term gives

$$\begin{aligned}
&\sum_{n=0}^{N} \sum_{\substack{K^n \in \mathcal{K}^n \\ e^0 \in \partial^0 K^n}} \frac{\beta_K^{n+1}}{2} \frac{|e^0||e_K^+|}{|\partial^0 K^n|} \left| \overline{\mu}_{K,e^0}^{n+1} - \mu_K^{n+1}(u_K^{n+1}) \right|^2 \\
&\leq \sum_{K^0 \in \mathcal{K}^0} |e_{K^0}^-| |V_K^0(\mu_K^0(u_K^0))| + \sum_{K^{N+1} \in \mathcal{K}^{N+1}} |e_{K^N}^+| |V_K^{N+1}(\mu_K^{N+1}(u_K^{N+1}))| \\
&\quad + \sum_{n=0}^{N} \sum_{K^n \in \mathcal{K}^n} |K^n| \Big(C_1 + C_2 |u_K^j| \Big) + \sum_{n=0}^{N} \sum_{\substack{K^n \in \mathcal{K}^n \\ e^0 \in \partial^0 K^n}} \frac{|e^0||e_K^+|}{|\partial^0 K^n|} R_{K,e^0}^{n+1}.
\end{aligned} \tag{1.50}$$

The last term is estimated using (1.26), (1.31), and the growth condition (1.5), yielding

$$\begin{aligned}
\sum_{n=0}^{N} \sum_{\substack{K^n \in \mathcal{K}^n \\ e^0 \in \partial^0 K^n}} \frac{|e^0||e_K^+|}{|\partial^0 K^n|} R_{K,e^0}^{n+1} &\leq \sum_{n=0}^{N} \sum_{K^n \in \mathcal{K}^n} \operatorname{Lip} V_K^{n+1} \int_{K^n} |\operatorname{div}_g f(u_K^n, p)| \, dV_K \\
&\leq \sum_{n=0}^{N} \sum_{K^n \in \mathcal{K}^n} |K^n| \Big(C_1 + C_2 |u_K^j| \Big).
\end{aligned}$$

Here, we have used that $\operatorname{Lip} V_K^{n+1}$ is uniformly bounded, which is an easy consequence of the corresponding bounds for the flux f. The result now follows from (1.48) and the L^∞ estimate in Lemma 1.8, which allows us to uniformly bound all of the terms on the right-hand side of (1.50). Note however that this bound depends, of course, on the entropy U. This completes the proof of Corollary 1.10. □

1.5.3 Global entropy inequality in space and time

In this paragraph, we deduce a global entropy inequality from the local entropy inequality (1.30). This is nothing but a discrete version of the entropy inequality used to define a weak entropy solution. Given a test-function ϕ defined on $\mathbf{M}$ we introduce its averages

$$\phi^n_{e^0} := \fint_{e^0} \phi(p) dV_{e^0},$$
$$\phi^n_{\partial^0 K} := \sum_{e^0 \in \partial^0 K^n} \frac{|e^0|}{|\partial^0 K^n|} \phi^n_{e^0} = \fint_{\partial^0 K^n} \phi(p)\, dV_{e^0}.$$

We are now ready to prove the global discrete entropy inequality, which is a discrete version of the entropy inequality in Definition 1.1.

Proposition 1.11. *Let* (U, F) *be a convex entropy pair, and let* ϕ *be a non-negative test-function. Then, the function* u^h *given by* (1.12) *satisfies the global entropy inequality*

$$\begin{aligned}
&- \sum_{n=0}^{\infty} \sum_{K^n \in \mathcal{K}^n} \int_{K^n} \mathrm{div}_g \Big(F(u^n_K, p)\phi(p)\Big)\, dV_K - \sum_{K \in \mathcal{K}^0} \int_{e^-_K} \phi^0_{\partial^0 K} g_p(F(u^0_K, p), \mathbf{n}_{K, e^-_K}) dV_{e^-_K} \\
&+ \sum_{n=0}^{\infty} \sum_{\substack{K^n \in \mathcal{K}^n \\ e^0 \in \partial^0 K^n}} \frac{|e^0|}{|\partial^0 K^n|} |e^+_K| \phi^n_{e^0} \Big(V^{n+1}_K (\tilde{\mu}^{n+1}_{K, e^0}) - V^{n+1}_K (\overline{\mu}^{n+1}_{K, e^0}) \Big) \\
&\le \sum_{n=0}^{\infty} \sum_{\substack{K^n \in \mathcal{K}^n \\ e^0 \in \partial^0 K^n}} \frac{|e^0|}{|\partial^0 K^n|} |e^+_K| (\phi^n_{\partial^0 K} - \phi^n_{e^0}) V^{n+1}(\overline{\mu}^{n+1}_{K, e^0}) \\
&\quad + \sum_{n=0}^{\infty} \sum_{\substack{K^n \in \mathcal{K}^n \\ e^0 \in \partial^0 K^n}} \int_{e^0} (\phi^n_{e^0} - \phi(p)) F_{e^0}(u^n_K, p)\, dV_{e^0} \\
&\quad - \sum_{n=0}^{\infty} \sum_{K^n \in \mathcal{K}^n} \int_{e^+_K} (\phi^n_{\partial^0 K} - \phi(p))\, g\Big(F(u^{n+1}_K, p) - F(u^n_K, p), \tilde{\mathbf{n}}_{K, e^+_K}(p)\Big)\, dV_{e^+_K}.
\end{aligned} \tag{1.51}$$

Proof. From the local entropy inequalities (1.30), we obtain

$$\begin{aligned}
&\sum_{\substack{K^n \in \mathcal{K}^n \\ e^0 \in \partial^0 K^n}} \frac{|e^0||e^+_K|}{|\partial^0 K^n|} \phi^n_{e^0} \Big(V^{n+1}_K (\overline{\mu}^{n+1}_{K, e^0}) - V^{n+1}_K \Big(\mu^{n+1}_K (u^n_K) \Big) \Big) \\
&\quad + \sum_{\substack{K^n \in \mathcal{K}^n \\ e^0 \in \partial^0 K^n}} |e^0| \phi^n_{e^0} \Big(\mathbf{Q}_{K^n, e^0}(u^n_K, u^n_{K_{e^0}}) - \mathbf{Q}_{K^n, e^0}(u^n_K, u^n_K) \Big) \\
&\le \sum_{\substack{K^n \in \mathcal{K}^n \\ e^0 \in \partial^0 K^n}} \frac{|e^0||e^+_K|}{|\partial^0 K^n|} \phi^n_{e^0} R^{n+1}_{K, e^0}.
\end{aligned} \tag{1.52}$$

Now, from the conservation property (1.15) we have that

$$\sum_{\substack{K^n\in\mathcal{K}^n\\ e^0\in\partial^0K^n}} |e^0|\phi^n_{e^0}\mathbf{Q}_{K^n,e^0}(u^n_K, u^n_{K_{e^0}}) = 0.$$

Also, from the consistency property (1.14), we find that

$$\begin{aligned}
\sum_{\substack{K^n\in\mathcal{K}^n\\ e^0\in\partial^0K^n}} \phi^n_{e^0}|e^0|\mathbf{Q}_{K^n,e^0}(u^n_K, u^n_K) &= \sum_{\substack{K^n\in\mathcal{K}^n\\ e^0\in\partial^0K^n}} \phi^n_{e^0}\int_{e^0} F_{e^0}(u^n_K, p)\, dV_{e^0}\\
&= \sum_{\substack{K^n\in\mathcal{K}^n\\ e^0\in\partial^0K^n}} \int_{e^0} \phi(p)F_{e^0}(u^n_K, p)\, dV_{e^0} + \sum_{\substack{K^n\in\mathcal{K}^n\\ e^0\in\partial^0K^n}} \int_{e^0}(\phi^n_{e^0} - \phi(p))F_{e^0}(u^n_K, p)\, dV_{e^0}.
\end{aligned}$$

Next, we have that

$$\begin{aligned}
&\sum_{\substack{K^n\in\mathcal{K}^n\\ e^0\in\partial^0K^n}} \frac{|e^0||e^+_K|}{|\partial^0K^n|}\phi^n_{e^0}V^{n+1}_K(\overline{\mu}^{n+1}_{K,e^0})\\
&= \sum_{\substack{K^n\in\mathcal{K}^n\\ e^0\in\partial^0K^n}} \frac{|e^0||e^+_K|}{|\partial^0K^n|}\phi^n_{\partial^0K}V^{n+1}_K(\overline{\mu}^{n+1}_{K,e^0}) + \sum_{\substack{K^n\in\mathcal{K}^n\\ e^0\in\partial^0K^n}} \frac{|e^0||e^+_K|}{|\partial^0K^n|}(\phi^n_{e^0} - \phi^n_{\partial^0K})V^{n+1}_K(\overline{\mu}^{n+1}_{K,e^0})\\
&\geq \sum_{K^n\in\mathcal{K}^n} |e^+_K|\phi^n_{\partial^0K}V^{n+1}_K(\mu^{n+1}_K(u^{n+1}_K)) + \sum_{\substack{K^n\in\mathcal{K}^n\\ e^0\in\partial^0K^n}} \frac{|e^0||e^+_K|}{|\partial^0K^n|}(\phi^n_{e^0} - \phi^n_{\partial^0K})V^{n+1}_K(\overline{\mu}^{n+1}_{K,e^0}).
\end{aligned}$$

Here, we have used that $V(v) \leq \sum_j \alpha_j V(v_j)$, for all convex functions V and convex combinations $v = \sum_j \alpha_j v_j$, specifically used for the convex function V^{n+1}_K and the convex combination (1.27). Also,

$$\sum_{\substack{K^n\in\mathcal{K}^n\\ e^0\in\partial^0K^n}} \frac{|e^0||e^+_K|}{|\partial^0K^n|}\phi^n_{e^0}V^{n+1}_K\Big(\mu^{n+1}_K(u^n_K)\Big) = \sum_{K\in\mathcal{K}^n} |e^+_K|\phi^n_{\partial^0K}V^{n+1}_K\Big(\mu^{n+1}_K(u^n_K)\Big).$$

Therefore, the inequality (1.52) becomes

$$\begin{aligned}
&\sum_{K^n\in\mathcal{K}^n} \phi^n_{\partial^0K}|e^+_K|\Big(V^{n+1}_K\Big(\mu^{n+1}_K(u^{n+1}_K)) - V^{n+1}_K\Big(\mu^{n+1}_K(u^n_K)\Big)\Big)\\
&\quad - \sum_{\substack{K^n\in\mathcal{K}^n\\ e^0\in\partial^0K^n}} \int_{e^0} \phi(p)F_{e^0}(u^n_K, p)\, dV_{e^0}\\
&\leq \sum_{\substack{K^n\in\mathcal{K}^n\\ e^0\in\partial^0K^n}} \frac{|e^0||e^+_K|}{|\partial^0K^n|}\phi^n_{e^0}R^{n+1}_{K,e^0} - \sum_{\substack{K^n\in\mathcal{K}^n\\ e^0\in\partial^0K^n}} \frac{|e^0||e^+_K|}{|\partial^0K^n|}(\phi^n_{e^0} - \phi^n_{\partial^0K})V^{n+1}_K(\overline{\mu}^{n+1}_{K,e^0})\\
&\quad + \sum_{\substack{K^n\in\mathcal{K}^n\\ e^0\in\partial^0K^n}} \int_{e^0}(\phi^n_{e^0} - \phi(p))F_{e^0}(u^n_K, p)\, dV_{e^0} =: A^h + B^h + C^h.
\end{aligned} \tag{1.53}$$

The first term in (1.53) can be written as

$$\begin{aligned}
&\sum_{K^n\in\mathcal{K}^n} \phi^n_{\partial^0 K}|e^+_K|\Big(V^{n+1}_K\Big(\mu^{n+1}_K(u^{n+1}_K)\Big) - V^{n+1}_K\Big(\mu^{n+1}_K(u^n_K)\Big)\Big)\\
&= \sum_{K^n\in\mathcal{K}^n}\int_{e^+_K}\phi(p)\, g\Big(F(u^{n+1}_K,p)-F(u^n_K,p), \tilde{\mathbf{n}}_{K,e^+_K}(p)\Big)\, dV_{e^+_K}\\
&\quad + \sum_{K^n\in\mathcal{K}^n}\int_{e^+_K}(\phi^n_{\partial^0 K}-\phi(p))\, g\Big(F(u^{n+1}_K,p)-F(u^n_K,p), \tilde{\mathbf{n}}_{K,e^+_K}(p)\Big)\, dV_{e^+_K}.
\end{aligned}$$

Combining this result with the identity

$$\begin{aligned}
&\int_K \mathrm{div}_g\Big(F(u,p)\phi(p)\Big)\, dV_K = \int_{\partial K}\phi(p)\, g(F(u,p), \tilde{\mathbf{n}}_{\partial K})dV_{\partial K}\\
&= \int_{e^+_K}\phi(p)\, g(F(u,p), \tilde{\mathbf{n}}_{K,e^+_K})\, dV_{e^+_K} + \int_{e^-_K}\phi(p)\, g(F(u,p), \tilde{\mathbf{n}}_{K,e^-_K})\, dV_{e^-_K}\\
&\quad + \sum_{e^0\in\partial^0 K}\int_{e^0}\phi(p)F_{e^0}(u,p)\, dV_{e^0}
\end{aligned}$$

(with $u = u^n_K$) and in view of (1.53) we see that

$$\begin{aligned}
&-\sum_{K^n\in\mathcal{K}^n}\int_{K^n}\mathrm{div}_g\Big(F(u^n_K,p)\phi(p)\Big)\, dV_K\\
&\le A^h + B^h + C^h\\
&\quad -\sum_{K^n\in\mathcal{K}^n}\Big(\int_{e^+_K}\phi(p)\, g\Big(F(u^{n+1}_K,p), \tilde{\mathbf{n}}_{K,e^+_K}(p)\Big)\, dV_{e^+_K} + \int_{e^-_K}\phi(p)\, g(F(u^n_K,p), \tilde{\mathbf{n}}_{K,e^-_K})\, dV_{e^-_K}\Big)\\
&\quad -\sum_{K^n\in\mathcal{K}^n}\int_{e^+_K}(\phi^n_{\partial^0 K}-\phi(p))\, g\Big(F(u^{n+1}_K,p)-F(u^n_K,p), \tilde{\mathbf{n}}_{K,e^+_K}(p)\Big)\, dV_{e^+_K}.
\end{aligned}$$

The inequality (1.51) is now obtained by summation in n. First, the (summed) terms A, B, C give the three terms on the right-hand side of (1.51) while, in the first sum above, terms cancel out two at a times it only remains the second term of the left-hand side of (1.51). This completes the proof of Proposition 2.13. □

1.6 Proof of convergence

This section contains a proof of the convergence of the finite volume method, and is based on the framework of measure-valued solutions to conservation laws, introduced by DiPerna [18] and extended to manifolds by Ben-Artzi and LeFloch [6]. The basic strategy will be to rely on the discrete entropy inequality (1.51) as well as on the entropy dissipation estimate (1.49), in order to check that any Young measure associated with the approximate solution is a measure-valued solution to the Cauchy problem under consideration. In turn, by the uniqueness result for measure-valued solutions it follows that, in fact, this solution is the unique weak entropy solution of the problem under consideration.

In the following, for the sake of simplicity, we denote by $\mathbf{M}$ our domain of discretization, which is not necessarily the whole manifold. Since the sequence u^h is uniformly bounded in $L^\infty(\mathbf{M})$, we can associate a subsequence and a Young measure $\nu : \mathbf{M} \to \text{Prob}(\mathbb{R})$, which is a family of probability measures in $\mathbb{R}$ parametrized by $p \in \mathbf{M}$. The Young measure allows us to determine all weak-$*$ limits of composite functions $a(u^h)$, for arbitrary real continuous functions a, according to the following property :

$$a(u^h) \overset{*}{\rightharpoonup} \langle \nu, a \rangle \quad \text{as } h \to 0 \tag{1.54}$$

where we use the notation $\langle \nu, a \rangle := \int_{\mathbb{R}} a(\lambda)\, d\nu(\lambda)$.

In view of the above property, the passage to the limit in the left-hand side of (1.51) is (almost) immediate. The uniqueness theorem [18, 6] tells us that once we know that ν is a measure-valued solution to the conservation law, we can prove that the support of each probability measure ν_p actually reduces to a single value $u(p)$, if the same is true on $\mathcal{H}_0$, that is, ν_p is the Dirac measure $\delta_{u(p)}$. It is then standard to deduce that the convergence in (1.54) is actually strong, and that, in particular, u^h converges strongly to u which in turn is the unique entropy solution of the Cauchy problem under consideration.

Lemma 1.12. *Let ν_p be the Young measure associated with the sequence u^h. Then, for every convex entropy pair (U, F) and every non-negative test-function ϕ defined on $\mathbf{M}$ with compact support, we have*

$$\begin{aligned} &- \int_{\mathbf{M}} \langle \nu_p, \operatorname{div}_g F(\cdot, p)\rangle \phi(p) + g\Big(\langle \nu_p, F(\cdot, p)\rangle, \nabla\phi\Big)\, dV_M \\ &- \int_{\mathcal{H}_0} \phi(p)\, g\Big(\langle \nu_p, F(\cdot, p)\rangle, \mathbf{n}_{\mathcal{H}_0}\Big)\, dV_{\mathcal{H}_0} + \int_{\mathbf{M}} \phi(p) \langle \nu_p, U'(\cdot) \operatorname{div}_g f(\cdot, p)\rangle\, dV_M \le 0. \end{aligned} \tag{1.55}$$

The following lemma is easily deduced from the corresponding result in the Euclidean space, by relying on a system of local coordinates. This result will be useful when analyzing the approximation.

Lemma 1.13. *Let $G : \mathbf{M} \to \mathbb{R}$ be a smooth function, and let e be a submanifold of $\mathbf{M}$. Then, there exists a point p_e (not necessarily in e), the center of mass of e, such that*

$$\left| \fint_e G(p) dV_e - G(p_e) \right| \le \operatorname{diam}(e)^2 \|G\|_{C^2(e)}.$$

We are now in position to complete the proof of the main theorem of this paper.

Proof of Theorem 1.5. Due to (1.55), we have for all convex entropy pairs (U, F),

$$\mathrm{div}_g\langle \nu, F(\cdot)\rangle - \langle \nu, (\mathrm{div}_g F)(\cdot)\rangle + \langle \nu, U'(\cdot)(\mathrm{div}_g f)(\cdot)\rangle \leq 0$$

in the sense of distributions in $\mathbf{M}$. Since on the initial hypersurface $\mathcal{H}_0$ the (trace of the) Young measure ν coincides with the Dirac mass δ_{u_0} (because u_0 is a bounded function), from the theory in [6] there exists a unique function $u \in L^\infty(\mathbf{M})$ such that the measure ν remains the Dirac mass δ_u for all Cauchy hypersurfaces $\mathcal{H}_t$, $0 \leq t \leq T$. Moreover, this implies that the approximations u^h converge strongly to u at least on compact sets. This concludes the proof. □

Proof of Lemma 1.12. The proof consists of passing inequality (1.51) to the limit and using property (1.54) of the Young measure. First, note that the first term on the left-hand side of inequality (1.51) converges immediately to the first integral term of (1.55). Next, take the second term of (1.51). Using the fact that $\phi^n_{\partial^0 K} - \phi(p) = \mathcal{O}(\tau + h)$, we see that this term converges to the second integral term in (1.55).

Next, we will prove that the third term on the left-hand side of (1.51) converges to the last term in (1.55). Observe first that

$$\tilde{\mu}^{n+1}_{K,e^0} - \overline{\mu}^{n+1}_{K,e^0} = \frac{1}{|e^+_K|}\int_{K^n} \mathrm{div}_g f(u^n_K, p)\, dV_K.$$

Therefore, we obtain

$$\begin{aligned}
&-\sum_{n=0}^{\infty} \sum_{\substack{K^n \in \mathcal{K}^n\\ e^0 \in \partial^0 K^n}} \frac{|e^0|}{|\partial^0 K^n|} |e^+_K| \phi^n_{e^0}\Big(V^{n+1}_K(\tilde{\mu}^{n+1}_{K,e^0}) - V^{n+1}_K(\overline{\mu}^{n+1}_{K,e^0})\Big)\\
&= \sum_{n=0}^{\infty} \sum_{\substack{K^n \in \mathcal{K}^n\\ e^0 \in \partial^0 K^n}} \frac{|e^0|}{|\partial^0 K^n|} \phi^n_{e^0}\Big(\partial_\mu V^{n+1}_K(\tilde{\mu}^{n+1}_{K,e^0}) \int_{K^n} \mathrm{div}_g f(u^n_K, p)\, dV_K + |e^+_K|\mathcal{O}(\tau^2)\Big)\\
&= \sum_{n=0}^{\infty} \sum_{\substack{K^n \in \mathcal{K}^n\\ e^0 \in \partial^0 K^n}} \frac{|e^0|}{|\partial^0 K^n|} \phi^n_{e^0}\Big(\Big(\partial_\mu V^{n+1}_K(\tilde{\mu}^{n+1}_{K,e^0}) - U'(u^{n+1}_K)\Big) \int_{K^n} \mathrm{div}_g f(u^n_K, p)\, dV_K\\
&\qquad\qquad + |e^+_K|\mathcal{O}(\tau^2) + U'(u^{n+1}_K) \int_{K^n} \mathrm{div}_g f(u^n_K, p)\, dV_K\Big).
\end{aligned}$$

Now, note that from the expression of V (see (1.32)),

$$\begin{aligned}
\partial_\mu V^{n+1}_K(\tilde{\mu}^{n+1}_{K,e^0}) - U'(u^{n+1}_K) &= U'((\mu^{n+1}_K)^{-1}(\tilde{\mu}^{n+1}_{K,e^0})) - U'(u^{n+1}_K)\\
&\leq \sup U'' \max_{n,K^n} \mathrm{Lip}(\mu^n_K)^{-1} |\tilde{\mu}^{n+1}_{K,e^0} - \mu^{n+1}_K|,
\end{aligned}$$

and so, using the L^∞ bound (1.39) and the growth condition (1.5), we find

$$\begin{aligned}
&\sum_{n=0}^{\infty} \sum_{\substack{K^n\in\mathcal{K}^n\\ e^0\in\partial^0K^n}} \frac{|e^0|}{|\partial^0 K^n|}\phi^n_{e^0}\Big(\partial_\mu V^{n+1}_K(\tilde\mu^{n+1}_{K,e^0}) - U'(u^{n+1}_K)\Big)\int_{K^n} \operatorname{div}_g f(u^n_K,p)\,dV_K\\
&\lesssim \sum_{n=0}^{\infty} \sum_{\substack{K^n\in\mathcal{K}^n\\ e^0\in\partial^0K^n}} \frac{|e^0|}{|\partial^0 K^n|}\phi^n_{e^0}|K^n||\tilde\mu^{n+1}_{K,e^0} - \mu^{n+1}_K|.
\end{aligned}$$

Applying the Cauchy-Schwarz inequality and the entropy dissipation estimate (1.49), we find that this term tends to zero with h. Note that property (1.48) is easily seen to be verified due to the smoothness of the functions V^n_K. We are left with the term

$$\sum_{n=0}^{\infty} \sum_{K^n\in\mathcal{K}^n} \phi^n_{\partial^0 K}U'(u^{n+1}_K)\int_{K^n} \operatorname{div}_g f(u^n_K,p)\,dV_K,$$

which is easily seen to be of the form

$$\sum_{n=0}^{\infty} \sum_{K^n\in\mathcal{K}^n} U'(u^n_K)\int_{K^n} \phi(p)\operatorname{div}_g f(u^n_K,p)\,dV_K + \mathcal{O}(h) \to \int_{\mathbf{M}} \langle \nu_p, U'(\cdot)\operatorname{div}_g f(\cdot,p)\rangle\, dV_M.$$

It remains to check that the terms on the right-hand side of (1.51) tend to zero with h. Namely, the first term on the right-hand side can be written as

$$\begin{aligned}
&\sum_{n=0}^{\infty} \sum_{\substack{K^n\in\mathcal{K}^n\\ e^0\in\partial^0K^n}} \frac{|e^0|}{|\partial^0 K^n|}|e^+_K|(\phi^n_{\partial^0 K} - \phi^n_{e^0})V^{n+1}(\overline\mu^{n+1}_{K,e^0})\\
&= \sum_{n=0}^{\infty} \sum_{\substack{K^n\in\mathcal{K}^n\\ e^0\in\partial^0K^n}} \frac{|e^0|}{|\partial^0 K^n|}|e^+_K|(\phi^n_{\partial^0 K} - \phi^n_{e^0})\Big(V^{n+1}(\overline\mu^{n+1}_{K,e^0}) - V^{n+1}(\mu^{n+1}_K(u^{n+1}_K))\Big)\\
&= o(1),
\end{aligned}$$

by the Cauchy-Schwarz inequality and the entropy dissipation estimate (1.49). Next, the second term on the right-hand side of (1.51) satisfies

$$\begin{aligned}
&\sum_{n=0}^{\infty} \sum_{\substack{K^n\in\mathcal{K}^n\\ e^0\in\partial^0K^n}} \int_{e^0} (\phi^n_{e^0} - \phi(p))F_{e^0}(u^n_K,p)\,dV_{e^0}\\
&= \sum_{n=0}^{\infty} \sum_{\substack{K^n\in\mathcal{K}^n\\ e^0\in\partial^0K^n}} \int_{e^0} (\phi^n_{e^0} - \phi(p))\Big(F_{e^0}(u^n_K,p) - \fint_{e^0} F_{e^0}(u^n_K,q)dq\Big)\,dV_{e^0}
\end{aligned}$$

which, in view of the regularity of ϕ and F, is bounded by $\sum_{n=0}^{\infty}\sum_{K^n\in\mathcal{K}^n} |\partial^0 K|\mathcal{O}(\tau_{K^n} + h)^2$. Using the CFL condition (1.19) and property (1.18), we can further bound this term by

$$\sum_{n=0}^{\infty} \sum_{K^n \in \mathcal{K}^n} |e_K^+| \mathcal{O}(\tau_{K^n}) \Big(\mathcal{O}(\tau_{K^n} + h) + \mathcal{O}(h^2/\tau_{K^n}) \Big) = o(1),$$

and so only the term

$$A^h(\phi) := -\sum_{n=0}^{N} \sum_{K^n \in \mathcal{K}^n} \int_{e_K^+} (\phi^n_{\partial^0 K} - \phi(p))\, g\Big(F(u_K^{n+1}, p) - F(u_K^n, p), \tilde{\mathbf{n}}_{K,e_K^+}(p)\Big)\, dV_{e_K^+}$$

remains to be controlled. Here, we will use that our triangulation is admissible, in the sense of Definition 1.4. First of all, by integrating by parts we rewrite it as

$$\begin{aligned} A^h(\phi) = \sum_{n=1}^{\infty} \sum_{K^n \in \mathcal{K}^n} & \int_{e_K^-} (\phi^{n-1}_{\partial^0 K} - \phi(p))\, g(F(u_K^n, p), \mathbf{n}_{K,e_K^-}(p))\, dV_{e_K^+} \\ & + \int_{e_K^+} (\phi^n_{\partial^0 K} - \phi(p))\, g(F(u_K^n, p), \mathbf{n}_{K,e_K^+}(p))\, dV_{e_K^+} \end{aligned}$$

plus a boundary term for $n = 0$ which easily tends to zero with h. Next, using Lemma 1.13 and equation (1.6), one may replace $\phi^n_{\partial^0 K}$ by $\phi(p^0_{K^n})$ and $\phi^-_{\partial^0 K}$ by $\phi(p^0_{K^{n-1}})$, where $p^0_{K^j}$ denotes the center of $\partial^0 K^j$, with an error term of the form $Ch\|\phi\|_{C^2}\|F\|_{L^\infty}$. Next, we replace (and similarly for e_K^-) $\phi(p)\, g(F(u_K^n, p), \mathbf{n}_{K,e_K^+}(p))$ with $\phi(p_K^+)\, g(F(u_K^n, p_K^+), \mathbf{n}_{K,e_K^+}(p_K^+))$. Using property (1.6), the corresponding error term is seen to be of the form $Ch\|\phi\|_{C^2}\|F\|_{C^2}$. The generic constants C do not depend on h nor ϕ.

We have

$$\begin{aligned} \Big|A^h(\phi)\Big| \le \Bigg| \sum_{n=1}^{\infty} \sum_{K^n \in \mathcal{K}^n} & |e_K^+| (\phi(p^0_{K^n}) - \phi(p^+_{K^n}))\, g\Big(F(u_K^n, p^+_{K^n}), \mathbf{n}_{K,e_K^+}(p^+_{K^n})\Big) \\ & + |e_K^-| (\phi(p^0_{K^-}) - \phi(p^+_{K^-}))\, g\Big(F(u_K^n, p^+_{K^-}), \mathbf{n}_{K,e_K^-}(p^+_{K^-})\Big) \Bigg| \\ & + h\|\phi\|_{C^2}\Big(\|F\|_{L^\infty} + \|F\|_{C^2}\Big). \end{aligned}$$

Now, performing a Taylor expansion of ϕ and using the definition of $\mathbf{w}_K$ (recall that $\mathbf{w}_K$ is the future-oriented vector at p_K^+ tangent to the geodesic connecting p_K^+ and p_K^0) we find, for instance,

$$\phi(p_K^0) - \phi(p_K^+) = g(\mathbf{w}_K, \nabla\phi(p_K^+)) + \mathcal{O}(h^2).$$

Therefore, by the definition of $\mathcal{E}(K)$ and using (1.20), we may express this conclusion by using the local deviation of the triangulation,

$$
\begin{aligned}
\left|A^h(\phi)\right| \leq & \Bigg| \sum_{n=1}^{\infty} \sum_{K^n \in \mathcal{K}^n} |e_K^+| \, g(\mathbf{w}_{K^n}, \nabla\phi(p_K^+)) \, g(F(u_K^n, p_K^+), \mathbf{n}_{K^n, e_K^+}(p_{K^n}^+)) \\
& \qquad + |e_K^-| \, g(\mathbf{w}_{K^-}, \nabla\phi(p_{K^-}^+)) \, g(F(u_K^n, p_{K^-}^+), \mathbf{n}_{K^n, e_K^-}(p_{K^-}^+)) \Bigg| \\
& + Ch\|\phi\|_{C^2}(\|F\|_{L^\infty} + \|F\|_{C^2}) \\
\leq & \sum_{K^n \in \mathcal{T}^h} \Big(|K|\mathcal{E}(K) - |K^-|\mathcal{E}(K^-) \Big)(\nabla\phi, F(u^h)) + Ch\|\phi\|_{C^2}(\|F\|_{L^\infty} + \|F\|_{C^2}) \\
\leq & \eta(h)\|\phi\|_{C^1}\|F\|_{L^\infty} + Ch\|\phi\|_{C^2}(\|F\|_{L^\infty} + \|F\|_{C^2}),
\end{aligned}
$$

which tends to zero since $\eta(h) \to 0$. This completes the proof of Lemma 1.12. □

Chapitre 2

Approche basée sur des champs de formes différentielles*

Approach based on differential forms

2.1 Introduction

The development of the mathematical theory (existence, uniqueness, qualitative behavior, approximation) of shock wave solutions to scalar conservation laws *defined on manifolds* is motivated by similar questions arising in compressible fluid dynamics. For instance, the shallow water equations of geophysical fluid dynamics (for which the background manifold is the Earth or, more generally, Riemannian manifold) and the Einstein-Euler equations in general relativity (for which the manifold metric is also part of the unknowns) provide important examples where the partial differential equations of interest are naturally posed on a (curved) manifold. Scalar conservation laws yield a drastically simplified, yet very challenging, mathematical model for understanding nonlinear aspects of shock wave propagation on manifolds.

In the present paper, given a (smooth) differential $(n+1)$-manifold M which we refer to as a *spacetime,* we consider the following class of *nonlinear conservation laws*

$$d(\omega(u)) = 0, \qquad u = u(x),\ x \in M. \tag{2.1}$$

Here, for each $\overline{u} \in \mathbb{R}$, $\omega = \omega(\overline{u})$ is a (smooth) *field of n-forms* on M which we refer to as the *flux field* of the conservation law (2.1).

Two special cases of (2.1) were recently studied in the literature. When $M = \mathbb{R}_+ \times N$ and the n-manifold N is endowed with a *Riemannian metric* h, the conservation law (2.1) is here equivalent to

$$\partial_t u + \mathrm{div}_h(b(u)) = 0, \qquad u = u(t, y),\ t \geq 0,\ y \in N.$$

*En collaboration avec P. G. LeFloch [34].

Here, div_h denotes the divergence operator associated with the metric h. In this case, the flux field is a *flux vector field* $b = b(\overline{u})$ *on the* n*-manifold* N and does not depend on the time variable. More generally, we may suppose that M is endowed with a *Lorentzian metric* g and, then, (2.1) takes the equivalent form

$$\mathrm{div}_g(a(u)) = 0, \qquad u = u(x),\ x \in M.$$

Observe that the flux $a = a(\overline{u})$ is now a vector field *on the* $(n + 1)$*-manifold* M.

Recall that, in the Riemannian or Lorentzian settings, the theory of weak solutions on manifolds was initiated by Ben-Artzi and LeFloch [6] and further developed in the follow-up papers by LeFloch and his collaborators [2, 3, 32, 33, 34]. Hyperbolic equations on manifolds were also studied by Panov in [40] with a vector field standpoint. The actual implementation of a finite volume scheme on the sphere was recently realized by Ben-Artzi, Falcovitz, and LeFloch [7].

In the present paper, we propose a new approach in which the conservation law is written in the form (2.1), that is, the flux $\omega = \omega(\overline{u})$ is defined as a *field of differential forms of degree* n. Hence, no geometric structure is a priori assumed on M, and the sole knowledge of the flux field structure is required. The fact that the equation (2.1) is a "conservation law" for the unknown quantity u can be understood by expressing Stokes theorem: for sufficiently smooth solutions u, at least, the conservation law (2.1) is equivalent to saying that the total flux

$$\int_{\partial\mathcal{U}} \omega(u) = 0, \qquad \mathcal{U} \subset M, \tag{2.2}$$

vanishes for every open subset $\mathcal{U}$ with smooth boundary. By relying on the conservation law (2.1) rather than the equivalent expressions in the special cases of Riemannian or Lorentzian manifolds, we are able to develop here a theory of entropy solutions to conservation laws posed on manifolds, which is technically and conceptually simpler but also provides a significant generalization of earlier works.

Recall that weak solutions to conservation laws contain shock waves and, for the sake of uniqueness, the class of such solutions must be restricted by an entropy condition (Lax [30]). This theory of conservation laws on manifolds is a generalization of fundamental works by Kruzkov [27], Kuznetsov [28], and DiPerna [18] who treated equations posed on the (flat) Euclidian space $\mathbb{R}^n$.

Our main result in the present paper is a generalization of the formulation and convergence of the finite volume method for general conservation law (2.1). In turn, we will establish the existence of a semi-group of entropy solutions which is contracting in a suitable distance.

The first difficulty is formulating the initial and boundary problem for (2.1) in the sense of distributions. A weak formulation of the boundary condition is proposed which takes into account the nonlinearity and hyperbolicity of the equation under consideration. We emphasize that our weak formulation applies to an arbitrary differential manifold. However, to proceed with the development of the well-posedness theory we then need to impose that the manifold satisfies a *global hyperbolicity condition,* which provides a global time-orientation and allow us to distinguish between "future" and "past" directions in the time-evolution. This assumption is standard in Lorentzian geometry for applications to general relativity. For simplicity in this paper, we then restrict attention to the case that the manifold is foliated by compact slices.

Second, we introduce a new version of the finite volume method (based on monotone numerical flux terms). The proposed scheme provides a natural discretization of the conservation law (2.1), which solely uses the n-volume form structure associated with the prescribed flux field ω.

Third, we derive several stability estimates satisfied by the proposed scheme, especially discrete versions of the entropy inequalities. As a corollary, we obtain a uniform control of the entropy dissipation measure associated with the scheme, which, however, is not sufficient by itself to the compactness of the sequence of approximate solutions.

The above stability estimates are sufficient to show that the sequence of approximate solutions generated by the finite volume scheme converges to an *entropy measure-valued solution* in the sense of DiPerna. To conclude our proof, we rely on a generalization of DiPerna's uniqueness theorem [18] and conclude with the existence of entropy solutions to the corresponding initial value problem.

In the course of this analysis, we also establish a *contraction property* for any two entropy solutions u, v, that is, given two hypersurfaces H, H' such that H' lies in the future of H,

$$\int_{H'} \mathbf{\Omega}(u_{H'}, v_{H'}) \le \int_H \mathbf{\Omega}(u_H, v_H). \tag{2.3}$$

Here, for all reals $\overline{u}, \overline{v}$, the n-form field $\mathbf{\Omega}(\overline{u}, \overline{v})$ is determined from the given flux field $\omega(\overline{u})$ and can be seen as a generalization (to the spacetime setting) of the notion of Kruzkov entropy $|\overline{u} - \overline{v}|$.

Recall that DiPerna's measure-valued solutions were used to establish the convergence of schemes by Szepessy [43, 44], Coquel and LeFloch [9, 10, 11], and Cockburn, Coquel, and LeFloch [12, 13]. For many related results and a review about the convergence techniques for hyperbolic problems, we refer to Tadmor [46] and Tadmor, Rascle, and Bagneiri [47]. Further hyperbolic models including also a coupling with elliptic equations and many applications were successfully investigated in the works by Kröner [25], and Eymard, Gallouet, and Herbin [20]. For higher-order schemes, see the paper by Kröner, Noelle, and Rokyta [26]. Also, an alternative approach to the convergence of finite volume schemes was later proposed by Westdickenberg and Noelle [50]. Finally, note that Kuznetsov's error estimate [12, 15] were recently extended to conservation laws on manifolds by LeFloch, Neves, and Okutmustur [33].

An outline of the paper is as follows. In Section 2.2, we introduce our definition of entropy solution which includes both initial-boundary data and entropy inequalities. The finite volume method is presented in Section 2.3, and discrete stability properties are then established in Section 2.4. The main statements are given at the beginning of Section 2.5, together with the final step of the convergence proof.

2.2 Conservation laws posed on a spacetime

2.2.1 A notion of weak solution

In this section we assume that M is an oriented, compact, differentiable $(n+1)$-manifold with boundary. Given an $(n+1)$-form α, its *modulus* is defined as the $(n+1)$-form

$$|\alpha| := |\overline{\alpha}|\, dx^0 \wedge \cdots \wedge dx^n,$$

where $\alpha = \overline{\alpha}\, dx^1 \wedge \cdots \wedge dx^n$ is written in an oriented frame determined from local coordinates $x = (x^\alpha) = (x^0, \dots, x^n)$. If H is a hypersurface, we denote by $i = i_H : H \to \mathbf{M}$ the canonical injection map, and by $i^* = i_H^*$ is the pull-back operator acting on differential forms defined on M.

On this manifold, we introduce a class of nonlinear hyperbolic equations, as follows.

Definition 2.1. *1. A* flux field *ω on the $(n+1)$-manifold M is a parametrized family $\omega(\overline{u}) \in \Lambda^n(M)$ of smooth fields of differential forms of degree n, that depends smoothly upon the real parameter $\overline{u}$.*

2. The conservation law *associated with a flux field ω and with unknown $u : M \to \mathbb{R}$ is*

$$d\Big(\omega(u)\Big) = 0, \tag{2.4}$$

where d denotes the exterior derivative operator and, therefore, $d\Big(\omega(u)\Big)$ is a field of differential forms of degree $(n+1)$ on M.

3. A flux field ω is said to grow at most linearly *if for every 1-form ρ on M*

$$\sup_{\overline{u}\in\mathbb{R}} \int_M \Big|\rho \wedge \partial_u \omega(\overline{u})\Big| < \infty. \tag{2.5}$$

With the above notation, by introducing local coordinates $x = (x^\alpha)$ we can write for all $\overline{u} \in \mathbb{R}$

$$\begin{aligned}
&\omega(\overline{u}) = \omega^\alpha(\overline{u})\, (\widehat{dx})_\alpha, \\
&(\widehat{dx})_\alpha := dx^0 \wedge \ldots \wedge dx^{\alpha-1} \wedge dx^{\alpha+1} \wedge \ldots \wedge dx^n.
\end{aligned}$$

Here, the coefficients $\omega^\alpha = \omega^\alpha(\overline{u})$ are smooth functions defined in the chosen local chart. Recall that the operator d acts on differential forms with arbitrary degree and that, given a p-form ρ and a p'-form ρ', one has $d(d\rho) = 0$ and $d(\rho \wedge \rho') = d\rho \wedge \rho' + (-1)^p \rho \wedge d\rho'$.

As it stands, the equation (2.4) makes sense for unknown functions that are, for instance, Lipschitz continuous. However, it is well-known that solutions to nonlinear hyperbolic equations need not be continuous and, consequently, we need to recast (2.4) in a weak form.

Given a *smooth* solution u of (2.4) we can apply Stokes theorem on any open subset $\mathcal{U}$ that is compactly included in M and has smooth boundary $\partial\mathcal{U}$. We obtain

$$0 = \int_{\mathcal{U}} d(\omega(u)) = \int_{\partial\mathcal{U}} i^*(\omega(u)). \tag{2.6}$$

Similarly, given any smooth function $\psi : M \to \mathbb{R}$ we can write

$$d(\psi\,\omega(u)) = d\psi \wedge \omega(u) + \psi\, d(\omega(u)),$$

where the differential $d\psi$ is a 1-form field. Provided u satisfies (2.4), we find

$$\int_M d(\psi\,\omega(u)) = \int_M d\psi \wedge \omega(u)$$

and, by Stokes theorem,

$$\int_M d\psi \wedge \omega(u) = \int_{\partial\mathbf{M}} i^*(\psi\omega(u)). \tag{2.7}$$

Note that a suitable orientation of the boundary ∂M is required for this formula to hold. This identity is satisfied by every smooth solution to (2.4) and this motivates us to reformulate (2.4) in the following weak form.

Definition 2.2 (Weak solutions on a spacetime)**.** *Given a flux field with at most linear growth ω, a function $u \in L^1(M)$ is called a* weak solution *to the conservation law* (2.4) *posed on the spacetime M if*

$$\int_M d\psi \wedge \omega(u) = 0$$

for every function $\psi : M \to \mathbb{R}$ compactly supported in the interior $\mathring{M}$.

The above definition makes sense since the function u is integrable and $\omega(\overline{u})$ has at most linear growth in $\overline{u}$, so that the $(n+1)$-form $d\psi \wedge \omega(u)$ is integrable on the compact manifold M.

2.2.2 Entropy inequalities

As is standard for nonlinear hyperbolic problems, weak solution must be further constrained by imposing initial, boundary, as well as entropy conditions.

Definition 2.3. *A (smooth) field of n-forms* $\Omega = \Omega(\overline{u})$ *is called a* (convex) entropy flux field *for the conservation law* (2.4) *if there exists a (convex) function* $U : \mathbb{R} \to \mathbb{R}$ *such that*

$$\Omega(\overline{u}) = \int_0^{\overline{u}} \partial_u U(\overline{v})\, \partial_u \omega(\overline{v})\, d\overline{v}, \qquad \overline{u} \in \mathbb{R}.$$

It is said to be also admissible *if, moreover,* $\sup |\partial_u U| < \infty$.

For instance, if one chooses the function $U(\overline{u}, \overline{v}) := |\overline{u} - \overline{v}|$, where $\overline{v}$ is a real parameter, the entropy flux field reads

$$\boldsymbol{\Omega}(\overline{u}, \overline{v}) := \operatorname{sgn}(\overline{u} - \overline{v})\, (\omega(\overline{u}) - \omega(\overline{v})), \tag{2.8}$$

which is a generalization to a spacetime of the so-called Kruzkov's entropy-entropy flux pair.

Based on the notion of entropy flux above, we can derive entropy inequalities in the following way. Given any smooth solution u to (2.4), by multiplying (2.4) by $\partial_u U(u)$ we obtain the additional conservation law

$$d(\Omega(u)) - (d\Omega)(u) + \partial_u U(u)(d\omega)(u) = 0.$$

However, for discontinuous solutions this identity can not be satisfied as an equality and, instead, we should impose that the entropy inequalities

$$d(\Omega(u)) - (d\Omega)(u) + \partial_u U(u)(d\omega)(u) \le 0 \tag{2.9}$$

hold in the sense of distributions for all admissible entropy pair (U, Ω). These inequalities can be justified, for instance, via the vanishing viscosity method, that is by searching for weak solutions that are realizable as limits of smooth solutions to the parabolic regularization of (2.4).

It remains to prescribe initial and boundary conditions. We emphasize that, without further assumption on the flux field (to be imposed shortly below), points along the boundary ∂M can not be distinguished and it is natural to prescribe the trace of the solution along the *whole* of the boundary ∂M. This is possible provided the boundary data, $u_B : \partial M \to \mathbb{R}$, is assumed by the solution in a suitably *weak sense*. Following Dubois and LeFloch [19], we use the notation

$$u\big|_{\partial M} \in \mathcal{E}_{U,\Omega}(u_B) \tag{2.10}$$

for all convex entropy pair (U, Ω), where for all reals $\overline{u}$

$$\mathcal{E}_{U,\Omega}(\overline{u}) := \Big\{ \overline{v} \in \mathbb{R} \;\Big|\; E(\overline{u}, \overline{v}) := \Omega(\overline{u}) + \partial_u U(\overline{u})(\omega(\overline{v}) - \omega(\overline{u})) \le \Omega(\overline{v}) \Big\}.$$

Recall that the boundary conditions for hyperbolic conservation laws (posed on the Euclidian space) were first studied by Bardos, Leroux, and Nedelec [5] in the class of solutions with bounded variation and, then, in the class of measured-valued solutions by Szepessy [45]. Later, a different approach was introduced by Cockburn, Coquel, and LeFloch [15] (see, in particular, the discussion p. 701 therein) in the course of their analysis of the finite volume methods, which was later expanded in Kondo and LeFloch [24]. An alternative and also powerful approach to the boundary conditions for conservation laws was independently introduced by Otto [39] and developed by followers. In the present paper, our proposed formulation of the initial and boundary value problem is a generalization of the works [15] and [24].

Definition 2.4 (Entropy solutions on a spacetime with boundary)**.** *Let $\omega = \omega(\overline{u})$ be a flux field with at most linear growth and $u_B \in L^1(\partial M)$ be a prescribed boundary function. A function $u \in L^1(M)$ is called an* entropy solution *to the boundary value problem* (2.4) *and* (2.10) *if there exists a bounded and measurable field of n-forms $\gamma \in L^1\Lambda^n(\partial M)$ such that*

$$\begin{aligned}\int_M \Big(d\psi \wedge \Omega(u) + \psi\,(d\Omega)(u) - \psi\,\partial_u U(u)(d\omega)(u)\Big)\\ + \int_{\partial \mathbf{M}} \psi_{|\partial M}\Big(i^*\Omega(u_B) + \partial_u U(u_B)\big(\gamma - i^*\omega(u_B)\big)\Big) \geq 0\end{aligned}$$

for every admissible convex entropy pair (U, Ω) and every smooth function $\psi : M \to \mathbb{R}_+$.

Observe that the above definition makes sense since each of the terms $d\psi \wedge \Omega(u)$, $(d\Omega)(u)$, $(d\omega)(u)$ belong to $L^1(M)$. The above definition can be generalized to encompass solutions within the much larger class of measure-valued mappings. Following DiPerna [18], we consider solutions that are no longer functions but *Young measures,* i.e, weakly measurable maps $\nu : M \to \text{Prob}(\mathbb{R})$ taking values within is the set of probability measures $\text{Prob}(\mathbb{R})$. For simplicity, we assume that the support supp ν is a compact subset of $\mathbb{R}$.

Definition 2.5. *Given a flux field $\omega = \omega(\overline{u})$ with at most linear growth and given a boundary function $u_B \in L^\infty(\partial M)$, one says that a compactly supported Young measure $\nu : M \to Prob(\mathbb{R})$ is an* entropy measure-valued solution *to the boundary value problem* (2.4), (2.10) *if there exists a bounded and measurable field of n-forms $\gamma \in L^\infty\Lambda^n(\partial M)$ such that the inequalities*

$$\begin{aligned}\int_M \Big\langle \nu, d\psi \wedge \Omega(\cdot) + \psi\,\Big(d(\Omega(\cdot)) - \partial_u U(\cdot)(d\omega)(\cdot)\Big)\Big\rangle\\ + \int_{\partial M} \psi_{|\partial M}\Big\langle \nu, \Big(i^*\Omega(u_B) + \partial_u U(u_B)\big(\gamma - i^*\omega(u_B)\big)\Big)\Big\rangle \geq 0\end{aligned}$$

hold for all convex entropy pair (U, Ω) and all smooth functions $\psi \geq 0$.

2.2.3 Global hyperbolicity and geometric compatibility

In general relativity, it is a standard assumption that the spacetime should be globally hyperbolic. This notion must be adapted to the present setting, since we do not have a Lorentzian structure, but solely the n-volume form structure associated with the flux field ω.

We assume here that the manifold M is foliated by hypersurfaces, say

$$M = \bigcup_{0 \le t \le T} H_t, \tag{2.11}$$

where each slice has the topology of a (smooth) n-manifold N with boundary. Topologically we have $M = [0, T] \times N$, and the boundary of M can be decomposed as

$$\begin{aligned} \partial M &= \mathcal{H}_0 \cup H_T \cup B, \\ B = (0, T) \times N &:= \bigcup_{0 < t < T} \partial H_t. \end{aligned} \tag{2.12}$$

The following definition imposes a non-degeneracy condition on the averaged flux on the hypersurfaces of the foliation.

Definition 2.6. *Consider a manifold M with a foliation* (2.11)-(2.12) *and let $\omega = \omega(\overline{u})$ be a flux field. Then, the conservation law* (2.4) *on the manifold M is said to satisfy the* global hyperbolicity condition *if there exist constants $0 < \underline{c} < \overline{c}$ such that for every non-empty hypersurface $e \subset H_t$, the integral $\int_e i^* \partial_u \omega(0)$ is positive and the function $\varphi_e : \mathbb{R} \to \mathbb{R}$,*

$$\varphi_e(\overline{u}) := ⨍_e i^* \omega(\overline{u}) = \frac{\int_e i^* \omega(\overline{u})}{\int_e i^* \partial_u \omega(0)}, \qquad \overline{u} \in \mathbb{R}$$

satisfies

$$\underline{c} \le \partial_u \varphi_e(\overline{u}) \le \overline{c}, \qquad \overline{u} \in \mathbb{R}. \tag{2.13}$$

The function φ_e represents the *averaged flux* along the hypersurface e. From now we assume that the conditions in Definition 2.6 are satisfied. It is natural to refer to $\mathcal{H}_0$ as an initial hypersurface and to prescribe an "initial data" $u_0 : \mathcal{H}_0 \to \mathbb{R}$ on this hypersurface and, on the other hand, to impose a boundary data u_B along the submanifold B. It will be convenient here to use the standard terminology of general relativity and to refer to H_t as *spacelike hypersurfaces*.

Under the global hyperbolicity condition (2.11)–(2.13), the initial and boundary value problem now takes the following form. The boundary condition (2.10) decomposes into an initial data

$$u_{\mathcal{H}_0} = u_0 \tag{2.14}$$

and a boundary condition

$$u\big|_B \in \mathcal{E}_{U,\Omega}(u_B). \tag{2.15}$$

Correspondingly, the condition in Definition 2.4 now reads

$$\int_M \Big(d\psi \wedge \Omega(u) + \psi\,(d\Omega)(u) - \psi\,\partial_u U(u)(d\omega)(u)\Big)$$
$$+ \int_B \psi_{|\partial M}\Big(i^*\Omega(u_B) + \partial_u U(u_B)\big(\gamma - i^*\omega(u_B)\big)\Big)$$
$$+ \int_{H_T} i^*\Omega(u_{H_T}) - \int_{\mathcal{H}_0} i^*\Omega(u_0) \geq 0.$$

Finally, we introduce:

Definition 2.7. *A flux field ω is called* geometry-compatible *if it is closed for each value of the parameter,*

$$(d\omega)(\overline{u}) = 0, \qquad \overline{u} \in \mathbb{R}. \tag{2.16}$$

This compatibility condition is natural since it ensures that constants are trivial solutions to the conservation law, a property shared by many models of fluid dynamics (such as the shallow water equations on a curved manifold). When (2.16) holds, then it follows from Definition 2.3 that every entropy flux field Ω also satisfies the condition

$$(d\Omega)(\overline{u}) = 0, \qquad \overline{u} \in \mathbb{R}.$$

In turn, the entropy inequalities (2.9) for a solution $u : M \to \mathbb{R}$ simplify drastically and take the form

$$d(\Omega(u)) \leq 0. \tag{2.17}$$

2.3 Finite volume method on a spacetime

2.3.1 Assumptions and formulation

From now on we assume that the manifold $M = [0, T] \times N$ is foliated by slices with *compact* topology N, and the initial data u_0 is taken to be a bounded function. We also assume that the global hyperbolicity condition holds and that the flux field ω is geometry-compatible, which simplifies the presentation but is not an essential assumption.

Let $\mathcal{T}^h = \bigcup_{K \in \mathcal{T}^h} K$ be a *triangulation* of the manifold M, that is, a collection of finitely many *cells (or elements)*, determined as the images of polyhedra of $\mathbb{R}^{n+1}$, satisfying the following conditions:

- The boundary ∂K of an element K is a piecewise smooth, n-manifold, $\partial K = \bigcup_{e \subset \partial K} e$ and contains exactly two spacelike faces, denoted by e_K^+ and e_K^-, and "vertical" elements

$$e^0 \in \partial^0 K := \partial K \setminus \Big\{e_K^+, e_K^-\Big\}.$$

- The intersection $K \cap K'$ of two distinct elements $K, K' \in \mathcal{T}^h$ is either a common face of K, K' or else a submanifold with dimension at most $(n-1)$.
- The triangulation is compatible with the foliation (2.11)-(2.12) in the sense that there exists a sequence of times $t_0 = 0 < t_1 < \ldots < t_N = T$ such that all spacelike faces are submanifolds of $H_n := H_{t_n}$ for some $n = 0, \ldots, N$, and determine a triangulation of the slices. We denote by $\mathcal{T}_0^h$ the set of all elements K which admit one face belonging to the initial hypersurface $\mathcal{H}_0$.

We define the measure $|e|$ of a hypersurface $e \subset M$ by

$$|e| := \int_e i^* \partial_u \omega(0). \tag{2.18}$$

This quantity is positive if e is sufficiently "close" to one of the hypersurfaces along which we have assumed the hyperbolicity condition (2.13). Provided $|e| > 0$ which is the case if e is included in one of the slices of the foliation, we associate to e the function $\varphi_e : \mathbb{R} \to \mathbb{R}$, as defined earlier. Recall the following hyperbolicity condition which holds along the triangulation since the spacelike elements are included in the spacelike slices:

$$\underline{c} \leq \partial_u \varphi_{e_K^\pm}(\overline{u}) \leq \overline{c}, \qquad K \in \mathcal{T}^h. \tag{2.19}$$

We introduce the finite volume method by formally averaging the conservation law (2.4) over each element $K \in \mathcal{T}^h$ of the triangulation, as follows. Applying Stokes theorem with a smooth solution u to (2.4), we get

$$0 = \int_K d(\omega(u)) = \int_{\partial K} i^* \omega(u).$$

Then, decomposing the boundary ∂K into its parts e_K^+, e_K^-, and $\partial^0 K$ we find

$$\int_{e_K^+} i^* \omega(u) - \int_{e_K^-} i^* \omega(u) + \sum_{e^0 \in \partial^0 K} \int_{e^0} i^* \omega(u) = 0. \tag{2.20}$$

Given the averaged values u_K^- along e_K^- and $u_{K_{e^0}}^-$ along $e^0 \in \partial^0 K$, we need an approximation u_K^+ of the average value of the solution u along e_K^+. To this end, the second term in (2.20) can be approximated by

$$\int_{e_K^-} i^* \omega(u) \approx \int_{e_K^-} i^* \omega(u_K^-) = |e_K^-| \varphi_{e_K^-}(u_K^-)$$

and the last term by

$$\int_{e^0} i^* \omega(u) \approx \mathbf{q}_{K,e^0}(u_K^-, u_{K_{e^0}}^-),$$

where the *total discrete flux* $\mathbf{q}_{K,e^0} : \mathbb{R}^2 \to \mathbb{R}$ (i.e., a scalar-valued function) must be prescribed.

Finally, the proposed version of the *finite volume method* for the conservation law (2.4) takes the form

$$\int_{e_K^+} i^*\omega(u_K^+) = \int_{e_K^-} i^*\omega(u_K^-) - \sum_{e^0 \in \partial^0 K} \mathbf{q}_{K,e^0}(u_K^-, u_{K_{e^0}}^-) \tag{2.21}$$

or, equivalently,

$$|e_K^+|\varphi_{e_K^+}(u_K^+) = |e_K^-|\varphi_{e_K^-}(u_K^-) - \sum_{e^0 \in \partial^0 K} \mathbf{q}_{K,e^0}(u_K^-, u_{K_{e^0}}^-). \tag{2.22}$$

We assume that the functions $\mathbf{q}_{K,e^0}$ satisfy the following natural assumptions for all $\overline{u}, \overline{v} \in \mathbb{R}$:

- *Consistency property :*

$$\mathbf{q}_{K,e^0}(\overline{u}, \overline{u}) = \int_{e^0} i^*\omega(\overline{u}). \tag{2.23}$$

- *Conservation property :*

$$\mathbf{q}_{K,e^0}(\overline{v}, \overline{u}) = -q_{K_{e^0},e^0}(\overline{u}, \overline{v}). \tag{2.24}$$

- *Monotonicity property :*

$$\partial_{\overline{u}} \mathbf{q}_{K,e^0}(\overline{u}, \overline{v}) \geq 0, \qquad \partial_{\overline{v}} \mathbf{q}_{K,e^0}(\overline{u}, \overline{v}) \leq 0. \tag{2.25}$$

We note that, in our notation, there is some ambiguity with the orientation of the faces of the triangulation. To complete the definition of the scheme we need to specify the discretization of the initial data and we define constant initial values $u_{K,0} = u_K^-$ (for $K \in \mathcal{T}_0^h$) associated with the initial slice $\mathcal{H}_0$ by setting

$$\int_{e_K^-} i^*\omega(u_K^-) := \int_{e_K^-} i^*\omega(u_0), \qquad e_K^- \subset \mathcal{H}_0. \tag{2.26}$$

Finally, we define a piecewise constant function $u^h : M \to \mathbb{R}$ by setting for every element $K \in \mathcal{T}^h$

$$u^h(x) = u_K^-, \qquad x \in K. \tag{2.27}$$

It will be convenient to introduce $N_K := \#\partial^0 K$, the total number of "vertical" neighbors of an element $K \in \mathcal{T}^h$, which we suppose to be uniformly bounded. For definiteness, we fix a finite family of local charts covering the manifold M, and we assume that the parameter h coincides with the largest diameter of faces $e_K^\pm$ of elements $K \in \mathcal{T}^h$, where the diameter is computed with the Euclidian metric expressed in the chosen local coordinates (which are fixed once for all and, of course, overlap in certain regions of the manifold).

For the sake of stability we will need to restrict the time-evolution and impose the following version of the Courant-Friedrich-Levy condition: for all $K \in \mathcal{T}^h$,

$$\frac{N_K}{|e_K^+|} \max_{e^0 \in \partial^0 K} \sup_u \left| \int_{e^0} \partial_u \omega(u) \right| < \inf_u \partial_u \varphi_{e_K^+}, \tag{2.28}$$

in which the supremum and infimum in u are taken over the range of the initial data.

We then assume the following conditions on the family of triangulations:

$$\lim_{h \to 0} \frac{\tau_{\max}^2 + h^2}{\tau_{\min}} = \lim_{h \to 0} \frac{\tau_{\max}^2}{h} = 0 \tag{2.29}$$

where $\tau_{\max} := \max_i(t_{i+1} - t_i)$ and $\tau_{\min} := \min_i(t_{i+1} - t_i)$. For instance, these conditions are satisfied if $\tau_{\max}$, $\tau_{\min}$, and h vanish at the same order.

Our main objective in the rest of this paper is to prove the convergence of the above scheme towards an entropy solution in the sense defined in the previous section.

2.3.2 A convex decomposition

Our analysis of the finite volume method relies on a decomposition of (2.22) into essentially one-dimensional schemes. This technique goes back to Tadmor [46], Coquel and LeFloch [9], and Cockburn, Coquel, and LeFloch [15].

By applying Stokes theorem to (2.16) with an arbitrary $\overline{u} \in \mathbb{R}$, we have

$$\begin{aligned} 0 &= \int_K d(\omega(\overline{u})) = \int_{\partial K} i^* \omega(\overline{u}) \\ &= \int_{e_K^+} i^* \omega(\overline{u}) - \int_{e_K^-} i^* \omega(\overline{u}) + \sum_{e^0 \in \partial^0 K} \mathbf{q}_{K,e^0}(\overline{u}, \overline{u}). \end{aligned}$$

Choosing $\overline{u} = u_K^-$, we deduce the identity

$$|e_K^+| \varphi_{e_K^+}(u_K^-) = |e_K^-| \varphi_{e_K^-}(u_K^-) - \sum_{e^0 \in \partial^0 K} \mathbf{q}_{K,e^0}(u_K^-, u_K^-), \tag{2.30}$$

which can be combined with (2.22) so that

$$\begin{aligned} &\varphi_{e_K^+}(u_K^+) \\ &= \varphi_{e_K^+}(u_K^-) - \sum_{e^0 \in \partial^0 K} \frac{1}{|e_K^+|} \Big(\mathbf{q}_{K,e^0}(u_K^-, u_{K_{e^0}}^-) - \mathbf{q}_{K,e^0}(u_K^-, u_K^-) \Big) \\ &= \sum_{e^0 \in \partial^0 K} \left(\frac{1}{N_K} \varphi_{e_K^+}(u_K^-) - \frac{1}{|e_K^+|} \Big(\mathbf{q}_{K,e^0}(u_K^-, u_{K_{e^0}}^-) - \mathbf{q}_{K,e^0}(u_K^-, u_K^-) \Big) \right). \end{aligned}$$

By introducing the intermediate values $\widetilde{u}^+_{K,e^0}$ given by

$$\varphi_{e_K^+}(\widetilde{u}^+_{K,e^0}) := \varphi_{e_K^+}(u_K^-) - \frac{N_K}{|e_K^+|}\Big(\mathbf{q}_{K,e^0}(u_K^-, u_{K_{e^0}}^-) - \mathbf{q}_{K,e^0}(u_K^-, u_K^-)\Big), \tag{2.31}$$

we arrive at the desired *convex decomposition*

$$\varphi_{e_K^+}(u_K^+) = \frac{1}{N_K} \sum_{e^0 \in \partial^0 K} \varphi_{e_K^+}(\widetilde{u}^+_{K,e^0}). \tag{2.32}$$

Given any entropy pair (U, Ω) and any hypersurface $e \subset M$ satisfying $|e| > 0$ we introduce the averaged entropy flux along e defined by

$$\varphi_e^\Omega(u) := \fint_e i^* \Omega(u).$$

Obviously, we have $\varphi_e^\omega(u) = \varphi_e(u)$.

Lemma 2.8. *For every convex entropy flux Ω one has*

$$\varphi^\Omega_{e_K^+}(u_K^+) \leq \frac{1}{N_K} \sum_{e^0 \in \partial^0 K} \varphi^\Omega_{e_K^+}(\widetilde{u}^+_{K,e^0}). \tag{2.33}$$

The proof below will actually show that the function $\varphi^\Omega_{e_K^+} \circ (\varphi^\omega_{e_K^+})^{-1}$ is convex.

Proof. It suffices to show the inequality for the entropy flux themselves, and then to average this inequality over e. So, we need to check:

$$\Omega(u_K^+) \leq \frac{1}{N_K} \sum_{e^0 \in \partial^0 K} \Omega(\widetilde{u}^+_{K,e^0}). \tag{2.34}$$

Namely, we have

$$\begin{aligned} &\frac{1}{N_K} \sum_{e^0 \in \partial^0 K} \Big(\Omega(\widetilde{u}^+_{K,e^0}) - \Omega(u_K^+)\Big) \\ &= \frac{1}{N_K} \sum_{e^0 \in \partial^0 K} \Big(\omega(u_K^+) - \omega(\widetilde{u}^+_{K,e^0})\Big) \partial_u U(u_K^+) + \frac{1}{N_K} \sum_{e^0 \in \partial^0 K} D_{K,e^0}, \end{aligned}$$

with

$$D_{K,e^0} := \int_0^1 \partial_{uu} U(u_K^+) \Big(\omega(\widetilde{u}^+_{K,e^0} + a(u_K^+ - \widetilde{u}^+_{K,e^0})) - \omega(\widetilde{u}^+_{K,e^0})\Big)\,(u_K^+ - \widetilde{u}^+_{K,e^0})\, da.$$

In the right-hand side of the above identity, the former term vanishes identically in view of (2.31) while the latter term is non-negative since $U(u)$ is convex in u and $\partial_u \omega$ is a positive n-form. □

2.4 Discrete stability estimates

2.4.1 Entropy inequalities

Using the convex decomposition (2.32), we can derive a discrete version of the entropy inequalities.

Lemma 2.9 (Entropy inequalities for the faces). *For every convex entropy pair $(\mathcal{U}, \Omega)$ and all $K \in \mathcal{T}^h$ and $e^0 \in \partial^0 K$, there exists a family of numerical entropy flux functions $\mathbf{Q}_{K,e^0} : \mathbb{R}^2 \to \mathbb{R}$ satisfying the following conditions for all $u, v \in \mathbb{R}$:*

- $\mathbf{Q}_{K,e^0}$ *is consistent with the entropy flux Ω:*

$$\mathbf{Q}_{K,e^0}(u,u) = \int_{e^0} i^* \Omega(u). \tag{2.35}$$

- *Conservation property:*

$$\mathbf{Q}_{K,e^0}(u,v) = -Q_{K_{e^0},e^0}(v,u). \tag{2.36}$$

- *Discrete entropy inequality: with the notation introduced earlier, the finite volume scheme satisfies*

$$\varphi^{\Omega}_{e_K^+}(\widetilde{u}^+_{K,e^0}) - \varphi^{\Omega}_{e_K^+}(u_K^-) + \frac{N_K}{|e_K^+|}\Big(\mathbf{Q}_{K,e^0}(u_K^-, u_{K_{e^0}}^-) - \mathbf{Q}_{K,e^0}(u_K^-, u_K^-)\Big) \le 0. \tag{2.37}$$

Combining Lemma 2.8 with the above lemma immediately implies:

Lemma 2.10 (Entropy inequalities for the elements). *For each $K \in \mathcal{T}^h$ one has the inequality*

$$|e_K^+|\left(\varphi^{\Omega}_{e_K^+}(u_K^+) - \varphi^{\Omega}_{e_K^+}(u_K^-)\right) + \sum_{e^0 \in \partial^0 K} \left(Q(u_K^-, u_{K_{e^0}}^-) - Q(u_K^-, u_K^-)\right) \le 0. \tag{2.38}$$

Proof of Lemma 2.9. Step 1. For $u, v \in \mathbb{R}$ and $e^0 \in \partial^0 K$ we introduce the notation

$$H_{K,e^0}(u,v) := \varphi_{e_K^+}(u) - \frac{N_K}{|e_K^+|}\Big(\mathbf{q}_{K,e^0}(u,v) - \mathbf{q}_{K,e^0}(u,u)\Big).$$

Observe that

$$H_{K,e^0}(u,u) = \varphi_{e_K^+}(u).$$

We claim that H_{K,e^0} satisfies the following properties:

$$\frac{\partial}{\partial u} H_{K,e^0}(u,v) \ge 0, \qquad \frac{\partial}{\partial v} H_{K,e^0}(u,v) \ge 0. \tag{2.39}$$

The proof of the second property is immediate by the monotonicity property (2.25), whereas, for the first one, we use the CFL condition (2.28) together with the monotonicity property (2.25). From the definition of $H_{K,e^0}(u,v)$, we observe that

$$H_{K,e^0}(u, u_{K_{e^0}}) = \Big(1 - \sum_{e^0 \in \partial^0 K} \alpha_{K,e^0}\Big)\varphi_{e_K^+}(u) + \sum_{e^0 \in \partial^0 K} \alpha_{K,e^0}\varphi_{e_K^+}(u_{K_{e^0}}),$$

where

$$\alpha_{K,e^0} := \frac{1}{|e_K^+|}\frac{\mathbf{q}_{K,e^0}(u, u_{K_{e^0}}) - \mathbf{q}_{K,e^0}(u,u)}{\varphi_{e_K^+}(u) - \varphi_{e_K^+}(u_{K_{e^0}})}.$$

This gives a convex combination of $\varphi_{e_K^+}(u)$ and $\varphi_{e_K^+}(u_{K_{e^0}})$. Indeed, by the monotonicity property (2.25) we have $\sum_{e^0 \in \partial^0 K} \alpha_{K,e^0} \geq 0$ and the CFL condition (2.28) gives us

$$\sum_{e^0 \in \partial^0 K} \alpha_{K,e^0} \leq \sum_{e^0 \in \partial^0 K} \frac{1}{|e_K^+|}\left|\frac{\mathbf{q}_{K,e^0}(u, u_{K_{e^0}}) - \mathbf{q}_{K,e^0}(u,u)}{\varphi_{e_K^+}(u) - \varphi_{e_K^+}(u_{K_{e^0}})}\right| \leq 1.$$

Step 2. It is sufficient to establish the entropy inequalities for the family of Kruzkov's entropies $\mathbf{\Omega}$. In connection with this choice, we introduce the numerical version of Kruzkov's entropy flux

$$\mathbf{Q}(u,v,c) := \mathbf{q}_{K,e^0}(u \vee c, v \vee c) - \mathbf{q}_{K,e^0}(u \wedge c, v \wedge c),$$

where $a \vee b = \max(a,b)$ and $a \wedge b = \min(a,b)$. Observe that $\mathbf{Q}_{K,e^0}(u,v)$ satisfies the first two properties of the lemma with the entropy flux replaced by the Kruzkov's family of entropies $\Omega = \mathbf{\Omega}$ defined in (2.8).

First, we observe

$$\begin{aligned}
&H_{K,e^0}(u \vee c, v \vee c) - H_{K,e^0}(u \wedge c, v \wedge c) \\
&= \varphi_{e_K^+}(u \vee c) - \frac{N_K}{|e_K^+|}\Big(\mathbf{q}_{K,e^0}(u \vee c, v \vee c) - \mathbf{q}_{K,e^0}(u \vee c, u \vee c)\Big) \\
&\quad - \Big(\varphi_{e_K^+}(u \wedge c) - \frac{N_K}{|e_K^+|}\Big(\mathbf{q}_{K,e^0}(u \wedge c, v \wedge c) - \mathbf{q}_{K,e^0}(u \wedge c, u \wedge c)\Big)\Big) \\
&= \varphi^{\Omega}_{e_K^+}(u,c) - \frac{N_K}{|e_K^+|}\Big(\mathbf{Q}(u,v,c) - \mathbf{Q}(u,u,c)\Big),
\end{aligned} \tag{2.40}$$

where we used

$$\varphi_{e_K^+}(u \vee c) - \varphi_{e_K^+}(u \wedge c) = \int_{e_K^+} i^*\mathbf{\Omega}(u,c) = \varphi^{\Omega}_{e_K^+}(u,c).$$

Second, we check that for $u = u_K^-$, $v = u_{K_{e^0}}^-$ and for any $c \in \mathbb{R}$

$$H_{K,e^0}(u_K^- \vee c, u_{K_{e^0}}^- \vee c) - H_{K,e^0}(u_K^- \wedge c, u_{K_{e^0}}^- \wedge c) \geq \varphi^{\Omega}_{e_K^+}(\widetilde{u}^+_{K,e^0}, c). \tag{2.41}$$

To prove (2.41) we observe that

$$
\begin{aligned}
&H_{K,e^0}(u,v)\vee H_{K,e^0}(\lambda,\lambda)\le H_{K,e^0}(u\vee\lambda, v\vee\lambda),\\
&H_{K,e^0}(u,v)\wedge H_{K,e^0}(\lambda,\lambda)\ge H_{K,e^0}(u\wedge\lambda, v\wedge\lambda),
\end{aligned}
$$

where H_{K,e^0} is monotone in both variables. Since $\varphi_{e_K^+}$ is monotone, we have

$$
\begin{aligned}
&H_{K,e^0}(u_K^-\vee c, u_{K_{e^0}}^-\vee c)-H_{K,e^0}(u_K^-\wedge c, u_{K_{e^0}}^-\wedge c)\\
&\ge \Big|H_{K,e^0}(u_K^-, u_{K_{e^0}}^-)-H_{K,e^0}(c,c)\Big| = \Big|\varphi_{e_K^+}(\widetilde{u}_{K,e^0}^+)-\varphi_{e_K^+}(c)\Big|\\
&=\operatorname{sgn}\Big(\varphi_{e_K^+}(\widetilde{u}_{K,e^0}^+)-\varphi_{e_K^+}(c)\Big)\Big(\varphi_{e_K^+}(\widetilde{u}_{K,e^0}^+)-\varphi_{e_K^+}(c)\Big)\\
&=\operatorname{sgn}\Big(\widetilde{u}_{K,e^0}^+-c\Big)\Big(\varphi_{e_K^+}(\widetilde{u}_{K,e^0}^+)-\varphi_{e_K^+}(c)\Big)=\varphi_{e_K^+}^{\Omega}(\widetilde{u}_{K,e^0}^+,c).
\end{aligned}
$$

Combining this identity with (2.40) (with $u=u_K^-$, $v=u_{K_{e^0}}^-$), we obtain the following inequality for the Kruzkov's entropies

$$
\varphi_{e_K^+}^{\Omega}(\widetilde{u}_{K,e^0}^+,c)-\varphi_{e_K^+}^{\Omega}(u_K^-,c)+\frac{N_K}{|e_K^+|}\Big(\mathbf{Q}(u,v,c)-\mathbf{Q}(u,u,c)\Big)\le 0.
$$

As already noticed, this inequality implies a similar inequality for all convex entropy flux fields and this completes the proof. □

If V is a convex function, then a *modulus of convexity* for V is any positive real $\beta<\inf V''$, where the infimum is taken over the range of data under consideration. We have seen in the proof of Lemma 2.8 that $\varphi_e^{\Omega}\circ(\varphi_e^{\omega})^{-1}$ is convex for every spacelike hypersurface e and every convex function U (involved in the definition of Ω).

Lemma 2.11 (Entropy balance inequality between two hypersurfaces)**.** *For $K\in\mathcal{T}^h$, let $\beta_{e_K^+}$ be a modulus of convexity for the function $\varphi_{e_K^+}^{\Omega}\circ\left(\varphi_{e_K^+}^{\omega}\right)^{-1}$ and set $\beta=\min_{K\in\mathcal{T}^h}\beta_{e_K^+}$. Then, for $i\le j$ one has*

$$
\begin{aligned}
&\sum_{K\in\mathcal{T}_{t_j}^h}|e_K^+|\varphi_{e_K^+}^{\Omega}(u_K^+)+\sum_{\substack{K\in\mathcal{T}_{[t_i,t_j)}^h\\ e^0\in\partial^0K}}\frac{\beta}{2N_K}|e_K^+|\Big|\widetilde{u}_{K,e^0}^+-u_K^+\Big|^2\\
&\le\sum_{K\in\mathcal{T}_{t_i}^h}|e_K^-|\varphi_{e_K^-}^{\Omega}(u_K^-),
\end{aligned}
\tag{2.42}
$$

where $\mathcal{T}_{t_i}^h$ is the subset of all elements K satisfying $e_K^-\in H_{t_i}$ while $\mathcal{T}_{[t_i,t_j)}^h:=\bigcup_{i\le k<j}\mathcal{T}_{t_k}^h$.

We observe that the numerical entropy flux terms no longer appear in (2.42).

Proof. Consider the discrete entropy inequality (2.37). Multiplying by $|e_K^+|/N_K$ and summing in $K \in \mathcal{T}^h$, $e^0 \in \partial^0 K$ gives

$$\begin{aligned}&\sum_{\substack{K\in\mathcal{T}^h\\ e^0\in\partial^0 K}} \frac{|e_K^+|}{N_K}\varphi^\Omega_{e_K^+}(\widetilde{u}^+_{K,e^0}) - \sum_{K\in\mathcal{T}^h} |e_K^+|\varphi^\Omega_{e_K^+}(u_K^-)\\ &+\sum_{\substack{K\in\mathcal{T}^h\\ e^0\in\partial^0 K}} \Big(\mathbf{Q}_{K,e^0}(u_K^-,u^-_{K_{e^0}}) - \mathbf{Q}_{K,e^0}(u_K^-,u_K^-)\Big) \le 0.\end{aligned} \tag{2.43}$$

Next, observe that the conservation property (2.36) gives

$$\sum_{\substack{K\in\mathcal{T}^h\\ e^0\in\partial^0 K}} \mathbf{Q}_{K,e^0}(u_K^-,u^-_{K_{e^0}}) = 0. \tag{2.44}$$

So (2.43) becomes

$$\begin{aligned}&\sum_{\substack{K\in\mathcal{T}^h\\ e^0\in\partial^0 K}} \frac{|e_K^+|}{N_K}\varphi^\Omega_{e_K^+}(\widetilde{u}^+_{K,e^0}) - \sum_{K\in\mathcal{T}^h} |e_K^+|\varphi^\Omega_{e_K^+}(u_K^-)\\ &-\sum_{\substack{K\in\mathcal{T}^h\\ e^0\in\partial^0 K}} \mathbf{Q}_{K,e^0}(u_K^-,u_K^-) \le 0.\end{aligned} \tag{2.45}$$

Now, if V is a convex function, and if $v = \sum_j \alpha_j v_j$ is a convex combination of v_j, then an elementary result on convex functions gives

$$V(v) + \frac{\beta}{2}\sum_j \alpha_j |v_j - v|^2 \le \sum_j \alpha_j V(v_j),$$

where $\beta = \inf V''$, the infimum being taken over all v_j. We apply this inequality with $v = \varphi_{e_K^+}(u_K^+)$ and $V = \varphi^\Omega_{e_K^+} \circ (\varphi^\omega_{e_K^+})^{-1}$, which is convex.

Thus, in view of the convex combination (2.32) and by multiplying the above inequality by $|e_K^+|$ and then summing in $K \in \mathcal{T}^h$, we obtain

$$\begin{aligned}&\sum_{\substack{K^n\in\mathcal{K}^n\\ e^0\in\partial^0 K^n}} |e_K^+|\varphi^\Omega_{e_K^+}(u_K^+) + \sum_{\substack{K^n\in\mathcal{K}^n\\ e^0\in\partial^0 K^n}} \frac{\beta}{2}\frac{|e_K^+|}{N_K}|\widetilde{u}^+_{K,e^0} - u_K^+|^2\\ &\le \sum_{\substack{K^n\in\mathcal{K}^n\\ e^0\in\partial^0 K^n}} \frac{|e_K^+|}{N_K}\varphi^\Omega_{e_K^+}(\widetilde{u}^+_{K,e^0}).\end{aligned}$$

Combining the result with (2.45), we find

$$\begin{aligned}&\sum_{K\in\mathcal{T}^h}|e_K^+|\varphi^{\Omega}_{e_K^+}(u_K^+)-\sum_{K\in\mathcal{T}^h}|e_K^+|\varphi^{\Omega}_{e_K^+}(u_K^-)+\sum_{\substack{K^n\in\mathcal{K}^n\\ e^0\in\partial^0K^n}}\frac{\beta}{2}\frac{|e_K^+|}{N_K}|\widetilde{u}^+_{K,e^0}-u_K^+|^2\\&\leq\sum_{\substack{K^n\in\mathcal{K}^n\\ e^0\in\partial^0K^n}}\mathbf{Q}_{K,e^0}(u_K^-,u_K^-).\end{aligned}\tag{2.46}$$

Using finally the identity

$$\begin{aligned}0&=\int_K d(\Omega(u_K^-))=\int_{\partial K}i^*\Omega(u_K^-)\\&=|e_K^+|\varphi^{\Omega}_{e_K^+}(u_K^-)-|e_K^-|\varphi^{\Omega}_{e_K^-}(u_K^-)+\sum_{e^0\in\partial^0K}\mathbf{Q}_{K,e^0}(u_K^-,u_K^-),\end{aligned}$$

we obtain the desired inequality, after further summation over all of the elements K within two arbitrary hypersurfaces. □

We apply Lemma 2.11 with a specific choice of entropy function U and obtain the following uniform estimate.

Lemma 2.12 (Global entropy dissipation estimate)**.** *The following global estimate of the entropy dissipation holds:*

$$\sum_{\substack{K\in\mathcal{T}^h\\ e^0\in\partial^0K}}\frac{|e_K^+|}{N_K}\left|\widetilde{u}^+_{K,e^0}-u_K^+\right|^2\lesssim C\int_{\mathcal{H}_0}i^*\Omega(u_0)\tag{2.47}$$

for some uniform constant $C>0$, which only depends upon the flux field and the sup-norm of the initial data, and where Ω is the n-form entropy flux field associated with the quadratic entropy function $U(u)=u^2/2$.

Proof. We apply the inequality (2.42) with the choice $U(u)=u^2$

$$0\geq\sum_{K\in\mathcal{T}^h}(|e_K^+|\varphi^{\Omega}_{e_K^+}(u_K^+)-|e_K^-|\varphi^{\Omega}_{e_K^-}(u_K^-))+\sum_{\substack{K\in\mathcal{T}^h\\ e^0\in\partial^0K}}\frac{\beta}{2}\frac{|e_K^+|}{N_K}\left|\widetilde{u}^+_{K,e^0}-u_K^+\right|^2.$$

After summing up in the "vertical" direction and keeping only the contribution of the elements $K\in\mathcal{T}_0^h$ on the initial hypersurface $\mathcal{H}_0$, we find

$$\sum_{\substack{K\in\mathcal{T}^h\\ e^0\in\partial^0K}}\frac{|e_K^+|}{N_K}\left|\widetilde{u}^+_{K,e^0}-u_K^+\right|^2\leq\frac{2}{\beta}\sum_{K\in\mathcal{T}_0^h}|e_K^-|\varphi^{\Omega}_{e_K^-}(u_{K,0}).$$

Finally, we observe that, for some uniform constant $C > 0$,

$$\sum_{K \in \mathcal{T}_0^h} |e_K^-| \varphi_{e_K^-}^{\Omega}(u_{K,0}) \leq C \int_{\mathcal{H}_0} i^* \Omega(u_0).$$

These expressions are essentially L^2 norm of the initial data, and the above inequality can be checked by fixing a reference volume form on the initial hypersurface $\mathcal{H}_0$ and using the discretization (2.26) of the initial data u_0. □

2.4.2 Global form of the discrete entropy inequalities

We now derive a global version of the (local) entropy inequality (2.37), i.e. we obtain a discrete version of the entropy inequalities arising in the very definition of entropy solutions.

One additional notation is necessary to handle "vertical face" of the triangulation: we fix a reference field of non-degenerate n-forms $\widetilde{\omega}$ on M which will be used to measure the "area" of the faces $e^0 \in \partial^0 K$. This is necessary in our convergence proof, only, not in the formulation of the finite volume method. So, for every $K \in \mathcal{T}^h$ we define

$$|e^0|_{\widetilde{\omega}} := \int_{e^0} i^* \widetilde{\omega} \quad \text{for faces } e^0 \in \partial^0 K \tag{2.48}$$

and the non-degeneracy condition means that $|e^0|_{\widetilde{\omega}} > 0$.

Given a test-function ψ defined on M and a face $e^0 \in \partial^0 K$ of some element, we introduce the following averages

$$\psi_{e^0} := \frac{\int_{e^0} \psi \, i^* \widetilde{\omega}}{\int_{e^0} i^* \widetilde{\omega}}, \qquad \psi_{\partial^0 K} := \frac{1}{N_K} \sum_{e^0 \in \partial^0 K} \psi_{e^0},$$

where, for the first time in our analysis, we use the reference n-volume form $\widetilde{\omega}$.

Lemma 2.13 (Global form of the discrete entropy inequalities). *Let Ω be a convex entropy flux field, and let ψ be a non-negative test-function supported away from the hypersurface $t = T$. Then, the finite volume approximations satisfy the global entropy inequality*

$$\begin{aligned} &- \sum_{K \in \mathcal{T}^h} \int_K d(\psi \Omega)(u_K^-) - \sum_{K \in \mathcal{T}_0^h} \int_{e_K^-} \psi \, i^* \Omega(u_{K,0}) \\ &\leq A^h(\psi) + B^h(\psi) + C^h(\psi), \end{aligned} \tag{2.49}$$

with

$$A^h(\psi) := \sum_{\substack{K^n\in\mathcal{K}^n\\ e^0\in\partial^0K^n}} \frac{|e_K^+|}{N_K}\Big(\psi_{\partial^0K}-\psi_{e^0}\Big)\Big(\varphi^{\Omega}_{e_K^+}(\widetilde{u}^+_{K,e^0})-\varphi^{\Omega}_{e_K^+}(u_K^+)\Big),$$

$$B^h(\psi) := \sum_{\substack{K^n\in\mathcal{K}^n\\ e^0\in\partial^0K^n}} \int_{e^0}\Big(\psi_{e^0}-\psi\Big)\, i^*\Omega(u_K^-),$$

$$C^h(\psi) := -\sum_{K\in\mathcal{T}^h}\int_{e_K^+}\Big(\psi_{\partial^0K}-\psi\Big)\Big(i^*\Omega(u_K^+)-i^*\Omega(u_K^-)\Big).$$

Proof. From the discrete entropy inequalities (2.37), we obtain

$$\begin{aligned}&\sum_{\substack{K^n\in\mathcal{K}^n\\ e^0\in\partial^0K^n}} \frac{|e_K^+|}{N_K}\psi_{e^0}\Big(\varphi^{\Omega}_{e_K^+}(\widetilde{u}^+_{K,e^0})-\varphi^{\Omega}_{e_K^+}(u_K^-)\Big)\\ &\quad+\sum_{\substack{K^n\in\mathcal{K}^n\\ e^0\in\partial^0K^n}} \psi_{e^0}\Big(\mathbf{Q}_{K,e^0}(u_K^-,u^-_{K_{e^0}})-\mathbf{Q}_{K,e^0}(u_K^-,u_K^-)\Big)\le 0.\end{aligned} \tag{2.50}$$

Thanks the conservation property (2.36), we have

$$\sum_{\substack{K^n\in\mathcal{K}^n\\ e^0\in\partial^0K^n}} \psi_{e^0}\mathbf{Q}_{K,e^0}(u_K^-,u^-_{K_{e^0}}) = 0$$

and, from the consistency property (2.35),

$$\begin{aligned}\sum_{\substack{K^n\in\mathcal{K}^n\\ e^0\in\partial^0K^n}} \psi_{e^0}\mathbf{Q}_{K,e^0}(u_K^-,u_K^-) &= \sum_{\substack{K^n\in\mathcal{K}^n\\ e^0\in\partial^0K^n}} \psi_{e^0}\int_{e^0} i^*\Omega(u_K^-)\\ &= \sum_{\substack{K^n\in\mathcal{K}^n\\ e^0\in\partial^0K^n}} \int_{e^0}\psi\, i^*\Omega(u_K^-) + \sum_{\substack{K^n\in\mathcal{K}^n\\ e^0\in\partial^0K^n}} \int_{e^0}(\psi_{e^0}-\psi) i^*\Omega(u_K^-).\end{aligned}$$

Next, we observe

$$\begin{aligned}&\sum_{\substack{K^n\in\mathcal{K}^n\\ e^0\in\partial^0K^n}} \frac{|e_K^+|}{N_K}\psi_{e^0}\,\varphi^{\Omega}_{e_K^+}(\widetilde{u}^+_{K,e^0})\\ &= \sum_{\substack{K^n\in\mathcal{K}^n\\ e^0\in\partial^0K^n}} \frac{|e_K^+|}{N_K}\psi_{\partial^0K}\varphi^{\Omega}_{e_K^+}(\widetilde{u}^+_{K,e^0}) + \sum_{\substack{K^n\in\mathcal{K}^n\\ e^0\in\partial^0K^n}} \frac{|e_K^+|}{N_K}(\psi_{e^0}-\psi_{\partial^0K})\varphi^{\Omega}_{e_K^+}(\widetilde{u}^+_{K,e^0})\\ &\ge \sum_{K\in\mathcal{T}^h}|e_K^+|\psi_{\partial^0K}\varphi^{\Omega}_{e_K^+}(u_K^+) + \sum_{\substack{K^n\in\mathcal{K}^n\\ e^0\in\partial^0K^n}} \frac{|e_K^+|}{N_K}(\psi_{e^0}-\psi_{\partial^0K})\varphi^{\Omega}_{e_K^+}(\widetilde{u}^+_{K,e^0}),\end{aligned}$$

where, we used the inequality (2.33) and the convex combination (2.32). In view of

$$\sum_{\substack{K^n\in\mathcal{K}^n\\ e^0\in\partial^0K^n}} \frac{|e_K^+|}{N_K}\psi_{e^0}\,\varphi^\Omega_{e_K^+}(u_K^-) = \sum_{K\in\mathcal{T}^h} |e_K^+|\psi_{\partial^0K}\,\varphi^\Omega_{e_K^+}(u_K^-),$$

the inequality (2.50) becomes

$$\begin{aligned}&\sum_{K\in\mathcal{T}^h}|e_K^+|\psi_{\partial^0K}\Big(\varphi^\Omega_{e_K^+}(u_K^+)-\varphi^\Omega_{e_K^+}(u_K^-)\Big)-\sum_{\substack{K^n\in\mathcal{K}^n\\ e^0\in\partial^0K^n}}\int_{e^0}\psi\, i^*\Omega(u_K^-)\\ &\le -\sum_{\substack{K^n\in\mathcal{K}^n\\ e^0\in\partial^0K^n}}\frac{|e_K^+|}{N_K}(\psi_{e^0}-\psi_{\partial^0K})\varphi^\Omega_{e_K^+}(\widetilde{u}^+_{K,e^0})+\sum_{\substack{K^n\in\mathcal{K}^n\\ e^0\in\partial^0K^n}}\int_{e^0}(\psi_{e^0}-\psi)\, i^*\Omega(u_K^-).\end{aligned}\tag{2.51}$$

Note that the first term in (2.51) can be written as

$$\begin{aligned}&\sum_{K\in\mathcal{T}^h}|e_K^+|\psi_{\partial^0K}\Big(\varphi^\Omega_{e_K^+}(u_K^+)-\varphi^\Omega_{e_K^+}(u_K^-)\Big)\\ &=\sum_{K\in\mathcal{T}^h}\int_{e_K^+}\psi(i^*\Omega(u_K^+)-i^*\Omega(u_K^-))+\sum_{K\in\mathcal{T}^h}\int_{e_K^+}(\psi_{\partial^0K}-\psi)\,(i^*\Omega(u_K^+)-i^*\Omega(u_K^-)).\end{aligned}$$

We can sum up (with respect to K) the identities

$$\begin{aligned}&\int_K d(\psi\Omega)(u_K^-)=\int_{\partial K}\psi\, i^*\Omega(u_K^-)\\ &=\int_{e_K^+}\psi\, i^*\Omega(u_K^-)-\int_{e_K^-}\psi\, i^*\Omega(u_K^-)+\sum_{e^0\in\partial^0K}\int_{e^0}\psi\, i^*\Omega(u_K^-)\end{aligned}$$

and combine them with the inequality (2.51). Finally, we arrive at the desired conclusion by noting that

$$\sum_{K\in\mathcal{T}^h}\Big(\int_{e_K^+}\psi\, i^*\Omega(u_K^+)-\int_{e_K^-}\psi\, i^*\Omega(u_K^-)\Big)=-\sum_{K\in\mathcal{T}^h_0}\int_{e_K^-}\psi\, i^*\Omega(u_{K,0}).$$

□

2.5 Convergence and well-posedness results

We are now in a position to establish:

Theorem 2.14 (Convergence of the finite volume method)**.** *Under the assumptions made in Section 2.3 and provided the flux field is geometry-compatible, the family of approximate solutions u^h generated by the finite volume scheme converges (as $h\to 0$) to an entropy solution of the initial value problem* (2.4), (2.14).

Our proof of convergence of the finite volume method can be viewed as a generalization to spacetimes of the technique introduced by Cockburn, Coquel and LeFloch [12, 13] for the (flat) Euclidean setting and already extended to Riemannian manifolds by Amorim, Ben-Artzi, and LeFloch [2] and to Lorentzian manifolds by Amorim, LeFloch, and Okutmustur [3].

We also deduce that:

Corollary 2.15 (Well-posedness theory on a spacetime)**.** *Let* $\mathbf{M} = [0,T] \times N$ *be a* $(n+1)$*-dimensional spacetime foliated by n-dimensional hypersurfaces* H_t *(*$t \in [0,T]$*) with compact topology N (cf.* (2.4)*). Let* ω *be a geometry-compatible flux field on* $\mathbf{M}$ *satisfying the global hyperbolicity condition* (2.13)*. An initial data* u_0 *being prescribed on* $\mathcal{H}_0$*, the initial value problem* (2.4)*,* (2.14) *admits an entropy solution* $u \in L^\infty(M)$ *which, moreover, has well-defined* L^1 *traces on any spacelike hypersurface of M. These solutions determines a (Lipschitz continuous) contracting semi-group in the sense that the inequality*

$$\int_{H'} i_{H'}^* \mathbf{\Omega}\Big(u_{H'}, v_{H'}\Big) \leq \int_{H} i_{H}^* \mathbf{\Omega}\Big(u_{H}, v_{H}\Big) \tag{2.52}$$

holds for any two hypersurfaces H, H' *such that* H' *lies in the future of* H*, and the initial condition is assumed in the weak sense*

$$\lim_{\substack{t\to 0\\ t>0}} \int_{H_t} i_{H_t}^* \mathbf{\Omega}\Big(u(t), v(t)\Big) = \int_{\mathcal{H}_0} i_{\mathcal{H}_0}^* \mathbf{\Omega}(u_0, v_0). \tag{2.53}$$

We can also extend a result originally established by DiPerna [18] (for conservation laws posed on the Euclidian space) within the broad class of entropy measure-valued solutions.

Theorem 2.16. *Let* ω *be a geometry-compatible flux field on a spacetime* $\mathbf{M}$ *satisfying the global hyperbolicity condition* (2.13)*. Then, any entropy measure-valued solution* ν *(see Definition 2.5) to the initial value problem* (2.4)*,* (2.14) *reduces to a Dirac mass at each point, more precisely*

$$\nu = \delta_u, \tag{2.54}$$

where $u \in L^\infty(\mathbf{M})$ *is the unique entropy solution to the same problem.*

We omit the details of the proof, since it is a variant of the Riemannian proof given in [6].

It remains to provide a proof of Theorem 2.14. Recall that a Young measure ν allows us to determine all weak-$*$ limits of composite functions $a(u^h)$ for all continuous functions a, as $h \to 0$,

$$a(u^h) \overset{*}{\rightharpoonup} \langle \nu, a \rangle = \int_{\mathbb{R}} a(\lambda)\, d\nu(\lambda). \tag{2.55}$$

Lemma 2.17 (Entropy inequalities for the Young measure)**.** *Let ν be a Young measure associated with the finite volume approximations u^h. Then, for every convex entropy flux field Ω and every non-negative test-function ψ supported away from the hypersurface $t = T$, one has*

$$\int_M \langle \nu, d\psi \wedge \Omega(\cdot) \rangle - \int_{\mathcal{H}_0} i^* \Omega(u_0) \leq 0. \tag{2.56}$$

Based on this lemma, we are now in position to complete the proof of Theorem 2.14. Thanks to (2.56), we have for all convex entropy pairs (U, Ω),

$$d\langle \nu, \Omega(\cdot) \rangle \leq 0$$

in the sense of distributions on M. On the initial hypersurface $\mathcal{H}_0$ the (trace of the) Young measure ν coincides with the Dirac mass δ_{u_0}. By Theorem 2.16 there exists a unique function $u \in L^\infty(M)$ (the entropy solution to the initial-value problem under consideration) such that the measure ν coincides with the Dirac mass δ_u. Moreover, this property also implies that the approximations u^h converge strongly to u, and this concludes the proof of the convergence of the finite volume scheme.

Proof of Lemma 2.17. The proof is a direct passage to the limit in the inequality (2.49), by using the property (2.55) of the Young measure. First of all, we observe that, the left-hand side of the inequality (2.49) converges to the left-hand side of (2.56). Indeed, since ω is geometry-compatible, the first term of interest

$$\sum_{K \in \mathcal{T}^h} \int_K d(\psi \Omega)(u_K^-) = \sum_{K \in \mathcal{T}^h} \int_K d\psi \wedge \Omega(u_K^-) = \int_M d\psi \wedge \Omega(u^h)$$

converges to $\int_M \langle \nu, d\psi \wedge \Omega(\cdot) \rangle$. On the other hand, the initial contribution

$$\sum_{K \in \mathcal{T}_0^h} \int_{e_K^-} \psi \, i^* \Omega(u_{K,0}) = \int_{\mathcal{H}_0} \psi \, i^* \Omega(u_0^h) \to \int_{\mathcal{H}_0} \psi \, i^* \Omega(u_0),$$

in which u_0^h is the initial discretization of the data u_0 and converges strongly to u_0 since the maximal diameter h of the element tends to zero.

It remains to check that the terms on the right-hand side of (2.49) vanish in the limit $h \to 0$. We begin with the first term $A^h(\psi)$. Taking the modulus of this expression, applying Cauchy-Schwarz inequality, and finally using the global entropy dissipation estimate (2.47), we obtain

$$
\begin{aligned}
|A^h(\psi)| &\leq \sum_{\substack{K^n\in\mathcal{K}^n\\ e^0\in\partial^0 K^n}} \frac{|e_K^+|}{N_K} |\psi_{\partial^0 K} - \psi| |\widetilde{u}^+_{K,e^0} - u_K^-| \\
&\leq \Big(\sum_{\substack{K^n\in\mathcal{K}^n\\ e^0\in\partial^0 K^n}} \frac{|e_K^+|}{N_K} |\psi_{\partial^0 K} - \psi|^2 \Big)^{1/2} \Big(\sum_{\substack{K^n\in\mathcal{K}^n\\ e^0\in\partial^0 K^n}} \frac{|e_K^+|}{N_K} |\widetilde{u}^+_{K,e^0} - u_K^-|^2 \Big)^{1/2} \\
&\leq \Big(\sum_{\substack{K^n\in\mathcal{K}^n\\ e^0\in\partial^0 K^n}} \frac{|e_K^+|}{N_K} (C\,(\tau_{\max} + h))^2 \Big)^{1/2} \Big(\int_{\mathcal{H}_0} i^*\Omega(u_0) \Big)^{1/2},
\end{aligned}
$$

hence

$$
|A^h(\psi)| \leq C'\,(\tau_{\max} + h) \Big(\sum_{K\in\mathcal{T}^h} |e_K^+| \Big)^{1/2} \leq C'' \frac{\tau_{\max} + h}{(\tau_{\min})^{1/2}}.
$$

Here, Ω is associated with the quadratic entropy and have used the fact that $|\psi_{\partial^0 K} - \psi| \leq C\,(\tau_{\max} + h)$. Our conditions (2.29) imply the upper bound for $A^h(\psi)$ tends to zero with h.

Next, we rely on the regularity of ψ and Ω and estimate the second term on the right-hand side of (2.49). By setting

$$
C_{e^0} := \frac{\int_{e^0} i^*\Omega(u_K^-)}{\int_{e^0} i^*\widetilde{\omega}},
$$

we obtain

$$
\begin{aligned}
|B^h(\psi)| &= \Big| \sum_{\substack{K^n\in\mathcal{K}^n\\ e^0\in\partial^0 K^n}} \int_{e^0} (\psi_{e^0} - \psi) \big(i^*\Omega(u_K^-) - C_{e^0} i^*\widetilde{\omega} \big) \Big| \\
&\leq \sum_{\substack{K^n\in\mathcal{K}^n\\ e^0\in\partial^0 K^n}} \sup_K |\psi_{e^0} - \psi| \int_{e^0} \Big| i^*\Omega(u_K^-) - C_{e^0} i^*\widetilde{\omega} \Big| \\
&\leq C \sum_{\substack{K^n\in\mathcal{K}^n\\ e^0\in\partial^0 K^n}} (\tau_{\max} + h)^2 \, |e^0|_{\widetilde{\omega}},
\end{aligned}
$$

hence

$$
|B^h(\psi)| \leq C \frac{(\tau_{\max} + h)^2}{h}.
$$

Again, our assumptions imply the upper bound for $B^h(\psi)$ tends to zero with h.

Finally, consider the last term in the right-hand side of (2.49)

$$
|C^h(\psi)| \leq \sum_{K\in\mathcal{T}^h} |e_K^+| \sup_K |\psi_{\partial^0 K} - \psi| \int_{e_K^+} |i^*\Omega(u_K^+) - i^*\Omega(u_K^-)|,
$$

using the modulus defined in the beginning of Section 2. In view of the inequality (2.34), we obtain

$$|C^h(\psi)| \leq C \sum_{\substack{K^n \in \mathcal{K}^n \\ e^0 \in \partial^0 K^n}} \frac{|e_K^+|}{N_K} |\psi_{\partial^0 K} - \psi| \left| \widetilde{u}_{K,e^0}^+ - u_K^- \right|,$$

and it is now clear that $C^h(\psi)$ satisfies the same estimate as the one we derived for $A^h(\psi)$. □

Partie II

Estimation d'erreur et mise en oeuvre

Chapitre 3

Estimation d'erreur pour les méthodes de volumes finis sur une variété*

Error estimate for finite volume methods on manifolds

3.1 Introduction and background

The mathematical theory of hyperbolic conservation laws posed on curved manifolds M was initiated by Ben-Artzi and LeFloch [6], and developed together with collaborators [2, 3, 7, 32, 34, 35]. For these equations, a suitable generalization of Kruzkov's theory has now been established and provides the existence and uniqueness of an entropy solution to the initial and boundary value problem for a large class of hyperbolic conservation laws and manifolds. The convergence of the finite volume schemes with monotone flux was also established for conservation laws posed on manifolds.

The purpose of the present paper is to show that the error estimate for finite volume methods, due to Cockburn, Coquel, and LeFloch [15, 13] in the Euclidian setting carries over to curved manifolds. To this end, we will need to revisit Kuznetzov's approximation theory [28, 29] and adapt the technique developed in [13]. One technical difficulty addressed here is the adaption of the standard "doubling of variables" technique to curved manifolds. We recover that the rate of error in the L^1 norm is of order $h^{1/4}$, where h is the maximal diameter of an element of the triangulation of the manifold, as discovered in [13].

Recall that the well-posedness theory for hyperbolic conservation laws

*En collaboration avec P. G. LeFloch et W. Neves [33].

posed on a compact manifold was established in [6], while the convergence of monotone finite volume schemes was proved in [2]. In both papers, DiPerna's measure-valued solutions [18] were used and can be viewed as a generalization of Kruzkov's theory [27]. In contrast, in the present paper we rely on Kuznetsov's theory, which allows us to bypass DiPerna's notion of measure-valued solutions. Indeed, our main result in this paper provides both an error estimate in the L^1 norm and, as a corollary, the actual convergence of the scheme to the entropy solution; this result can be used to establish the existence of this entropy solution.

For another approach to conservation laws on manifolds we refer to Panov [40] and for high-order numerical methods to Rossmanith, Bale, and LeVeque [41] and the references therein. Concerning the Euclidian case $M = \mathbb{R}^n$ we want to mention that the work by Cockburn, Coquel, and LeFloch [15, 13] (submitted and distributed in 1990 and 1991, respectively) was followed by important developments and applications by Kröner [25] and Eymard, Gallouet, and Herbin [20] to various hyperbolic problems including also elliptic equations. In [15], the technique of convergence using measure-valued solutions goes back to pioneering works by Szepessy [43, 44] and Coquel and LeFloch [9, 10, 11]. Concerning the error estimates we also refer to Lucier [36, 37], as well as to Bouchut and Perthame [8] where the Kuznetsov theory is revisited.

An outline of this paper follows. In the rest of the present section we present some background on conservation laws on manifolds and briefly recall the corresponding well-posedness theory. Then in Section 2 we present the class of schemes under consideration together with the error estimate. Sections 3 and 4 contain estimates for various terms arising in the decomposition of the L^1 distance between the exact and the approximate solutions. The proof of the main theorem is given at the beginning of Section 4.

3.2 Conservation laws on a manifold

Let (M, g) be a connected, compact, n-dimensional, smooth manifold endowed with a smooth metric g, that is, a smooth and non-degenerate 2-covariant tensor field: for each $x \in M$, g_x is a scalar product on the tangent space T_xM at x. For any tangent vectors $X, Y \in T_xM$, we use the notation $g_x(X, Y) = \langle X, Y\rangle_g$ and $|X|_g := \langle X, X\rangle_g^{1/2}$. We denote by d_g the associated distance function and by $dv_g = dv_M$ the volume measure determined by the metric. Moreover, we denote by ∇_g the Levi-Civita connection associated with g. The divergence operator div_g of a vector field is defined intrinsically as the trace of its covariant derivative. It follows from the Gauss-Green formula that for every smooth vector field and any smooth open subset $S \subset M$

$$\int_S \mathrm{div}_g f \, dv_M = \int_{\partial S} \langle f, n\rangle_g \, dv_{\partial S},$$

where ∂S is the boundary of S, n is the outward unit normal along ∂S, and $dv_{\partial S}$ is the induced measure on ∂S.

Consider local coordinates (x^i) together with the associated basis of tangent vectors $\{\mathrm{e}_i\} = \{\partial_i\}$ and covectors $\{\mathrm{e}^i\}$. The differential of a function $u : M \to \mathbb{R}$ is the differential form $du = (du)_i\,\mathrm{e}^i = \frac{\partial u}{\partial x^i} e^i$, where the summation convention over repeated indices is used. The vector field $\nabla_g u$ associated with du is given by $\nabla_g u = (\nabla_g u)^i\;\; \mathrm{e}_i = g^{ij}\,(du)_j e_i$, where (g^{ij}) is the inverse of the matrix $(g_{ij}) = \Big(\langle \mathrm{e}_i, \mathrm{e}_j\rangle_g\Big)$. The covariant derivative of a vector field X is a $(1,1)$-tensor field whose coordinates are denoted by $(\nabla_g X)^j_k$. The following formula for the divergence of a smooth vector field will be useful:

$$\begin{aligned}\mathrm{div}_g\Big(f(u,x)\Big) &= du(\partial_u f(u,x)) + \Big(\,\mathrm{div}_g f\Big)(u,x)\\ &= \partial_u f^i\,\frac{\partial u}{\partial x^i} + \frac{1}{\sqrt{|g|}}\partial_i(\sqrt{|g|}\;f^i).\end{aligned}$$

We will use the following standard notation for function spaces defined on M. For $p \in [1,\infty]$ the usual norm of a function h in the Lebesgue space $L^p(M;g)$ is denoted by $\|h\|_{L^p(M;g)}$ and, when $p = \infty$, we also write $\|h\|_\infty$. For any $f \in L^1_{\mathrm{loc}}(M;g)$ and any open subset $N \subset M$ we use the notation

$$⨍_N f(y)\;dv_g(y) := |N|_g^{-1}\int_N f(y)\;dv_g(y), \qquad |N|_g := \int_N dv_g.$$

3.2.1 Well-posedness theory

We are interested in the following initial-value problem posed on the manifold (M,g)

$$u_t + \mathrm{div}_g\Big(f(u,\cdot)\Big) = 0 \qquad \text{on } \mathbb{R}_+ \times M, \tag{3.1}$$

$$u(0,\cdot) = u_0 \qquad \text{on } M, \tag{3.2}$$

where $u : \mathbb{R}_+ \times M \to \mathbb{R}$ is the unknown and the flux $f = f_x(u) = f(\overline{u},x)$ is a smooth vector field which is defined for all $x \in M$ and also depends smoothly upon the real parameter $\overline{u}$. The initial data in (3.2) is assumed to be measurable and bounded, i.e. $u_0 \in L^\infty(M)$. Moreover, f satisfies the following growth condition

$$\max_{x\in M}|\Big(\,\mathrm{div}_g f\Big)(u,x)| \le C + C'\,|u|, \qquad u \in \mathbb{R} \tag{3.3}$$

for some constants $C, C' > 0$.

Definition 3.1. *A pair (U,F) is called an entropy pair if $U : \mathbb{R} \to \mathbb{R}$ is a Lipschitz continuous function and $F = F(u,x)$ is a vector field such that, for almost all $\bar{u} \in \mathbb{R}$ and all $x \in M$,*

$$\partial_u F(\bar{u},x) = \partial_u U(\bar{u})\partial_u f(\bar{u},x).$$

If U is also convex, then (U,F) is called a convex entropy pair.

The most important example of convex entropy pairs is the family of Kruzkov's entropies, defined for $u, c \in \mathbb{R}$ by

$$\begin{aligned}\big(U(u,c), F_x(u,c)\big) &:= \big(|u-c|,\ \mathrm{sgn}(u-c)(f_x(u)-f_x(c))\big)\\ &= \big((u\vee c - u\wedge c),\ f_x(u\vee c)-f_x(u\wedge c)\big),\end{aligned} \tag{3.4}$$

where $u \vee c = \max\{u, c\}$, $u \wedge c = \min\{u, c\}$.

Definition 3.2. *A function $u \in L^\infty(\mathbb{R}_+ \times M)$ is called an entropy solution to the initial value problem* (3.1)-(3.2)*, if for every entropy pair* (U, F) *and all smooth functions $\phi = \phi(t, x) \geq 0$ compactly supported in* $[0, \infty) \times M$*,*

$$\begin{aligned}&\iint_{\mathbb{R}_+\times M} \big(U(u)\,\phi_t + \langle F_x(u), \nabla_g\phi\rangle_g\big)\,dv_g dt\\ &+ \iint_{\mathbb{R}_+\times M} G_x(u)\phi(t,x)\,dv_g dt + \int_M U(u_0(x))\phi(0)\,dv_g \geq 0,\end{aligned} \tag{3.5}$$

where $G_x(u) := (\mathrm{div}_g F_x)(u) - \partial_u U(u)(\mathrm{div}_g f_x)(u)$.

For instance, with Kruzkov's entropies the above definition becomes (for all $c \in \mathbb{R}$)

$$\begin{aligned}&\iint_{\mathbb{R}_+\times M} \big(U(u,c)\,\phi_t + \langle F_x(u,c), \nabla_g\phi\rangle_g\big)\,dv_g dt\\ &- \iint_{\mathbb{R}_+\times M} \mathrm{sgn}(u-c)\,(\mathrm{div}_g f)(c,x)\,\phi\,dv_g dt + \int_M |u_0 - c|\,\phi(0)\,dv_g \geq 0.\end{aligned}$$

The well-posed theory for the initial value problem (3.1)-(3.2) was established in Ben-Artzi and LeFloch [6].

In the present paper, we are interested in the discretization of the problem (3.1)-(3.2) in the case that the initial data is bounded and has finite total variation

$$u_0 \in L^\infty(M) \cap BV(M; g). \tag{3.6}$$

In particular, it is established in [6] that in the case of bounded initial data, the following variant of the maximum principle is established:

$$\|u(t)\|_{L^\infty(M)} \leq C_0(T, g) + C_0'(T, g)\,\|u(s)\|_{L^\infty(M)}, \qquad 0 \leq s \leq t \leq T,$$

where the constants $C_0, C_0' > 0$ depend on T and the metric g.

Recall the definition of the total variation of a function $w : M \to \mathbb{R}$

$$\mathrm{TV}_g(w) := \sup_{\|\phi\|_\infty \leq 1} \int_M w\ \mathrm{div}_g\phi\,dv_g,$$

where ϕ describes all C^1 vector fields with compact support. We denote by

$$BV(M;g) = \{u \in L^1(M;g) \,/\, \mathrm{TV}_g(u) < \infty\},$$

the space of all functions with finite total variation on M. It is well-known that (provided g is sufficiently smooth) the imbedding $BV(M;g) \subset L^1(M;g)$ is compact.

In fact, an important property of entropy solutions to (3.1)-(3.2) is the following one: u has finite total variation for all times $t \geq 0$ if (3.6) holds and, moreover,

$$\mathrm{TV}_g(u(t)) \leq C_1(T,g) + C_1'(T,g)\ \mathrm{TV}_g(u(s)), \qquad 0 \leq s \leq t \leq T,$$

where the constants $C_1, C_1' > 0$ depend on T and g; see [6] for details. Of course, this implies a control of the flux of the equation

$$\sup_{t \geq 0} \int_M \left| \mathrm{div}_g \left(f(u(t,\cdot),\cdot) \right) \right| dv_g \leq C\ \mathrm{TV}_g(u_0).$$

However, as noted in [2], this inequality can be derived more directly from the conservation laws and one checks that the constant C is independent of both T and g and only depend on the largest wave speed arising in the problem.

3.3 Statement of the main result

3.3.1 Family of geodesic triangulations

For $\tau > 0$, we consider the uniform mesh $t_n := n\,\tau$ $(n = 0, 1, 2, \ldots)$ on the half-line $\mathbb{R}_+$. For $h > 0$ we denote by $\mathcal{T}^h$ a triangulation of the given manifold M which is made of non-overlapping and non-empty curved polyhedra $K \subset M$, whose vertices in ∂K are joined by geodesic faces. We assume that, if two distinct elements $K_1, K_2 \in \mathcal{T}^h$ have a non-empty intersection, say I, then either I is a geodesic face of both K_1, K_2 or $\mathcal{H}^{n-1}(I) = 0$, where $\mathcal{H}^{n-1}$ denotes the $(n-1)$-dimensional Hausdorff measure.

The boundary ∂K of K consists of the set of all faces e of K. We denote by K_e the unique element distinct from K sharing the face e with K. The outward unit normal to an element K at some point $x \in e$ is denoted by $\mathbf{n}_{e,K}(x) \in T_x M$. Finally, $|K|$ and $|e|$ represent the n- and $(n-1)$-dimensional Hausdorff measures of K and e, respectively. We set

$$p_K := \sum_{e \in \partial K} |e|$$

and for each $K \in \mathcal{T}^h$ the diameter h_K of K is

$$h_K := \sup_{x,y \in K} d_g(x,y).$$

We set

$$h := \sup\{h_K : K \in \mathcal{T}^h\},$$

which is assumed to tend to zero along a sequence of geodesic triangulations. We also assume that there exist constants $\gamma_1, \gamma_2 > 0$ such that

$$\gamma_1^{-1} h \leq \tau \leq \gamma_1 h \tag{3.7}$$

and

$$\gamma_2^{-1} |K| \leq h_K \, p_K \leq \gamma_2 |K| \tag{3.8}$$

for all $K \in \mathcal{T}^h$. This condition implies that (as $h \to 0$)

$$\tau \to 0, \qquad h^2 \tau^{-1} \to 0.$$

Finally, we set $T = \tau \, n_T$ for every integer n_T.

3.3.2 Numerical flux-functions

As in the Euclidean case, the finite volume method can be introduced by formally averaging the conservation law (3.1) over an element $K \in \mathcal{T}^h$, applying the Gauss-Green formula, and finally discretizing the time derivative with a two-point scheme. First, we define a right-continuous, piecewise constant function: for $n = 0, 1, \ldots$,

$$u^h(t, x) = u_K^n \qquad (t, x) \in [t_n, t_{n+1}) \times M, \tag{3.9}$$

where

$$u_K^n := \fint_K u(t_n, x) \, dv_g(x),$$

and

$$u_K^0 := \fint_K u_0(x) dv_g(x). \tag{3.10}$$

Then, in view of (3.1) we write

$$\begin{aligned} 0 &= \frac{d}{dt} \fint_K u(t, x) \, dv_g(x) + \fint_K \mathrm{div}_g \, f(u(t, x), x) \, dv_g(x) \\ &\approx \frac{u_K^{n+1} - u_K^n}{\tau} + \frac{1}{|K|} \sum_{e \in \partial K} \int_e \langle f(u(t, y)), n_{e,K}(y) \rangle_g \, d\Gamma_g(y). \end{aligned}$$

We introduce flux-functions $f_{e,K} : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ and write

$$\fint_e \langle f(w(u_K^n, u_{K_e}^n), y), n_{e,K}(y) \rangle_g \, d\Gamma_g(y) \approx f_{e,K}(u_K^n, u_{K_e}^n).$$

The discrete flux are assumed to satisfy the following properties:

- *Consistency property :* for $u \in \mathbb{R}$,

$$f_{e,K}(u,u) = \fint_e \langle f(u,y), \mathbf{n}_{e,K}(y)\rangle_g \, d\Gamma_g(y). \tag{3.11}$$

- *Conservation property :* for $u, v \in \mathbb{R}$,

$$f_{e,K}(u,v) + f_{e,K_e}(v,u) = 0. \tag{3.12}$$

- *Monotonicity property:*

$$\frac{\partial}{\partial u} f_{e,K} \geq 0, \qquad \frac{\partial}{\partial v} f_{e,K} \leq 0. \tag{3.13}$$

Then, we formulate the finite volume approximation as follows:

$$u_K^{n+1} := u_K^n - \frac{\tau}{|K|} \sum_{e \in \partial K} |e| \; f_{e,K}(u_K^n, u_{K_e}^n) \qquad (n = 0, 1, \ldots). \tag{3.14}$$

For the sake of stability of the numerical method, we impose a CFL stability condition:

$$\tau \sup_{K \in \mathcal{T}^h} \frac{p_K}{|K|} \; \mathrm{Lip}(f) \leq 1, \tag{3.15}$$

where $\mathrm{Lip}(f)$ is the Lipschitz constant of f.

3.3.3 Main theorem

The main result of the present paper is as follows.

Theorem 3.3 (Error estimate for the finite volume scheme on manifolds)**.** *Let $u : \mathbb{R}_+ \times M \to \mathbb{R}$ be the entropy solution associated with the initial value problem* (3.1)-(3.2) *for an initial data $u_0 \in L^\infty(M) \cap BV(M; g)$. Let u^h be the approximate solution defined by* (3.9) *and* (3.14)*. Then, for each $T > 0$ there exist constants*

$$\mathrm{C}_0 = \mathrm{C}_0(T, g, \|u_0\|_{L^\infty}), \quad \mathrm{C}_1 = \mathrm{C}_1(T, g, \mathrm{TV}_g(u_0)),$$
$$\mathrm{C}_2 = \mathrm{C}_2(T, g, \|u_0\|_{L^2(M;g)})$$

such that for all $t \in [0, T]$

$$\begin{aligned} &\|u^h(t) - u(t)\|_{L^1(M;g)} \\ &\leq \Big(\mathrm{C}_0 \; |M|_g + \mathrm{C}_1\Big)\, h + \Big(\mathrm{C}_0 \; |M|_g^{1/2} + \Big(\mathrm{C}_0 \; \mathrm{C}_1\Big)^{1/2}\Big) |M|_g^{1/2} \; h^{1/2} \\ &\quad + \Big(\Big(\mathrm{C}_0 \; \mathrm{C}_2\Big)^{1/2} |M|_g^{1/2} + \Big(\mathrm{C}_1 \; \mathrm{C}_2\Big)^{1/2}\Big) |M|_g^{1/4} \; h^{1/4}. \end{aligned}$$

The rest of the present paper will be devoted to the proof of this theorem, which will follow from a suitable generalization of the arguments introduced earlier in Cockburn, Coquel, and LeFloch [13].

Remark 3.4. 1. It is sufficient to establish Theorem 3.3 for smoother initial data. When u_0 is measurable and bounded on M one can then show the existence of weak solutions to the initial value problem (3.1)-(3.2) as follows. Let $u_0^h \in L^\infty(M) \cap BV(M)$ be such that

$$\lim_{h\to 0} u_0^h = u_0 \qquad \text{in } L^1(M; g).$$

Solving the corresponding problem (3.1)-(3.2) for the regularized initial data, we deduce from Theorem 3.3 that the approximation solutions $\{u^h\}_{h>0}$ form a Cauchy sequence in L^1. Moreover, in view of the (discrete) maximum principle established later in this paper and for every $T > 0$, these solutions are uniformly bounded in $L^\infty((0,T) \times M)$. Consequently, there is a function $u \in L^\infty$, such that

$$\lim_{h\to 0} u^h = u \qquad \text{in the } L^1 \text{ norm}.$$

Finally, we note that Definition 3.2 is stable in the L^1 norm.

2. An immediate consequence of the L^1-contraction property is an estimate of the modulus of continuity in time, that is

$$\|u^h(t) - u^h(s)\|_{L^1(M;g)} \le C\, h + C'\, |t-s|,$$

where the constants $C > 0$, $C' \ge 1$ may depend on T as well as the metric g.

3.3.4 Discrete entropy inequalities

We will rely on a discrete version of the entropy inequality formulated by expressing u_K^{n+1} as a convex combination of essentially one-dimensional schemes. For each $K \in \mathcal{T}^h$, $e \in \partial K$, we define

$$\tilde{u}_{K,e}^{n+1} := u_K^n - \frac{\tau p_K}{|K|}\Big(f_{e,K}(u_K^n, u_{K_e}^n) - f_{e,K}(u_K^n, u_K^n)\Big) \tag{3.16}$$

and set

$$u_{K,e}^{n+1} := \tilde{u}_{K,e}^{n+1} - \frac{\tau p_K}{|K|} f_K^n, \tag{3.17}$$

where

$$f_K^n := \frac{1}{p_K} \sum_{e\in\partial K} |e| f_{e,K}(u_K^n, u_K^n).$$

Therefore, in agreement with (3.14) we find

$$u_K^{n+1} = \frac{1}{p_K} \sum_{e\in\partial K} |e|\; u_{K,e}^{n+1}. \tag{3.18}$$

Remark 3.5. The error estimate will be derived from the family of Kruzkov's entropies. Recall that any smooth entropy $\eta(u)$ can be recovered by the family of Kruzkov's entropies, that is

$$\eta(u) = \frac{1}{2}\int_{\mathbb{R}} \partial_u^2 \eta(\xi)\, U(u,\xi)\, d\xi.$$

The result follows for any entropy by a standard regularization argument. Moreover, if (η, q) is any convex entropy pair, then

$$q_x(u) = \frac{1}{2}\int_{\mathbb{R}} \partial_u^2 \eta(\xi)\, F_x(u,\xi)\, d\xi.$$

Define the numerical family of Kruzkov's entropy-flux as

$$F_{e,K}(u,v,c) := f_{e,K}(u \vee c, v \vee c) - f_{e,K}(u \wedge c, v \wedge c). \tag{3.19}$$

Given any convex entropy pair (U, F), define the numerical entropy-flux $F_{e,K}(u,v)$ associated to F by

$$F_{e,K}(u,v) := \frac{1}{2}\int_{\mathbb{R}} \partial_u^2 U(\xi)\, F_{e,K}(u,v,\xi)\, d\xi.$$

Hence, from the condition of the discrete flux we see that $F_{e,K}(u,v)$ satisfies

- For $u \in \mathbb{R}$,
$$F_{e,K}(u,u) = \fint_e \langle F_y(u), \mathbf{n}_{e,K}(y)\rangle_g\, d\Gamma_g(y).$$
- For $u, v \in \mathbb{R}$,
$$F_{e,K}(u,v) + F_{e,K_e}(v,u) = 0.$$

These properties are inherited from the corresponding properties for the numerical family of Kruzkov's entropy-flux.

We are in a position to derive the discrete entropy inequalities. For each $K \in \mathcal{T}^h$, $e \in \partial K$ and $u, v \in \mathbb{R}$, we define

$$H_{e,K}(u,v) := u - \varpi_K\Big(f_{e,K}(u,v) - f_{e,K}(u,u)\Big),$$

where

$$\varpi_K := \frac{\tau p_K}{|K|}.$$

Hence from (3.16), $H_{e,K}(u_K^n, u_{K_e}^n) = \tilde{u}_{K,e}^{n+1}$ and by definition of $H_{e,K}$, we have

$$\partial_u H_{e,K}(u,v) \geq 0, \quad \partial_v H_{e,K}(u,v) \geq 0.$$

The last inequality is an immediate consequence of the monotonicity of $f_{e,K}(u,v)$. The former follows from this property and the CFL condition. Moreover, we observe that

$$\begin{aligned}
&H_{e,K}(u \vee \lambda, v \vee \lambda) - H_{e,K}(u \wedge \lambda, v \wedge \lambda) \\
&= \Big(u \vee \lambda - \varpi_K \left(f_{e,K}(u \vee \lambda, v \vee \lambda) - f_{e,K}(u \vee \lambda, u \vee \lambda)\right)\Big) \\
&\quad - \Big(u \wedge \lambda - \varpi_K \left(f_{e,K}(u \wedge \lambda, v \wedge \lambda) - f_{e,K}(u \wedge \lambda, u \wedge \lambda)\right)\Big) \\
&= (u \vee \lambda - u \wedge \lambda) - \varpi_K \left(f_{e,K}(u \vee \lambda, v \vee \lambda) - f_{e,K}(u \wedge \lambda, v \wedge \lambda)\right) \\
&\quad + \varpi_K \left(f_{e,K}(u \vee \lambda, u \vee \lambda) - f_{e,K}(u \wedge \lambda, u \wedge \lambda)\right) \\
&= U(u,\lambda) - \varpi_K \Big(F_{e,K}(u,v,\lambda) - F_{e,K}(u,u,\lambda)\Big),
\end{aligned} \tag{3.20}$$

where we have used (3.19). Now, since $H_{e,K}(u,v)$ is an increasing function in both variables, we have

$$\begin{aligned}
H_{e,K}(u,v) \vee H_{e,K}(\lambda,\lambda) &\leq H_{e,K}(u \vee \lambda, v \vee \lambda), \\
H_{e,K}(u,v) \wedge H_{e,K}(\lambda,\lambda) &\geq H_{e,K}(u \wedge \lambda, v \wedge \lambda),
\end{aligned}$$

hence,

$$\begin{aligned}
&H_{e,K}(u \vee \lambda, v \vee \lambda) - H_{e,K}(u \wedge \lambda, v \wedge \lambda) \\
&\geq \Big(H_{e,K}(u,v) \vee H_{e,K}(\lambda,\lambda)\Big) - \Big(H_{e,K}(u,v) \wedge H_{e,K}(\lambda,\lambda)\Big) \\
&= U(H_{e,K}(u,v),\lambda).
\end{aligned} \tag{3.21}$$

Consequently, from (3.20), (3.21) taking $u = u_K^n$ and $v = u_{K_e}^n$, we obtain

$$U(\tilde{u}_{K,e}^{n+1}) - U(u_K^n) + \frac{\tau p_K}{|K|}\Big(F_{e,K}(u_K^n, u_{K_e}^n) - F_{e,K}(u_K^n, u_K^n)\Big) \leq 0.$$

Therefore, we have proved:

Lemma 3.6 (Entropy inequalities for the finite volume scheme)**.** *Let (U,F) be a convex entropy pair. Then, there exists a family of Lipschitz functions $F_{e,K} : \mathbb{R}^2 \to \mathbb{R}$, called numerical entropy-flux associated to F, satisfying the following conditions:*

- Consistency property : *for $u \in \mathbb{R}$,*

$$F_{e,K}(u,u) = \fint_e \langle F_y(u), \mathbf{n}_{e,K}(y)\rangle_g \, d\Gamma_g(y). \tag{3.22}$$

- Conservation property : *for $u, v \in \mathbb{R}$,*

$$F_{e,K}(u,v) + F_{e,K_e}(v,u) = 0. \tag{3.23}$$

- Discrete entropy inequality:

$$U(\tilde{u}_{K,e}^{n+1}) - U(u_K^n) + \frac{\tau p_K}{|K|}\Big(F_{e,K}(u_K^n, u_{K_e}^n) - F_{e,K}(u_K^n, u_K^n)\Big) \le 0. \tag{3.24}$$

From (3.17) and (3.24), we can write the discrete entropy inequality in terms of $u_{K,e}^{n+1}$ and u_K^n, that is

$$U(u_{K,e}^{n+1}) - U(u_K^n) + \frac{\tau p_K}{|K|}\Big(F_{e,K}(u_K^n, u_{K_e}^n) - F_{e,K}(u_K^n, u_K^n)\Big) \le D_{K,e}^{n+1}, \tag{3.25}$$

where $D_{K,e}^{n+1} = U(u_{K,e}^{n+1}) - U(\tilde{u}_{K,e}^{n+1})$.

To end this section we also recall the discrete maximum principle established in Amorim, Ben-Artzi and LeFloch [2]: for $n = 0, 1, \dots, n_T$,

$$\max_{K\in\mathcal{T}^h} |u_K^n| \le \Big(\tilde{C}_0(T) + \max_{K\in\mathcal{T}^h} |u_K^0|\Big)\,\tilde{C}_0'(T),$$

for some constants $\tilde{C}_0(T), \tilde{C}_0'(T) > 0$.

3.4 Derivation of the error estimate

3.4.1 Fundamental inequality

From now on it will be convenient to use the notation $Q_T := [0,T]\times M$. In this section we derive a basic approximation inequality on the manifold M, that is, we derive a generalization of the Kuznetsov's approximation inequality for the L^1 distance

$$\|u^h(T) - u(T)\|_{L^1(M;g)}.$$

Before proceeding we need introduce some special test-functions and make some preliminary observations.

Let $\varphi : \mathbb{R} \to \mathbb{R}$ be any C^∞ function such that supp$\varphi \subset [0,1]$, $\varphi \ge 0$, and $\int \varphi = 1$. For each $t' \in \mathbb{R}$, $x' \in M$ be fixed, each $\delta, \epsilon > 0$ and all $t \in \mathbb{R}$, $x \in M$, we define

$$\rho_\delta(t;t') := \frac{1}{\delta}\,\varphi\Big(\frac{|t-t'|^2}{\delta^2}\Big), \qquad \psi_\epsilon(x;x') := \frac{1}{\epsilon^n}\,\varphi\Big(\frac{(d_g(x,x'))^2}{\epsilon^2}\Big), \tag{3.26}$$

where we use the Riemannian distance. Observe that $\rho_\delta(t;t') = \rho_\delta(t';t)$, $\psi_\epsilon(x;x') = \psi_\epsilon(x';x)$, and

$$\int_{\mathbb{R}} \rho_\delta(t;t')\,dt = 1, \qquad \int_M \psi_\epsilon(x;x')\,dv_g(x) = 1.$$

Clearly, ψ_ϵ is a Lipschitz function on M with compact support contained in the geodesic ball of radius ϵ, hence $\psi_\epsilon \in W^{1,\infty}(M)$ and by Rademacher's theorem [21] it is differentiable almost everywhere. Moreover, there exists a constant $C > 0$, such that for $\mathcal{L}^1-$a.e. $t \in \mathbb{R}$ and $\mathcal{H}^n-$a.e. $x \in M$

$$\begin{aligned} |\rho_\delta(t;t')| &\leq \frac{C}{\delta}, \qquad |\partial_t \rho_\delta(t;t')| \leq \frac{C}{\delta^2}, \\ |\psi_\epsilon(x;x')| &\leq \frac{C}{\epsilon^n}, \qquad |\nabla_g \psi_\epsilon(x;x')| \leq \frac{C}{\epsilon^{n+1}}. \end{aligned} \tag{3.27}$$

If $v : M \to \mathbb{R}$ is a locally integrable function on M, then there exists a sequence of smooth functions $\{v^m\}$ defined on M, such that

$$\lim_{m\to\infty} v^m = v \qquad \text{on } L^1(M;g).$$

As in Kruzkov [27], in the case of manifolds we can establish the following approximation result.

Lemma 3.7. *Given a bounded and measurable function $v : M \to \mathbb{R}$ and $\epsilon > 0$, the function*

$$V_\epsilon := \int_M \int_M \psi_\epsilon(x;x')\, |v(x) - v(x')|\, dv_g(x)\, dv_g(x')$$

satisfies

$$\lim_{\epsilon\to 0} V_\epsilon = 0.$$

Proof. First, consider the case that v is smooth on M. Let x, x' be two points on M and, $\gamma : [0,1] \to M$ be a minimizing geodesic with $\gamma(0) = x$ and $\gamma(1) = x'$. Therefore, for some $\xi \in (0,1)$ one can write

$$\begin{aligned} |v(x) - v(x')| = |v(\gamma(1)) - v(\gamma(0))| &= \Big|\frac{d}{dt}(v \circ \gamma)(\xi)\Big| \\ &= |\langle du(\gamma(\xi)), \Big(\frac{d\gamma}{dt}\Big)(\xi)\rangle| \leq |\nabla_g v(\gamma(\xi))|\, \Big|\frac{d\gamma}{dt}(\xi)\Big| \\ &\leq \|\nabla_g v\|_\infty\, d_g(x,x'). \end{aligned}$$

Then, applying the inequality above and using (3.27), we obtain

$$\begin{aligned} V_\epsilon &= \int_M \int_M \psi_\epsilon(x;x')\, |v(x) - v(x')|\, dv_g(x')\, dv_g(x) \\ &\leq \int_M \frac{C}{\epsilon^n}\, \|\nabla_g v\|_\infty V_g\Big(B_x(\epsilon)\Big)\, \epsilon\, dv_g(x) \\ &\leq C\,|M|\, \|\nabla_g v\|_\infty\, \epsilon, \end{aligned}$$

where $V_g\Big(B_x(\epsilon)\Big)$ denotes the volume of the geodesic ball of center x and radius ϵ, and the constant $C > 0$ does not depend on ϵ.

Finally, suppose that v is measurable and bounded on M. Since M is compact, v is integrable on M and there exists a sequence of smooth functions $\{v^m\}$ defined on M converging to v in $L^1(M;g)$. We then conclude with a routine approximation argument. □

For each $p, p' \in Q_T$, $p := (t, x)$, $p' := (t', x')$, we consider the special test function ϕ defined by

$$\phi(p\,; p') := \rho_\delta(t; t')\, \psi_\epsilon(x; x').$$

Therefore, as $\delta, \epsilon \to 0$, the support of ϕ is concentrated on the set $\{p = p'\}$. For convenience, we introduce the following piecewise approximation of the functions ρ_δ and ψ_ϵ. For $n = 0, 1, \dots$, we define $\tilde\rho(t; t')$, $\tilde\rho'(t; t')$ as

$$\begin{aligned} \tilde\rho_\delta(t; t') &:= \rho_\delta(t_{n+1}; t') \qquad (t \in [t_n, t_{n+1})), \\ \tilde\rho'_\delta(t; t') &:= \rho_\delta(t; t'_{n+1}) \qquad (t' \in [t'_n, t'_{n+1})), \end{aligned} \tag{3.28}$$

and for all $K \in \mathcal{T}^h$, we define $\tilde\psi_\epsilon(x; x')$, $\tilde\psi'_\epsilon(x; x')$ by the averages of $\psi_\epsilon(x; x')$ along each interface, that is

$$\begin{aligned} \tilde\psi_\epsilon(x; x') &:= \fint_{\partial K} \psi_\epsilon(y; x')\, d\Gamma_g(y) \quad (x \in K, x' \in M), \\ \tilde\psi'_\epsilon(x; x') &:= \fint_{\partial K} \psi_\epsilon(x; y')\, d\Gamma_g(y') \quad (x \in M, x' \in K). \end{aligned} \tag{3.29}$$

The following estimate will be useful.

Lemma 3.8. *The functions ψ_ϵ, $\tilde\psi_\epsilon$ be defined by* (3.26)*, and* (3.29)*, respectively, satisfy the estimate*

$$\sup_{x \in M} \int_M |\tilde\psi_\epsilon(x; x') - \psi_\epsilon(x; x')|\, dv_g(x') \le C\, \frac{h}{\epsilon},$$

where $C > 0$ does not depend on $\epsilon, h > 0$.

Proof. Given K, let x', x be two points on M and K, respectively. Analogously to the proof of Lemma 3.7, we could write

$$\begin{aligned} |\tilde\psi_\epsilon(x; x') - \psi_\epsilon(x; x')| &\le \fint_{\partial K} |\psi_\epsilon(y; x') - \psi_\epsilon(x; x')|\, d\Gamma_g(y) \\ &\le |\nabla_g \psi_\epsilon(c; x')|\, h \le C \frac{h}{\epsilon^{n+1}}, \end{aligned}$$

where

$$|\nabla_g \psi_\epsilon(c; x')| = \sup_{y \in \partial K} |\nabla_g \psi_\epsilon(y; x')|.$$

Now, we integrate the above inequality on M and obtain

$$\int_M |\tilde\psi_\epsilon(x; x') - \psi_\epsilon(x; x')|\, dv_g(x') \le \frac{C}{\epsilon^{n+1}}\, h\, V_g\big(B_c(\epsilon)\big) \le C\, \frac{h}{\epsilon}.$$

□

Moreover, we define the corresponding approximations ϕ^h, $\phi^{h'}$, $\partial_t^h\phi$, $\partial_t^{h'}\phi$, $\nabla_g^h\phi$ and $\nabla_g^{h'}\phi$ of the exact value, time derivative and covariant derivative of the function ϕ, respectively:

$$\phi^h(p\,;p') := \tilde{\rho}_\delta(t;t')\,\psi_\epsilon(x;x'), \qquad \phi^{h'}(p\,;p') := \tilde{\rho}'_\delta(t;t')\,\psi_\epsilon(x;x'),$$
$$\partial_t^h\phi(p\,;p') := \partial_t\rho_\delta(t;t')\,\tilde{\psi}_\epsilon(x;x'), \qquad \partial_t^{h'}\phi(p\,;p') := \partial_{t'}\rho_\delta(t;t')\,\tilde{\psi}'_\epsilon(x;x'),$$

and

$$\nabla_g^h\phi(p\,;p') := \tilde{\rho}_\delta(t;t')\,\nabla_g\psi_\epsilon(x;x'),$$
$$\nabla_g^{h'}\phi(p\,;p') := \tilde{\rho}'_\delta(t;t')\,\nabla'_g\psi_\epsilon(x;x').$$

Analogously, for convenient we introduce a piecewise constant approximation of the exact solution u, that is

$$\tilde{u}(t,x) := u(t_n,x), \qquad (t\in[t_n,t_{n+1}),x\in M). \tag{3.30}$$

Remark 3.9. 1. Note that u represents the entropy solution to problem (3.1)-(3.2), while u^h denotes the piecewise-constant approximate solution (3.9) given by the schema (3.14). By definition we have $\tilde{u}^h = u^h$.

2. The zero-order approximations of the test function ϕ, that is, ϕ^h and $\phi^{h'}$ are due to the explicit dependence of the flux function with the spacial variable.

3. One denotes by $\partial_{t'}$, ∇'_g respectively the time derivative and covariant derivative with respect to t' and x' variables.

Next, let us define the approximate entropy dissipation form

$$E^h_{\delta,\epsilon}(u,u^h) := \iint_{Q_T} \Theta^h_{\delta,\epsilon}(u,u^h(t',x');t',x')\,dv_g(x')dt',$$

where

$$\begin{aligned}
&\Theta^h_{\delta,\epsilon}(u,u^h(t',x');t',x')\\
&\quad = -\iint_{Q_T}\Big(|\tilde{u}(t,x)-u^h(t',x')|\,\partial_t^h\phi\\
&\qquad + \Big(\operatorname{sgn}(u(t,x)-u^h(t',x'))\langle f(u(t,x),x)-f(u^h(t',x'),x),\nabla_g^h\phi\rangle\Big)\Big)\,dv_g(x)dt\\
&\qquad + \iint_{Q_T}\operatorname{sgn}(u(t,x)-u^h(t',x'))\,(\operatorname{div}_g f)(u^h(t',x'),x)\,\phi^h\,dv_g(x)dt\\
&\qquad - \int_M |u(0,x)-u^h(t',x')|\phi(0)\,dv_g(x) + \int_M |u(T,x)-u^h(t',x')|\phi(T)\,dv_g(x).
\end{aligned} \tag{3.31}$$

Here, the term $\Theta^h_{\delta,\epsilon}(u,u^h;t',x')$ is a measure of the entropy dissipation associated with the entropy solution u. Observe that $\tilde{u}$ defined by (3.30) appears in the first term of the right-hand side of (3.31). This is due to the fact that the time derivative of u^h needs special treatment, as was observed in [15].

Analogously, reversing the role of u and u^h, we define

$$E^h_{\delta,\epsilon}(u^h, u) := \iint_{Q_T} \Theta^h_{\delta,\epsilon}(u^h, u(t,x); t, x)\, dv_g(x) dt,$$

where

$$\begin{aligned}
&\Theta^h_{\delta,\epsilon}(u^h, u(t,x); t, x) \\
&:= -\iint_{Q_T} \Big(|u^h(t',x') - u(t,x)| \partial^{h'}_t \phi \\
&\quad + \Big(\operatorname{sgn}(u^h(t',x') - u(t,x)) \langle f(u^h(t',x'), x') - f(u(t,x), x'), \nabla^{h'}_g \rangle \Big) \Big) dv_g(x') dt' \\
&\quad + \iint_{Q_T} \operatorname{sgn}(u^h(t',x') - u(t,x))\ (\operatorname{div}'_g f)(u(t,x), x')\ \phi^{h'}\, dv_g(x') dt' \\
&\quad - \int_M |u^h(0,x') - u(t,x)| \phi(0)\, dv_g(x') + \int_M |u^h(T,x') - u(t,x)| \phi(T)\, dv_g(x').
\end{aligned}$$

Observing that $\partial_{t'} \rho_\delta(t;t') = -\partial_t \rho_\delta(t;t')$, $\nabla'_g \psi_\epsilon(x;x') = -\nabla_g \psi_\epsilon(x;x')$ and adding the terms $E_{\delta,\epsilon}(u, u^h)$ and $E_{\delta,\epsilon}(u^h, u)$, we get the following decomposition:

$$E^h_{\delta,\epsilon}(u, u^h) + E^h_{\delta,\epsilon}(u^h, u) = R^h_{\delta,\epsilon}(u, u^h) - S^h_{\delta,\epsilon}(u, u^h), \tag{3.32}$$

where

$$\begin{aligned}
R^h_{\delta,\epsilon}(u, u^h) := &\iint_{Q_T} \int_M |u^h(T,x') - u(t,x)|\ \phi(t,x;T,x') dv_g(x') dv_g(x) dt \\
&+ \iint_{Q_T} \int_M |u^h(t',x') - u(T,x)|\ \phi(T,x;t',x') dv_g(x) dv_g(x') dt' \\
&- \iint_{Q_T} \int_M |u^h(0,x') - u(t,x)|\ \phi(t,x;0,x') dv_g(x') dv_g(x) dt \\
&- \iint_{Q_T} \int_M |u^h(t',x') - u(0,x)|\ \phi(0,x;t',x') dv_g(x) dv_g(x') dt',
\end{aligned} \tag{3.33}$$

and

$$
\begin{aligned}
&S^h_{\delta,\epsilon}(u,u^h)\\
&:= \iint_{Q_T}\iint_{Q_T} \Big(|\tilde{u}(t,x) - u^h(t',x')|\ \tilde{\psi}_\epsilon(x;x')\\
&\qquad - |u(t,x) - u^h(t',x')|\ \tilde{\psi}'_\epsilon(x;x')\Big)\,\partial_t\rho_\delta(t;t')\,dv_g(x')dt'dv_g(x)dt\\
&\quad + \iint_{Q_T}\iint_{Q_T} \operatorname{sgn}(u(t,x) - u^h(t',x'))\Big(\Big\langle\Big(f(u(t,x),x) - f(u^h(t',x'),x)\Big)\ \tilde{\rho}_\delta(t;t')\\
&\qquad - \Big(f(u(t,x),x') - f(u^h(t',x'),x')\Big)\ \tilde{\rho}'_\delta(t;t'), \nabla_g\psi_\epsilon(x;x')\Big\rangle_g\\
&\qquad + (\operatorname{div}'_g f)(u(t,x),x')\phi^{h'} - (\operatorname{div}_g f)(u^h(t',x'),x)\phi^h\Big)\,dv_g(x')dt'dv_g(x)dt.
\end{aligned}
\tag{3.34}
$$

Passing to the limit as $\delta, \epsilon \to 0$, we expect that $R^h_{\delta,\epsilon}(u,u^h)$ converges to

$$
\int_M |u^h(T,x) - u(T,x)|\ dv_g(x) - \int_M |u^h(0,x) - u(0,x)|\ dv_g(x),
$$

and, if $\tilde{u}$ is replaced by u and the exact differentials in time-space are used, then the term $S^h_{\delta,\epsilon}(u,u^h)$ is expected to converge to zero.

Finally, we obtain the basic approximation inequality which is derived as a lower bound for the term $R^h_{\delta,\epsilon}(u,u^h)$.

Proposition 3.10 (Basic approximation inequality)**.** *The L^1 distance between the approximate and the exact solution satisfies*

$$
\begin{aligned}
\int_M |u^h(T,x) - u(T,x)|\ dv_g(x) \le C \int_M |u^h(0,x) - u(0,x)| dv_g(x) + C\Big(1 + \mathrm{TV}_g(u_0)\Big)\,(\epsilon + \delta)\\
+ C \sup_{0\le t\le T}\Big(S^h_{\delta,\epsilon}(u,u^h) + E^h_{\delta,\epsilon}(u,u^h) + E^h_{\delta,\epsilon}(u^h,u)\Big),
\end{aligned}
$$

where the constant $C > 0$ may depend on T and also on the metric g, but do not depend on h, ϵ, τ, and δ.

Proof. First, we write (3.33) as

$$
R^h_{\delta,\epsilon}(u,u^h) = R_1 + R_2 + R_3 + R_4,
$$

with obvious notation. For R_2, we simply observe that $R_2 \ge 0$. To estimate R_1, we consider the following decomposition

$$
\begin{aligned}
|u^h(T,x') - u(t,x)| &= |u^h(T,x') - u(T,x')| - \Big(|u^h(T,x') - u(T,x')| - |u^h(T,x') - u(T,x)|\Big)\\
&\quad - \Big(|u^h(T,x') - u(T,x)| - |u^h(T,x') - u(t,x)|\Big)\\
&\ge |u^h(T,x') - u(T,x')| - |u(T,x) - u(T,x')| - |u(t,x) - u(T,x)|.
\end{aligned}
$$

Then, using this decomposition in the expression R_1, we have

$$\begin{aligned}R_1 =& \int_Q \int_M |u^h(T,x') - u(t,x)| \, \phi(t,x;T,x') dv_g(x') dv_g(x) dt \\ \geq& \frac{1}{2} \int_M |u^h(T,x') - u(T,x')| dv_g(x') \\ &- \frac{1}{2}\Big(\big(C + C \ \mathrm{TV}_g(u_0)\big)\, \epsilon + \big(C + C \ \mathrm{TV}_g(u_0)\big)\, \delta\Big).\end{aligned}$$

Analogously, we obtain

$$\begin{aligned}R_3 \geq& -\frac{1}{2} \int_M |u^h(0,x') - u(0,x')| \ dv_g(x') \\ &- \frac{1}{2}\Big(\big(C + C \ \mathrm{TV}_g(u_0)\big)\, \epsilon + \big(C + C \ \mathrm{TV}_g(u_0)\big)\, \delta\Big),\end{aligned}$$

and

$$\begin{aligned}R_4 \geq& - \iint_Q \rho_\delta(t') |u^h(t',x') - u(t',x')| \ dv_g(x') dt' \\ &- \frac{1}{2}\Big(\big(C + C \ \mathrm{TV}_g(u_0)\big)\, \epsilon + \big(C + C \ \mathrm{TV}_g(u_0)\big)\, \delta\Big).\end{aligned}$$

Hence adding all these inequalities, we get

$$\begin{aligned}2R^h_{\delta,\epsilon}(u,u^h) \geq& \int_M |u^h(T,x') - u(T,x')| \ dv_g(x') - \int_M |u^h(0,x') - u(0,x')| \ dv_g(x') \\ &- 2 \iint_Q \rho_\delta(t') \ |u^h(t',x') - u(t',x')| \ dv_g(x') dt' \\ &- 3\Big(\big(C + C \ \mathrm{TV}_g(u_0)\big)\, \epsilon + \big(C + C \ \mathrm{TV}_g(u_0)\big)\, \delta\Big).\end{aligned}$$

Finally, by a simple algebraic manipulation we deduce from the above inequality that

$$\int_M |u^h(T,x') - u(T,x')| \ dv_g(x') \leq A + 2 \iint_Q \rho_\delta(t') |u^h(t',x') - u(t',x')| \ dv_g(x') dt',$$

where

$$\begin{aligned}A =& \int_M |u^h(0,x') - u(0,x')| \ dv_g(x') \\ &+ 3 \Big(\big(C_1(T) + C_1'(T) \ \mathrm{TV}_g(u_0)\big)\, \epsilon + \big(\tilde{C}_1(T) + \tilde{C}_1'(T) \ \mathrm{TV}_g(u_0)\big)\, \delta\Big) \\ &+ 2 \Big(S^h_{\delta,\epsilon}(u,u^h) + E^h_{\delta,\epsilon}(u,u^h) + E^h_{\delta,\epsilon}(u,u^h)\Big).\end{aligned}$$

Then, applying the Gronwall's inequality, we get

$$\int_M |u^h(T,x') - u(T,x')| \ dv_g(x') \leq 3 \ A.$$

□

3.4.2 Dealing with the lack of symmetry

In this subsection we estimate the lack of symmetry in the term $S^h_{\delta,\epsilon}(u,u^h)$.

Proposition 3.11 (Estimate of $S^h_{\delta,\epsilon}(u,u^h)$)**.** *The following inequality holds*

$$S^h_{\delta,\epsilon}(u,u^h) \leq C\Big(1+\|u_0\|_{L^\infty}\Big)\,\frac{h}{\epsilon}\,|M| + C\Big(1+TV_g(u_0)\Big)\Big(\frac{\tau}{\delta}+\frac{h}{\epsilon}+\epsilon\Big),$$

where the constant $C>0$ may depend on T and the metric g, but do not depend on h, ϵ, τ, and δ.

Proof. **Step 1.** From (3.34) we write

$$S^h_{\delta,\epsilon}(u,u^h) = S_1 + S_2,$$

with obvious notation. Consider the decomposition $S_1 = S_1' + S_1''$:

$$S_1' = \iint_{Q_T}\iint_{Q_T}\Big(|\tilde{u}(t,x)-u^h(t',x')| - |u(t,x)-u^h(t',x')|\Big) \\ \tilde{\psi}_\epsilon(x;x')\,\partial_t\rho_\delta(t;t')\,dv_g(x')dt'\;dv_g(x)dt,$$

$$S_1'' = \iint_{Q_T}\iint_{Q_T}|u(t,x)-u^h(t',x')|\,\partial_t\rho_\delta(t;t') \\ \Big(\tilde{\psi}_\epsilon(x;x')-\tilde{\psi}_\epsilon'(x;x')\Big)\,dv_g(x')dt'\;dv_g(x)dt.$$

Using (3.28),(3.29) and the definition of $\tilde{u}$, it follows that

$$\begin{aligned}|S_1'| &\leq \iint_{Q_T}|\tilde{u}(t,x)-u(t,x)|\int_M\tilde{\psi}_\epsilon(x;x')dv_g(x')\int_0^T|\partial_t\rho_\epsilon(t;t')|dt'\;dv_g(x)dt \\ &\leq T\,\frac{C}{\delta}\sup_{t\in[0,T]}\|\tilde{u}(t)-u(t)\|_{L^1(M;g)}\sup_{K\in\mathcal{T}^h}\fint_{\partial K}\int_M\psi_\epsilon(x;x')dv_g(x')d\Gamma(x) \\ &\leq C\,T\,\frac{\tau}{\delta}\Big(C+C\;\mathrm{TV}_g(u_0)\Big).\end{aligned}$$

On the other hand, to estimate S_1'' we integrate by parts

$$
\begin{aligned}
S_1'' = &- \iint_{Q_T} \iint_{Q_T} \operatorname{sgn}(u(t,x) - u^h(t',x'))\, u_t(t,x)\, \rho_\delta(t;t') \\
&\qquad\qquad \Big(\tilde{\psi}_\epsilon(x;x') - \tilde{\psi}'_\epsilon(x;x')\Big)\, dv_g(x')dt'\; dv_g(x)dt \\
&+ \int_M \iint_{Q_T} |u(T,x) - u^h(t',x')|\; \rho_\delta(T;t') \\
&\qquad\qquad \Big(\tilde{\psi}_\epsilon(x;x') - \tilde{\psi}'_\epsilon(x;x')\Big)\, dv_g(x')dt'\; dv_g(x) \\
&- \int_M \iint_{Q_T} |u(0,x) - u^h(t',x')|\; \rho_\delta(t') \\
&\qquad\qquad \Big(\tilde{\psi}_\epsilon(x;x') - \tilde{\psi}'_\epsilon(x;x')\Big)\, dv_g(x')dt'\; dv_g(x) \\
\le & \iint_{Q_T} \iint_{Q_T} |u_t(t,x)|\; \rho_\delta(t;t')\, \Big|\tilde{\psi}_\epsilon(x;x') - \tilde{\psi}'_\epsilon(x;x')\Big|\, dv_g(x')dt'\; dv_g(x)dt \\
&+ \int_M \iint_{Q_T} \Big(|u(T,x) - u^h(t',x')|\rho_\delta(T;t') + |u(0,x) - u^h(t',x')|\; \rho_\delta(t')\Big) \\
&\qquad\qquad \Big|\tilde{\psi}_\epsilon(x;x') - \tilde{\psi}'_\epsilon(x;x')\Big|\, dv_g(x')dt'\; dv_g(x).
\end{aligned}
$$

Hence, we conclude that

$$
|S_1''| \le C\, \frac{h}{\epsilon}\Big(T\,\Big(C + C\ \mathrm{TV}_g(u_0)\Big) + |M|\,\Big(C + C\,\|u_0\|_{L^\infty}\Big)\Big).
$$

Here, with some abuse of notation we have written $u_t\, dv_g(x')$ to denote the integration with respect to the measure u_t.

Step 2. Finally, in order to estimate S_2 we observe that

$$
\begin{aligned}
S_2 = & \iint_{Q_T} \iint_{Q_T} \operatorname{sgn}(u(t,x) - u^h(t',x')) \\
&\Big(\Big(\Big\langle \sigma_{x'}(f_x(u(t,x)) - f_{x'}(u(t,x)), \nabla_g \psi_\epsilon \Big\rangle_g + (\operatorname{div}'_g f)_{x'}(u(t,x))\psi_\epsilon\Big)\tilde{\rho}_\delta \\
&- \Big(\Big\langle \sigma_{x'}(f_x(u^h(t',x')) - f_{x'}(u^h(t',x')), \nabla_g \psi_\epsilon \Big\rangle_g + (\operatorname{div}_g f)_x(u^h(t',x'))\psi_\epsilon\Big)\tilde{\rho}_\delta \\
&+ \langle f_{x'}(u(t,x)) - f_{x'}(u^h(t',x')), \nabla_g \psi_\epsilon\rangle_g \Big(\tilde{\rho}_\delta - \tilde{\rho}'_\delta\Big) \\
&- \Big(\operatorname{div}'_g f\Big)_{x'}(u(t,x))\Big(\phi^h - \phi^{h'}\Big)\Big)\, dv_g(x')dt' dv_g(x)dt,
\end{aligned}
$$

where $\sigma_{x'}(f_x(u(t,x))$ is the parallel transport of the vector $f_x(u(t,x))$ from the point x to x'. We use that f is a smooth vector field on M. For $x \in M$ fixed, $u \in \mathbb{R}$ fixed and $\epsilon > 0$ sufficiently small, using the Landau notation $O(\cdot)$ we write

$$\begin{aligned}&\langle \sigma_{x'}(f_x(u)) - f_{x'}(u), \nabla_g \psi_\epsilon(x;x')\rangle_g + \Big(\operatorname{div}'_g f\Big)(u,x')\, \psi_\epsilon(x;x')\\
&= \langle \Big(\nabla'_g f\Big)_{x'}(u)\, k(x;x'), \nabla_g \psi_\epsilon(x,x')\rangle_g + \Big(\operatorname{tr}\nabla'_g f\Big)(u,x')\, \psi_\epsilon(x,x') + O(d_g^2(x,x'))\\
&= \langle \Big(\nabla'_g f\Big)_{x'}(u), \nabla_g\Big(k(x;x')\, \psi_\epsilon(x;x')\Big)\rangle_g + O(\epsilon^2),\end{aligned}$$

where $k(x;x')$ is the tangent vector at x of the minimizing geodesic from x to x'. Analogously, we have

$$\begin{aligned}&\langle \sigma_{x'}(f_x(u^h)) - f(u^h,x'), \nabla_g \psi_\epsilon(x;x')\rangle_g + \Big(\operatorname{div}_g f\Big)(u^h,x)\, \psi_\epsilon(x;x')\\
&= \langle \Big(\nabla'_g f\Big)(u^h,x'), \nabla_g\Big(k(x;x')\, \psi_\epsilon(x;x')\Big)\rangle_g + O(\epsilon^2).\end{aligned}$$

Now, denoting the Lipschitz $(1,1)-$tensor field I as

$$I(t,x;t',x') := \Big(\nabla'_g F_{x'}\Big)(u(t,x), u^h(t',x')),$$

we have $|S_2| \le |S'_2| + |S''_2| + O(\epsilon)$, where

$$\begin{aligned}S'_2 &= \iint_{Q_T}\iint_{Q_T} \Big\langle I(t,x;t',x'), \nabla_g\Big(k(x;x')\, \psi_\epsilon(x;x')\Big)\Big\rangle_g\, \tilde\rho_\delta\, dv_g(x')dt'dv_g(x)dt,\\
S''_2 &= \iint_{Q_T}\iint_{Q_T} \operatorname{sgn}(u-u^h)\Big(\Big\langle f_{x'}(u(t,x)) - f_{x'}(u^h(t',x')), \nabla_g\psi_\epsilon\Big\rangle_g\Big(\tilde\rho_\delta - \tilde\rho'_\delta\Big)\\
&\qquad - \Big(\operatorname{div}'_g f\Big)_{x'}(u(t,x))\Big(\phi^h - \phi^{h'}\Big)\Big)\, dv_g(x')dt'dv_g(x)dt.\end{aligned}$$

The term S''_2 is estimated as the final step of item 2, we focus on S'_2 term. Applying the Gauss-Green formula with respect to the x variable, it follows that

$$\iint_{Q_T}\iint_{Q_T} \Big\langle I(t,x';t',x'), \nabla_g\Big(k(x;x')\, \psi_\epsilon(x;x')\Big)\Big\rangle_g \tilde\rho_\delta\, dv_g(x')dt'dv_g(x)dt = 0.$$

Therefore, subtracting the above expressions from S'_2, we obtain

$$|S'_2| \le C\, T\, \Big(C + C\, \mathrm{TV}_g(u_0)\Big)\, \epsilon,$$

where, we have used the fact that, due to the compactness of M and the regularity of the flux function, the function $\nabla_g f(u,x)$ is uniformly Lipschitz continuous for u in a compact set. Combining this result with the estimation of S''_2, that is

$$|S''_2| \le C\, T\, \Big(\frac{\tau}{\delta}\Big(C + C\, \mathrm{TV}_g(u_0)\Big) + \frac{h}{\epsilon}\Big(C + C\|u_0\|_{L^\infty}\Big)|M|_g\Big),$$

we complete the proof of the proposition. $\square$

3.4.3 Entropy production for the exact solution

We now consider the approximate entropy dissipation associated with the exact solution.

Proposition 3.12 (Estimate of the quantity $E^h_{\delta,\epsilon}(u,u^h)$). *The following inequality holds*

$$E^h_{\delta,\epsilon}(u,u^h) \le C\,|M|_g\left(1+\|u_0\|_{L^\infty}\right) + C\left(1+\mathrm{TV}_g(u_0)\right)\frac{h}{\epsilon},$$

where $C>0$ *may depend on* T *and the metric* g, *but do not depend on* h, ϵ, τ, *and* δ.

Proof. 1. For each $c\in\mathbb{R}$ and in the sense of distributions we have

$$U_t(u(t,x),c) + \mathrm{div}_g\,F_x(u(t,x),c) + \mathrm{sgn}(u(t,x)-c)(\mathrm{div}_g\,f)(c,x) \le 0.$$

Now, we set $c = u^h(t',x')$ for each $(t',x')\in[0,T]\times M$ fixed. Since u is an entropy solution to (3.1), for $n=0,1,\dots$, we have

$$\begin{aligned}
&\int_M \Big(U\big(u(t_{n+1},x),u^h(t',x')\big) - U\big(u(t_n,x),u^h(t',x')\big)\Big)\psi_\epsilon(x;x')dv_g(x)\\
&-\int_{t_n}^{t_{n+1}}\int_M \langle F_x(u(t,x),u^h(t',x')),\nabla_g\psi_\epsilon(x;x')\rangle_g\,dv_g(x)dt\\
&+\int_{t_n}^{t_{n+1}}\int_M \mathrm{sgn}(u(t,x)-u^h(t',x'))(\mathrm{div}_g\,f)(u^h(t',x'),x)\,\psi_\epsilon(x;x')\,dv_g(x)dt \le 0.
\end{aligned}$$

2. Next, we multiply this inequality by $\tilde\rho_\delta(t;t') = \rho_\delta(t_{n+1};t')$ and summing the first term in time, it follows that

$$\begin{aligned}
-\sum_{n=0}^{n_T-1}&\int_M U(u(t_n,x),u^h)\Big(\rho_\delta(t_{n+1};t')-\rho_\delta(t_n;t')\Big)\psi_\epsilon(x;x')dv_g(x)\\
&-\iint_{Q_T}\langle F_x(u(t,x),u^h),\nabla_g\psi_\epsilon(x;x')\rangle\tilde\rho_\delta(t;t')\,dv_g(x)dt\\
&+\iint_{Q_T}\mathrm{sgn}(u(t,x)-u^h)(\mathrm{div}_g\,f)_x(u^h)\,\psi_\epsilon(x;x')\tilde\rho_\delta(t;t')\,dv_g(x)dt\\
&+\int_M U(u(T,x),u^h)\rho_\delta(T;t')\psi_\epsilon(x;x')dv_g(x)\\
&-\int_M U(u(0,x),u^h)\rho_\delta(t')\psi_\epsilon(x;x')dv_g(x) \le 0.
\end{aligned}$$

By the definition (3.30) of $\tilde u$ we have the identity

$$\begin{aligned}
&\int_M U(u(t_n,x),u^h)\Big(\rho_\delta(t_{n+1};t')-\rho_\delta(t_n;t')\Big)\psi_\epsilon(x;x')dv_g(x)\\
&\qquad=\int_{t_n}^{t_{n+1}}\int_M U(\tilde u(t,x),u^h)\partial_t\rho_\delta(t;t')\psi_\epsilon(x;x')\,dv_g(x)dt.
\end{aligned}$$

Therefore, $E^h_{\delta,\epsilon}(u, u^h)$ is bounded above by

$$\iint_{Q_T} \iint_{Q_T} U(\tilde{u}(t,x), u^h)\partial_t \rho_\delta(t;t') \Big(\psi_\epsilon(x;x') - \tilde{\psi}_\epsilon(x;x')\Big)\, dv_g(x)dt\ dv_g(x')dt'.$$

3. By integrating by parts the above equation with respect to t, it follows that

$$\begin{aligned}
I :=& \iint_{Q_T} \iint_{Q_T} U(\tilde{u}(t,x), u^h)\partial_t \rho_\delta(t;t')\Big(\psi_\epsilon - \tilde{\psi}_\epsilon\Big)\, dv_g(x)dt\ dv_g(x')dt' \\
=& -\sum_{n=0}^{n_T-1} \int_M \iint_{Q_T} \Big(U(u(t_{n+1},x), u^h) - U(u(t_n,x), u^h)\Big)\, (\psi_\epsilon - \tilde{\psi}_\epsilon) \\
& \qquad \rho_\delta(t_{n+1};t') dv_g(x')dt'\ dv_g(x) \\
& + \int_M \iint_{Q_T} U(u(T,x), u^h)\Big(\psi_\epsilon - \tilde{\psi}_\epsilon\Big)\rho_\delta(T;t')\, dv_g(x')dt'\ dv_g(x) \\
& - \int_M \iint_{Q_T} U(u(0,x), u^h)\Big(\psi_\epsilon - \tilde{\psi}_\epsilon\Big)\rho_\delta(t')\, dv_g(x')dt'\ dv_g(x),
\end{aligned}$$

and thus

$$\begin{aligned}
I \leq& \sum_{n=0}^{n_T-1} \tau \int_M \iint_{Q_T} |u_t|\, \rho_\delta(t_{n+1};t')\, |\psi_\epsilon(x;x') - \tilde{\psi}_\epsilon(x;x')| dv_g(x')dt'\ dv_g(x) \\
& + \int_M \iint_{Q_T} \Big(|u(T,x) - u^h(t',x')|\rho_\delta(T;t') + |u(0,x) - u^h(t',x')|\rho_\delta(t')\Big). \\
& \qquad |\psi_\epsilon(x;x') - \tilde{\psi}_\epsilon(x;x')|\, dv_g(x')dt'\ dv_g(x) \\
\leq& C\, \frac{h}{\epsilon}\Big(T\,\Big(C + C\ \mathrm{TV}_g(u_0)\Big) + |M|_g\,\Big(C + C\,\|u_0\|_{L^\infty}\Big)\Big).
\end{aligned}$$

□

3.5 Entropy production for the approximate solutions

It remains to control the approximate entropy dissipation form for the approximate solution.

Proposition 3.13 (Estimate of the quantity $E^h_{\delta,\epsilon}(u^h, u)$)**.** *The following inequality holds*

$$\begin{aligned}
E^h_{\delta,\epsilon}(u^h, u) \leq& C\Big(1 + \|u_0\|_{L^\infty}\Big)\Big(\frac{h}{\epsilon} + \tau + \epsilon\Big)|M|_g \\
& + C\Big(1 + \|u_0\|_{L^2(M;g)}\Big)\,\frac{h^{1/2}}{\epsilon}\ |M|_g^{1/2},
\end{aligned}$$

where $C > 0$ may depend on T and the metric g, but do not depend on h, ϵ, τ, and δ.

Once this estimate is established we can complete the proof of the main theorem, as follows.

Proof of Theorem 3.3. 1. First, by (4.17) there exists $\gamma_1 > 0$, such that $\tau \le \gamma_1\, h$. Moreover, without loss of generality we can take $\delta = \epsilon$. Therefore, combining Propositions 3.10-3.13 together and denoting

$$\begin{aligned}
&A_0(T) := C\Big(1 + \|u_0\|_{L^\infty}\Big), \qquad A_1(T) := C\Big(1 + \mathrm{TV}_g(u_0)\Big)\\
&A_2(T) := C\Big(1 + \|u_0\|_{L^2(M;g)}\Big),
\end{aligned}$$

we obtain

$$\|u^h(T) - u(T)\|_{L^1(M;g)} \le \frac{C}{2}\Big(L_{-1}\,\epsilon^{-1} + 2\,L_0 + L_1\,\epsilon\Big),$$

where

$$\begin{aligned}
L_{-1} &:= A_0(T)\,|M|_g\,h + A_1(T)\,h + A_2(T)\,|M|_g^{1/2}\,h^{1/2},\\
L_0 &:= A_0(T)\,|M|_g\,h + \|u^h(0) - u(0)\|_{L^1(M;g)},\\
L_1 &:= A_0(T)\,|M|_g + A_1(T).
\end{aligned}$$

Then, minimizing with respect to ϵ, we obtain

$$\|u^h(T) - u(T)\|_{L^1(M;g)} \le C\Big(\sqrt{L_{-1}\,L_1} + L_0\Big).$$

2. Next, proceeding as in the proof of Lemma 3.7 and by (3.9), that is, $u^h(0,x) = u_K^0$, we have

$$\begin{aligned}
\|u_K^0 - u_0\|_{L^1(M;g)} &\le \int_M \fint_K |u_0(z) - u_0(x)| dv_g(z)\, dv_g(x)\\
&\le \|\nabla_g u_0\|_{L^1(M;g)}\,h = \mathrm{TV}_g(u_0)\,h.
\end{aligned}$$

Consequently, it follows that

$$\begin{aligned}
&\|u^h(T) - u(T)\|_{L^1(M;g)}\\
&\le \Big(A_0(T)|M|_g + A_1(T)\Big)\,C\,h\\
&\quad + \Big(A_0(T)|M|_g^{1/2} + \Big(A_0(T)\,A_1(T)\Big)^{1/2}\Big)\,C\,|M|_g^{1/2}\,h^{1/2}\\
&\quad + \Big(\Big(A_0(T)\,A_2(T)\Big)^{1/2}|M|_g^{1/2} + \Big(A_1(T)\,A_2(T)\Big)^{1/2}\Big)\,C\,|M|_g^{1/4}\,h^{1/4},
\end{aligned}$$

which completes the proof of Theorem 3.3. □

Proof of Proposition 3.13. 1. Fix $K \in \mathcal{T}^h$ and $e \in \partial K$. For $(t,x) \in [0,T] \times M$ fixed, we set $c = u(t,x)$ and $u^h(t',x') = u^n_K$, for $(t',x') \in [t'_n, t'_{n+1}) \times M$, $(n = 0, 1, \dots)$ and we define

$$\tilde{\psi}'_{\epsilon,e}(x;x') := \fint_e \psi_\epsilon(x;y')\, d\Gamma_g(y').$$

Therefore, by Definition 3.29 and analogously to Lemma 3.8, it follows that

$$\begin{aligned}\tilde{\psi}'_\epsilon(x;x') &= \sum_{e \in \partial K} \frac{|e|}{p_K}\, \tilde{\psi}'_{\epsilon,e}(x;x'),\\ \int_M |\tilde{\psi}'_\epsilon(x;x') - \tilde{\psi}'_{\epsilon,e}(x;x')|\, dv_g(x) &\leq C\, \frac{h}{\epsilon},\end{aligned}$$

where the positive constant C does not depend on $h, \epsilon > 0$. Now, we write the local entropy inequality (3.25) for K and, since it is also valid for K_e, we obtain respectively

$$\frac{|K|}{p_K}\Big(U(u^{n+1}_{K,e}, c) - U(u^n_K, c)\Big) + \tau\Big(F_{e,K}(u^n_K, u^n_{K_e}, c) - F_{e,K}(u^n_K, u^n_K, c)\Big) \leq \frac{|K|}{p_K} D^{n+1}_{K,e},$$

$$\frac{|K_e|}{p_{K_e}}\Big(U(u^{n+1}_{K_e,e}, c) - U(u^n_{K_e}, c)\Big) + \tau\Big(F_{e,K_e}(u^n_{K_e}, u^n_K, c) - F_{e,K_e}(u^n_{K_e}, u^n_{K_e}, c)\Big) \leq \frac{|K_e|}{p_{K_e}} D^{n+1}_{K_e,e}.$$

We sum the two above inequalities and from (3.22) and (3.23), we obtain

$$\begin{aligned}&\frac{|K|}{p_K}\Big(U(u^{n+1}_{K,e}, c) - U(u^n_K, c)\Big) + \frac{|K_e|}{p_{K_e}}\Big(U(u^{n+1}_{K_e,e}, c) - U(u^n_{K_e}, c)\Big)\\ &\quad + \tau \fint_e \langle F_{y'}(u^n_{K_e}, c) - F_{y'}(u^n_K, c), \mathbf{n}_{e,K}(y')\rangle_g\, d\Gamma_g(y')\\ &\quad \leq \frac{|K|}{p_K} D^{n+1}_{K,e} + \frac{|K_e|}{p_{K_e}} D^{n+1}_{K_e,e}.\end{aligned} \tag{3.35}$$

2. We multiply inequality (3.35) by $|e|\ \tilde{\psi}'_{\epsilon,e}$ and sum over all $e \in \partial K$ and $K \in \mathcal{T}^h$:

$$\begin{aligned}&\sum_{\substack{e \in \partial K\\ K \in \mathcal{T}^h}} \frac{|K|}{p_K} U(u^{n+1}_{K,e}, c)\ |e|\ \tilde{\psi}'_{\epsilon,e} - \sum_{K \in \mathcal{T}^h} |K|\ U(u^n_K, c)\ \tilde{\psi}'_\epsilon\\ &\qquad - \tau \sum_{K \in \mathcal{T}^h} \int_{\partial K} \langle F_{y'}(u^n_K, c), \mathbf{n}_{e,K}(y')\rangle_g\, \tilde{\psi}'_{\epsilon,e}(x,x')\, d\Gamma_g(y')\\ &\quad \leq \sum_{\substack{e \in \partial K\\ K \in \mathcal{T}^h}} \frac{|K|}{p_K}\ |e|\ \tilde{\psi}'_{\epsilon,e}\ D^{n+1}_{K,e},\end{aligned} \tag{3.36}$$

where we have used

$$\sum_{\substack{e\in\partial K\\ K\in\mathcal{T}^h}} \frac{|K_e|}{p_{K_e}} U(u^{n+1}_{K_e,e}, c)\ \tilde{\psi}'_{\epsilon,e} = \sum_{\substack{e\in\partial K\\ K\in\mathcal{T}^h}} \frac{|K|}{p_K} U(u^{n+1}_{K,e}, c)\ \tilde{\psi}'_{\epsilon,e},$$
$$\sum_{\substack{e\in\partial K\\ K\in\mathcal{T}^h}} \frac{|K_e|}{p_{K_e}} U(u^{n}_{K_e,e}, c)\ \tilde{\psi}'_{\epsilon,e} = \sum_{e\in\partial K K\in\mathcal{T}^h} \frac{|K|}{p_K} U(u^{n}_{K,e}, c)\ \tilde{\psi}'_{\epsilon,e},$$

and

$$\sum_{\substack{e\in\partial K\\ K\in\mathcal{T}^h}} \int_e \langle F_{y'}(u^n_{K_e}, c), \mathbf{n}_{e,K}(y')\rangle_g\ \tilde{\psi}'_{\epsilon,e}\, d\Gamma_g(y')$$
$$= -\sum_{\substack{e\in\partial K\\ K\in\mathcal{T}^h}} \int_e \langle F_{y'}(u^n_{K}, c), \mathbf{n}_{e,K}(y')\rangle_g\ \tilde{\psi}'_{\epsilon,e}\, d\Gamma_g(y').$$

Since u^{n+1}_K is a convex combination of $u^{n+1}_{K,e}$ and the Kruzkov's entropy U is convex, we have by Jensen's inequality

$$\sum_{K\in\mathcal{T}^h} |K|\ U(u^{n+1}_K)\ \tilde{\psi}'_\epsilon \le \sum_{\substack{e\in\partial K\\ K\in\mathcal{T}^h}} |K|\ \frac{|e|}{p_K}\ U(u^{n+1}_{K,e})\ \tilde{\psi}'_\epsilon.$$

Therefore, from (3.36) we obtain

$$\begin{aligned}
&\sum_{K\in\mathcal{T}^h} |K|\Big(U(u^{n+1}_K, c) - U(u^n_K, c)\Big)\ \tilde{\psi}'_\epsilon - \tau \sum_{K\in\mathcal{T}^h} \int_{\partial K} \langle F_{y'}(u^n_K, c), \mathbf{n}_{e,K}(y')\rangle_g\ \psi_\epsilon(x; y')\ d\Gamma_g(y')\\
&\quad \le \sum_{\substack{e\in\partial K\\ K\in\mathcal{T}^h}} \frac{|K|}{p_K}\ |e|\ \tilde{\psi}'_{\epsilon,e}\ D^{n+1}_{K,e} + \sum_{\substack{e\in\partial K\\ K\in\mathcal{T}^h}} \frac{|K|}{p_K}\ |e|\ U(u^{n+1}_{K,e}, c)\left(\tilde{\psi}'_\epsilon - \tilde{\psi}'_{\epsilon,e}\right)\\
&\qquad + \tau \sum_{\substack{e\in\partial K\\ K\in\mathcal{T}^h}} \int_e \langle F_{y'}(u^n_K, c), \mathbf{n}_{e,K}(y')\rangle_g \left(\tilde{\psi}'_{\epsilon,e}(x; x') - \psi_\epsilon(x; y')\right) d\Gamma_g(y').
\end{aligned} \tag{3.37}$$

3. Applying Gauss-Green's formula it follows that

$$\begin{aligned}
&\int_{\partial K} \langle F_{y'}(u^n_K, c), \mathbf{n}_{e,K}(y')\rangle_g\ \psi_\epsilon(x; y')\ d\Gamma_g(y')\\
&= \int_K \Big(\operatorname{div}'_g F_{x'}\Big)(u^n_K, c)\psi_\epsilon(x; x')\ dv_g(x') + \int_K \langle F_{x'}(u^n_K, c), \nabla'_g \psi_\epsilon(x; x')\rangle_g\ dv_g(x').
\end{aligned}$$

Then, from (3.37) we deduce that

$$\begin{aligned}
&\sum_{K\in\mathfrak{T}^h} |K|\Big(U(u_K^{n+1},c)-U(u_K^n,c)\Big)\,\tilde\psi'_\epsilon - \int_{t'_n}^{t'_{n+1}}\int_M \langle F_{x'}(u^h(t',x'),c),\nabla'_g\psi_\epsilon(x;x')\rangle_g\,dv_g(x')\\
&\quad-\int_{t'_n}^{t'_{n+1}}\int_M \Big(\operatorname{div}'_g F_{x'}\Big)(u^h(t',x'),c)\psi_\epsilon(x;x')\,dv_g(x')\\
&\le \sum_{\substack{e\in\partial K\\ K\in\mathfrak{T}^h}} \frac{|K|}{p_K}\,|e|\,\tilde\psi'_{\epsilon,e}\,D_{K,e}^{n+1} + \sum_{\substack{e\in\partial K\\ K\in\mathfrak{T}^h}} \frac{|K|}{p_K}\,|e|\,U(u_{K,e}^{n+1},c)\,\Big(\tilde\psi'_\epsilon-\tilde\psi'_{\epsilon,e}\Big)\\
&\quad+\int_{t'_n}^{t'_{n+1}}\sum_{\substack{e\in\partial K\\ K\in\mathfrak{T}^h}}\int_e \langle F_{y'}(u_K^n,c),\mathbf{n}_{e,K}(y')\rangle_g\Big(\tilde\psi'_{\epsilon,e}(x;x')-\psi_\epsilon(x;y')\Big)\,d\Gamma_g(y').
\end{aligned}$$

By algebraic manipulation, we see that the expression

$$\begin{aligned}
&\sum_{K\in\mathfrak{T}^h} |K|\Big(U(u_K^{n+1},c)-U(u_K^n,c)\Big)\,\tilde\psi'_\epsilon\\
&\quad-\int_{t'_n}^{t'_{n+1}}\int_M \langle F_{x'}(u^h(t',x'),c),\nabla'_g\psi_\epsilon(x;x')\rangle_g\,dv_g(x')dt'\\
&\quad+\int_{t'_n}^{t'_{n+1}}\int_M \operatorname{sgn}(u^h(t',x')-c)\,\Big(\operatorname{div}'_g f\Big)(c,x')\psi_\epsilon(x;x')\,dv_g(x')dt'
\end{aligned}$$

is bounded above by

$$\begin{aligned}
&\le \int_{t'_n}^{t'_{n+1}}\int_M \operatorname{sgn}(u^h(t',x')-c)\,\Big(\operatorname{div}'_g f\Big)(u^h(t',x'),x')\psi_\epsilon(x;x')\,dv_g(x')dt'\\
&\quad+\sum_{\substack{e\in\partial K\\ K\in\mathfrak{T}^h}} \frac{|K|}{p_K}\,|e|\,\tilde\psi'_{\epsilon,e}\,D_{K,e}^{n+1} + \sum_{\substack{e\in\partial K\\ K\in\mathfrak{T}^h}} \frac{|K|}{p_K}\,|e|\,U(u_{K,e}^{n+1},c)\,\Big(\tilde\psi'_\epsilon-\tilde\psi'_{\epsilon,e}\Big)\\
&\quad+\int_{t'_n}^{t'_{n+1}}\sum_{\substack{e\in\partial K\\ K\in\mathfrak{T}^h}}\int_e \langle F_{y'}(u_K^n,c),\mathbf{n}_{e,K}(y')\rangle_g\Big(\tilde\psi'_{\epsilon,e}(x;x')-\psi_\epsilon(x;y')\Big)\,d\Gamma_g(y').
\end{aligned}$$

Now, we multiply this inequality by $\tilde\rho'_\delta(t;t') = \rho_\delta(t;t'_{n+1})$ and summing with respect to time variable, i.e. t', we obtain that the expression

$$
\begin{aligned}
&-\sum_{n=0}^{n_T-1}\int_M U(u_K^n,c)\Big(\rho_\delta(t;t'_{n+1})-\rho_\delta(t;t'_n)\Big)\,\tilde{\psi}'_\epsilon(x;x')\,dv_g(x')\\
&-\iint_{Q_T}\langle F_{x'}(u^h(t',x'),c),\nabla'_g\psi_\epsilon(x;x')\rangle_g\,\tilde{\rho}'_\delta(t;t')\,dv_g(x')dt'\\
&+\iint_{Q_T}\operatorname{sgn}(u^h(t',x')-c)\,\Big(\operatorname{div}'_g f\Big)(c,x')\psi_\epsilon(x;x')\,\tilde{\rho}'_\delta(t;t')\,dv_g(x')dt'\\
&+\int_M U(u^h(T,x'),c)\rho_\delta(t;T)\,\tilde{\psi}'_\epsilon(x;x')\,dv_g(x')\\
&-\int_M U(u^h(0,x'),c)\rho_\delta(t)\,\tilde{\psi}'_\epsilon(x;x')\,dv_g(x')
\end{aligned}
$$

is bounded above by

$$
\begin{aligned}
\le &\iint_{Q_T}\operatorname{sgn}\Big(u^h(t',x')-c\Big)\,\Big(\operatorname{div}'_g f\Big)(u^h(t',x'),x')\psi_\epsilon(x;x')\,\tilde{\rho}'_\delta(t;t')\,dv_g(x')dt'\\
&+\sum_{n=0}^{n_T-1}\sum_{\substack{e\in\partial K\\ K\in\mathcal{T}^h}}\frac{|K|}{p_K}\,|e|\,\tilde{\psi}'_{\epsilon,e}\,\tilde{\rho}'_\delta(t;t')\,D_{K,e}^{n+1}\\
&+\sum_{n=0}^{n_T-1}\sum_{\substack{e\in\partial K\\ K\in\mathcal{T}^h}}\frac{|K|}{p_K}\,|e|\,U(u_{K,e}^{n+1},c)\,\Big(\tilde{\psi}'_\epsilon-\tilde{\psi}'_{\epsilon,e}\Big)\,\tilde{\rho}'_\delta(t;t')\\
&+\int_0^T\sum_{\substack{e\in\partial K\\ K\in\mathcal{T}^h}}p_K\,\frac{|e|}{p_K}\fint_e\langle F_{y'}(u_K^n,c),\mathbf{n}_{e,K}(y')\rangle_g\Big(\tilde{\psi}'_{\epsilon,e}(x;x')-\psi_\epsilon(x;y')\Big)\,\tilde{\rho}'_\delta(t;t')\,d\Gamma_g(y')dt'.
\end{aligned}
$$

4. Following the lines of proof of Proposition 3.12, we can also derive the identity

$$
\begin{aligned}
&\int_M U(u_K^n,c)\Big(\rho_\delta(t;t'_{n+1})-\rho_\delta(t;t'_n)\Big)\tilde{\psi}'_\epsilon(x;x')dv_g(x')\\
&=\int_{t'_n}^{t'_{n+1}}\int_M U(u^h(t',x'),c)\,\partial_{t'}\rho_\delta(t;t')\,\tilde{\psi}'_\epsilon(x;x')\,dv_g(x')dt'.
\end{aligned}
$$

Then, the term $E^h_{\delta,\epsilon}(u^h, u)$ is bounded above by

$$
\begin{aligned}
&\iint_{Q_T} \Big(\sum_{n=0}^{n_T-1} \sum_{\substack{e\in\partial K \\ K\in\mathcal{T}^h}} \frac{|K|}{p_K} \, |e| \; \tilde\rho'_\delta \; \tilde\psi'_{\epsilon,e} \; D^{n+1}_{K,e} \Big) dv_g(x)dt \\
&+ \iint_{Q_T} \Big(\sum_{n=0}^{n_T-1} \sum_{\substack{e\in\partial K \\ K\in\mathcal{T}^h}} \frac{|K|}{p_K} \, |e| \; U(u^{n+1}_{K,e}, c) \left(\tilde\psi'_\epsilon - \tilde\psi'_{\epsilon,e}\right) \tilde\rho'_\delta(t;t') \Big) dv_g(x)dt \\
&+ \iint_{Q_T} \int_M \Big(|u^h(T,x') - u(t,x)| \rho_\delta(t;T) + |u^h(0,x') - u(t,x)| \; \rho_\delta(t) \Big) \\
&\qquad \left| \psi_\epsilon(x;x') - \tilde\psi'_\epsilon(x;x') \right| \, dv_g(x') \; dv_g(x)dt \\
&+ \iint_{Q_T} \int_0^T \sum_{K\in\mathcal{T}^h} \int_K \left| \Big(\operatorname{div}'_g f \Big)(u^n_K, x') - \Big(\operatorname{div}'_g f \Big)(u^n_K, x) \right| \\
&\qquad \tilde\rho'_\delta(t;t') \; \psi_\epsilon(x;x') \; dv_g(x')dt' \; dv_g(x)dt \\
&+ \iint_{Q_T} \int_0^T \sum_{K\in\mathcal{T}^h} \int_{\partial K} \left| F_{y'}(u^n_K, c) - \sigma_{y'}(F_y(u^n_K, c)) \right|_g \\
&\qquad \left| \tilde\psi'_\epsilon(x;x') - \psi_\epsilon(x;y') \right| \; \tilde\rho'_\delta(t;t') \; d\Gamma_g(y')dt'dv_g(x)dt.
\end{aligned}
\tag{3.38}
$$

5. We write (3.38) as $E_1 + E_2 + E_3 + E_4 + E_5$ with obvious notation. In order to estimate E_1 we recall that

$$
D^{n+1}_{K,e} = U(u^{n+1}_{K,e}, c) - U(\tilde u^{n+1}_{K,e}, c),
$$

and

$$
u^{n+1}_{K,e} - \tilde u^{n+1}_{K,e} = -\tau \fint_K \Big(\operatorname{div}'_g f \Big)(u^n_K, x') dv_g(x').
$$

Therefore, since U is convex from Aleksandrov's theorem it has second derivative a.e. (see [21]), and we can write

$$
D^{n+1}_{K,e} = \tau \Big(-\partial_u U(\tilde u^{n+1}_{K,e}, c) \fint_K \Big(\operatorname{div}'_g f \Big)(u^n_K, x') \, dv_g(x') + O(\tau) \Big).
$$

Then, we have

$$
\begin{aligned}
E_1 \le & \iint_{Q_T} \int_0^T \sum_{K\in\mathcal{T}^h} \int_K \left| \Big(\operatorname{div}'_g f \Big)(u^n_K, x') - \Big(\operatorname{div}'_g f \Big)(u^n_K, x) \right| \tilde\rho'_\delta \; \tilde\psi'_\epsilon \; dv_g(x')dt' \\
& + C \, T \, |M|_g \Big(1 + \|u_0\|_{L^\infty} \Big) \, \tau \\
\le & \; C' \; T \, |M|_g \Big(1 + \|u_0\|_{L^\infty} \Big) \, (\tau + \epsilon),
\end{aligned}
$$

where we have used the fact that for every compact K the function $\nabla_g\Big(\operatorname{div}_g f\Big)$ is uniformly bounded in $K \times M$. Now to estimate E_2, we observe that

$$\sum_{n=0}^{n_T-1} \sum_{\substack{e\in\partial K\\ K\in\mathcal{T}^h}} \frac{|K|}{p_K}\, |e|\; U(u_K^{n+1}, c)\left(\tilde{\psi}'_\epsilon - \tilde{\psi}'_{\epsilon,e}\right) = 0.$$

Moreover, from [2] we recall the uniform bound

$$\sum_{n=0}^{n_T-1} \sum_{\substack{e\in\partial K\\ K\in\mathcal{T}^h}} \frac{|K|}{p_K}\, |e|\; |u_{K,e}^{n+1} - u_K^{n+1}|^2 \le \|u_0\|^2_{L^2(M;g)} + C''$$

for some the constant $C'' > 0$.

Hence, we have

$$\begin{aligned} E_2 &= \iint_{Q_T} \Big(\sum_{n=0}^{n_T-1} \sum_{\substack{e\in\partial K\\ K\in\mathcal{T}^h}} \frac{|K|}{p_K}\, |e| \left(U(u_{K,e}^{n+1}, c) - U(u_K^{n+1}, c)\right) \\ &\qquad \left(\tilde{\psi}'_\epsilon - \tilde{\psi}'_{\epsilon,e}\right) \tilde{\rho}'_\delta(t;t')\Big) dv_g(x)dt \\ &\le \sum_{n=0}^{n_T-1} \sum_{\substack{e\in\partial K\\ K\in\mathcal{T}^h}} \frac{|K|}{p_K}\, |e| \left|u_{K,e}^{n+1} - u_K^{n+1}\right| \int_M \left|\tilde{\psi}'_\epsilon - \tilde{\psi}'_{\epsilon,e}\right| dv_g(x) \\ &\le C\frac{\gamma_1}{\epsilon} \sum_{n=0}^{n_T-1} \sum_{\substack{e\in\partial K\\ K\in\mathcal{T}^h}} \frac{|K|}{p_K}\, |e| \left|u_{K,e}^{n+1} - u_K^{n+1}\right| \tau, \end{aligned}$$

where we have used (4.17). Applying Cauchy-Schwartz's inequality, we obtain

$$\begin{aligned} E_2 &\le C\, \frac{\gamma_1}{\epsilon} \left(T\, |M|_g\right)^{1/2} \Big(\sum_{n=0}^{n_T-1} \sum_{\substack{e\in\partial K\\ K\in\mathcal{T}^h}} \frac{|K|}{p_K}\, |e| \left|u_{K,e}^{n+1} - u_K^{n+1}\right|^2 \tau\Big)^{1/2} \\ &\le C\ \sqrt{\gamma_1} \left(T\, |M|_g\right)^{1/2} \frac{h^{1/2}}{\epsilon} \left(\|u_0\|^2_{L^2(M;g)} + C(T)\right)^{1/2} \\ &\le C_2 \left(1 +\ \|u_0\|_{L^2(M;g)}\right) \frac{h^{1/2}}{\epsilon}\ |M|_g^{1/2}. \end{aligned}$$

The terms E_3 and E_4 are estimated in the same way that we have already done, that is

$$\begin{aligned} E_3 &\le C\, |M|_g \Big(1 + \|u_0\|_{L^\infty}\Big) \frac{h}{\epsilon}, \\ E_4 &\le C\, T\, |M|_g \Big(1 + \|u_0\|_{L^\infty}\Big)\, \epsilon. \end{aligned}$$

Finally, we estimate the last term, that is

$$\begin{aligned} E_5 &= \iint_{Q_T} \int_0^T \sum_{K \in \mathfrak{T}^h} |K| \frac{p_K}{|K|} \fint_{\partial K} \left| F_{y'}(u_K^n, c) - F_y(u_K^n, c) \right| \\ &\qquad \left| \tilde{\psi}'_\epsilon(x; x') - \psi_\epsilon(x; y') \right| \tilde{\rho}'_\delta(t; t')\, d\Gamma_g(y') dt' dv_g(x) dt \\ &\leq C\, T\, |M|_g \Big(1 + \|u_0\|_{L^\infty} \Big) \frac{h}{\epsilon}, \end{aligned}$$

where we have used the condition (3.8). □

Chapitre 4

Version relativiste de l'équation de Burgers*

The relativistic version of Burgers equation

4.1 Introduction

We consider scalar, hyperbolic balance laws posed on an $(N+1)$-dimensional curved spacetime (M, ω) endowed with a volume form,

$$\operatorname{div}^{\omega}(T(v)) = S(v), \tag{4.1}$$

whose unknown function is the scalar field $v : M \to \mathbb{R}$ and where $\operatorname{div}^{\omega}$ denotes the divergence operator associated with ω. The given vector field $T = T(v)$ is defined on the manifold M, and depends upon v as a parameter. The manifold (with boundary) M is assumed to be foliated by hypersurfaces, that is,

$$M = \bigcup_{t \geq 0} H_t, \tag{4.2}$$

such that each slice H_t is an N-dimensional manifold endowed with a normal 1-form field N_t and has the same topology as the initial slice H_0. Global hyperbolicity of the spacetime and the equation (4.1) is ensured by assuming that the function

$$v \mapsto T^0(v) := \langle N_t, T(v) \rangle \quad \text{is strictly increasing.} \tag{4.3}$$

Furthermore, the right-hand side $S = S(v)$ of (4.1) is a given scalar field defined on M and depending upon v as a parameter.

In this paper, we investigate the class of hyperbolic equations (4.1)–(4.3) which should be viewed as a simplified model for the dynamics of relativistic

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compressible fluids. Specifically, our first aim below will be to impose that (4.1) satisfies the *Lorentz invariance property* satisfied by relativistic compressible fluids, and identify a unique (up to normalization) balance law which could be viewed as a relativistic generalization to the classical Burgers equation known as a simplified model of non-relativistic fluids. In short, we may refer to our new equation as the *relativistic version of Burgers equation,* especially we will see that the non-relativistic version is recovered when the light speed converges to infinity.

We will see that, for a suitable choice of the vector field $T(v)$, the balance law (4.1) exhibits many of the mathematical properties (hyperbolicity, genuine nonlinearity, stationary solutions) and the challenging difficulties (shock wave, long-time asymptotics) encountered with the full Euler system of compressible fluids. Important, in the context of hyperbolic equations posed on a curved background, the proposed model provides a good set-up to develop, test, and compare together, shock-capturing methods of numerical approximation.

In a second part of this paper, we introduce a finite volume scheme for the approximation of weak solutions to general balance laws (4.1), which we then apply to our relativistic version of Burgers equation. By construction, the proposed scheme is fully consistent with the divergence form of the equation and therefore applies to weak solutions containing, for instance, propagating shock waves. Numerical experiments are presented with various choices of flux field and volume forms, and demonstrate the convergence of the proposed finite volume scheme, and its relevance for computing long-time asymptotics of (possibly discontinuous) solutions in a curved background.

4.2 The relativistic generalization of Burgers equation

4.2.1 Derivation of a Lorentz invariant model

In this section, we search for flux fields $T(v)$ for which solutions to the equation (4.1) satisfy a Lorentz invariant property similar to the one satisfied by relativistic fluids. Without loss of generality and in order to simplify the derivation we now assume that $N = 1$, $S(v) \equiv 0$, and that the manifold $M = [0, +\infty) \times \mathbb{R}$ covered by a single coordinate chart (x^0, x^1) with $\omega = dx^0 dx^1$, so that (4.1) takes the form of a conservation law

$$\partial_0 T^0(v) + \partial_1 T^1(v) = 0,$$

with $\partial_0 = \partial/\partial x^0$ and $\partial_1 = \partial/\partial x^1$. In addition, in the present section, we also assume that the functions $T^0 = T^0(v)$ and $T^1 = T^1(v)$ are independent of (x^0, x^1). The general geometric set-up will be discussed later in Section 4.3.

Recall that the Lorentz transformations $(x^0, x^1) \mapsto (\overline{x}^0, \overline{x}^1)$ is defined by

$$\begin{aligned} \overline{x}^0 &:= \gamma_\epsilon(V)\,(x^0 - \epsilon^2 V x^1), \\ \overline{x}^1 &:= \gamma_\epsilon(V)\,(-V\,x^0 + x^1), \qquad \gamma_\epsilon(V) = \left(1 - \epsilon^2\,V^2\right)^{-1/2}, \end{aligned} \tag{4.4}$$

where $\epsilon \in (-1, 1)$ denotes the inverse of the (normalized) speed of light, and $\gamma_\epsilon(V)$ is the so-called Lorentz factor associated with a given speed $V \in (-1/\epsilon, 1/\epsilon)$. These equations can be inverted to give

$$\begin{aligned} x^0 &= \gamma_\epsilon(V)\,(\overline{x}^0 + \epsilon^2 V \overline{x}^1), \\ x^1 &= \gamma_\epsilon(V)\,(V\,\overline{x}^0 + \overline{x}^1). \end{aligned}$$

Instead of V, it is often convenient to use the modified velocity $U \in \mathbb{R}$ defined by

$$e^{\epsilon U} := \gamma_\epsilon(V)\,(1 + \epsilon\,V) = \left(\frac{1 + \epsilon V}{1 - \epsilon V}\right)^{1/2},$$

and to rewrite the Lorentz transformations in the more compact form $\overline{x}^0 \pm \epsilon\,\overline{x}^1 = e^{\mp\epsilon U}\,(x^0 \pm \epsilon\,x^1)$ or, equivalently,

$$\begin{aligned} \overline{x}^0 &= \cosh(\epsilon U)\,x^0 - \epsilon\,\sinh(\epsilon U)\,x^1, \\ \overline{x}^1 &= -\frac{1}{\epsilon}\sinh(\epsilon U)\,x^0 + \cosh(\epsilon U)\,x^1, \end{aligned} \tag{4.5}$$

where

$$\cosh(\epsilon U) = \gamma_\epsilon(V), \quad \sinh(\epsilon U) = \epsilon V\,\gamma_\epsilon(V).$$

Recall that Lorentz transformations (together with spatial rotations if $N \geq 2$) form the so-called Lorentz group of all isometries characterized by the condition that the length element of the Minkowski metric is preserved, that is,

$$-\epsilon^{-2}\,(\overline{x}^0)^2 + (\overline{x}^1)^2 = -\epsilon^{-2}\,(x^0)^2 + (x^1)^2.$$

Recall also that the relativistic Euler equations of compressible fluids are invariant under Lorentz transformations (4.4). More precisely, given a speed V, the fluid velocity component v in the coordinate system (x^0, x^1) is related to the component $\overline{v}$ in the coordinates $(\overline{x}^0, \overline{x}^1)$ thanks to the Lorentz transformations (4.4) which yields

$$\overline{v} = \frac{v - V}{1 - \epsilon^2 V\,v}. \tag{4.6}$$

Clearly, in the non-relativistic limit, corresponding to $\epsilon \to 0$, one recovers the Galilean transformations (with now V describing $\mathbb{R}$)

$$\overline{x}^0 = x^0, \quad \overline{x}^1 = -V\,x^0 + x^1, \quad \overline{v} = v - V \qquad \text{when } \epsilon = 0. \tag{4.7}$$

Theorem 4.1 (The relativistic version of Burgers equation)**.** *The conservation law*

$$\partial_0 T^0(v) + \partial_1 T^1(v) = 0, \tag{4.8}$$

is invariant under Lorentz transformations if and only if after suitable normalization (discussed below) one has

$$T^0(v) = \frac{v}{\sqrt{1-\epsilon^2 v^2}}, \qquad T^1(v) = \frac{1}{\epsilon^2}\Big(\frac{1}{\sqrt{1-\epsilon^2 v^2}} - 1\Big), \tag{4.9}$$

where the scalar field v takes its value in $(-1/\epsilon, 1/\epsilon)$.

Proof. The proof is done in three steps.

Step 1. Changing coordinates and construction of $H_\pm(v)$.

Considering the conservation law (4.8) in (x^0, x^1) coordinates, we have by Lorentz transformations

$$\frac{\partial \overline{x}^0}{\partial x^0} = \frac{1}{\sqrt{1-\epsilon^2 V^2}} = \gamma_\epsilon(V), \qquad \frac{\partial \overline{x}^0}{\partial x^1} = \frac{-V}{\sqrt{1-\epsilon^2 V^2}} = -V\,\gamma_\epsilon(V),$$
$$\frac{\partial \overline{x}^1}{\partial x^0} = \frac{-\epsilon^2 V}{\sqrt{1-\epsilon^2 V^2}} = -\epsilon^2 V \gamma(V), \qquad \frac{\partial \overline{x}^1}{\partial x^1} = \frac{1}{\sqrt{1-\epsilon^2 V^2}} = \gamma_\epsilon(V).$$

An application of Chain rule gives

$$\partial_0 T^0 = \frac{\partial T^0}{\partial x^0} = \frac{\partial T^0}{\partial \overline{x}^0}\frac{\partial \overline{x}^0}{\partial x^0} + \frac{\partial T^0}{\partial \overline{x}^1}\frac{\partial \overline{x}^1}{\partial x^0} = \frac{\partial T^0}{\partial \overline{x}^0}\gamma - \frac{\partial T^0}{\partial \overline{x}^1}\gamma\, V,$$
$$\partial_1 T^1 = \frac{\partial T^1}{\partial x^1} = \frac{\partial T^1}{\partial \overline{x}^0}\frac{\partial \overline{x}^0}{\partial x^1} + \frac{\partial T^1}{\partial \overline{x}^1}\frac{\partial \overline{x}^1}{\partial x^1} = -\epsilon^2\frac{\partial T^1}{\partial \overline{x}^0}\gamma\, V + \frac{\partial T^1}{\partial \overline{x}^1}\gamma.$$

Then substituting these in main equation, it follows that

$$0 = \partial_0 T^0 + \partial_1 T^1 = \partial_0(\gamma\, T^0 - \epsilon^2\,\gamma\, V\, T^1) + \partial_1(-V\,\gamma\, T^0 + \gamma\, T^1), \tag{4.10}$$

which has the same structure as (4.8) in $(\overline{x}^0, \overline{x}^1)$ coordinates if

$$\begin{aligned} \gamma\Big(T^0(v) - \epsilon^2 V\, T^1(v)\Big) &= T^0\Big(\frac{v-V}{1-\epsilon^2 v\, V}\Big) + C_1, \\ \gamma\Big(-VT^0(v) + T^1(v)\Big) &= T^1\Big(\frac{v-V}{1-\epsilon^2 v\, V}\Big) + C_2, \end{aligned} \tag{4.11}$$

is satisfied, which is the desired invariance property. In the rest of proof the exact forms of T^0 and T^1 are studied.

We start by normalizing the fluxes so that $T^0(0) = T^1(0) = 0$, and that the constants $C_1 = -T^0(-V)$, $C_2 = -T^1(-V)$ depending upon V. Next multiplying the second equation of (4.11) by $\pm\epsilon$ and summing with the first one, it follows that

$$\gamma\,(1 \mp \epsilon V)(T^0 \pm \epsilon\, T^1)(v) = (T^0 \pm \epsilon\, T^1)\Big(\frac{v-V}{1-\epsilon^2\, vV}\Big) - (T^0 \pm \epsilon\, T^1)(-V).$$

Next assigning $(T^0 \pm \epsilon\, T^1) =: H_\pm$, the last equation is rewritten by

$$H_\pm(v) = \sqrt{\frac{1 \pm \epsilon V}{1 \mp \epsilon V}}\Bigg(H_\pm\Big(\frac{v-V}{1-\epsilon^2\, vV}\Big) - H_\pm(-V)\Bigg). \tag{4.12}$$

Step 2. Defining u, $\phi_\epsilon(u)$ et $\phi_\epsilon(U)$ to establish a general formulation.

We define

$$u := \frac{1}{2\epsilon}\ln\Big(\frac{1+\epsilon v}{1-\epsilon v}\Big), \quad \phi_\epsilon(u) := v, \quad \phi_\epsilon(U) := V$$

so that

$$v = \frac{1}{\epsilon}\frac{e^{2\epsilon u}-1}{e^{2\epsilon u}+1} = \phi_\epsilon(u), \quad \overline{v} = \frac{1}{\epsilon}\frac{e^{2\epsilon(u-U)}-1}{e^{2\epsilon(u-U)}+1} = \phi_\epsilon(u-U), \quad \sqrt{\frac{1 \pm \epsilon V}{1 \mp \epsilon V}} = e^{\pm\,\epsilon U}.$$

By inserting these in (4.12), it follows that

$$H_\pm\Big(\phi_\epsilon(u)\Big) - H_\pm\Big(\phi_\epsilon(0)\Big) = e^{\pm\,\epsilon U}\Big(H_\pm\Big(\phi_\epsilon(u-U)\Big) - H_\pm\Big(\phi_\epsilon(-U)\Big)\Big). \tag{4.13}$$

Step 3. Derivation of (4.13) and obtaining T^0 and T^1.

To avoid abuse of notation, we take $H, \phi, e^{\epsilon U}$ instead of $H_\pm, \phi_\epsilon, e^{\pm\epsilon U}$, respectively. We derive (4.13) with respect to u

$$H'\Big(\phi(u)\Big)\phi'(u) = e^{\epsilon U}H'\Big(\phi(u-U)\Big)\phi'(u-U),$$

and with respect to U

$$H''\Big(\phi(u-U)\Big)\Big(\phi'(u-U)\Big)^2 = \epsilon H'\Big(\phi(u-U)\Big)\phi'(u-U) - \phi''(u-U)H'\Big(\phi(u-U)\Big).$$

Finally letting $U = 0$ it follows that

$$H''\Big(\phi(u)\Big)\Big(\phi'(u)\Big)^2 = \epsilon H'\Big(\phi(u)\Big)\phi'(u) - \phi''(u)H'\Big(\phi(u)\Big).$$

We define $K(u) := H\Big(\phi_\epsilon(u)\Big)$ and substitute in previous equation to obtain

$$K''(u) = \epsilon K'(u),$$

which has the solution

$$K(u) = c\frac{e^{\epsilon u}}{\epsilon} + \tilde{c}, \qquad c \text{ and } \tilde{c} \text{ are constants.}$$

It follows that

$$H_+\Big(\phi(u)\Big) = c_1\frac{e^{\epsilon u}}{\epsilon} + c_2, \qquad H_-\Big(\phi(u)\Big) = \tilde{c_1}\frac{e^{-\epsilon u}}{-\epsilon} + \tilde{c_2},$$

where $c_1, c_2, \tilde{c_1}$ and $\tilde{c_2}$ are constants. We can choose these constants so that

$$H_\pm\Big(\phi(u)\Big) = \frac{e^{\pm\epsilon u} - 1}{\pm\epsilon}.$$

Recalling from Step1 that $H_\pm = T^0 \pm \epsilon T^1$ with $H_\pm\Big(\phi(u)\Big) = H_\pm(v)$, then

$$T^0 = \frac{H_+ + H_-}{2}, \quad T^1 = \frac{H_+ - H_-}{2\epsilon},$$

from which we can deduce

$$T^0(v) = T^0\Big(\phi(u)\Big) = \frac{e^{\epsilon u} - e^{-\epsilon u}}{2\epsilon} = \frac{1}{\epsilon}\sinh(\epsilon u) = u + O(\epsilon^2 u^3),$$
$$T^1(v) = T^1\Big(\phi(u)\Big) = \frac{e^{\epsilon u} + e^{-\epsilon u} - 2}{2\epsilon^2} = \frac{1}{\epsilon^2}\Big(\cosh(\epsilon u) - 1\Big) = \frac{u^2}{2} + O(\epsilon^2 u^4).$$

Observe that T^0 and T^1 are linear and quadratic, respectively, and that, in the limiting case $\epsilon \to 0$, one obtains the inviscid Burger equation. Moreover, substituting $u = \frac{1}{2\epsilon}\ln\Big(\frac{1+\epsilon v}{1-\epsilon v}\Big)$ in $T^0\Big(\phi(u)\Big)$ and $T^1\Big(\phi(u)\Big)$ one gets

$$T^0(v) = \frac{v}{\sqrt{1-\epsilon^2 v^2}}, \qquad T^1(v) = \frac{1}{\epsilon^2}\Big(\frac{1}{\sqrt{1-\epsilon^2 v^2}} - 1\Big),$$

which is the desired result.

□

4.2.2 Hyperbolicity and convexity properties

In the following theorem, we propose an equivalent version of conservation law (4.8) satisfying certain properties in the relativistic and non-relativistic cases.

Theorem 4.2 (Properties of the relativistic Burgers equation)**.**

1. *The map* $w = T^0(v) = \frac{v}{\sqrt{1-\epsilon^2 v^2}} \in \mathbb{R}$ *is increasing and one-to-one from* $(-1/\epsilon, 1/\epsilon)$ *onto* $\mathbb{R}$.

2. *In terms of the new unknown $w \in \mathbb{R}$, the equation* (4.8) *is equivalent to*

$$\begin{aligned} &\partial_0 w + \partial_1 f_\epsilon(w) = 0, \\ &f_\epsilon(w) = \frac{1}{\epsilon^2}\Big(\pm \sqrt{1+\epsilon^2 w^2} - 1\Big), \end{aligned} \tag{4.14}$$

where the flux f_ϵ is strictly convex, (or strictly concave), and, therefore, the conservation law (4.8) *is genuinely nonlinear in the sense that*

$$\frac{\partial_v T^1(v)}{\partial_v T^0(v)} = \partial_w f_\epsilon(w)$$

is strictly increasing (decreasing) in the variable $T^0(v)$.

3. *In the non-relativistic limit $\epsilon \to 0$, one recovers the inviscid Burger equation*

$$\partial_0 u + \partial_1 (u^2/2) = 0, \tag{4.15}$$

where $u \in \mathbb{R}$.

The proposed equation (4.14) retains several key features of the relativistic Euler equations:

- Like the conservation of mass-energy in the Euler system, (4.14) has a conservative form.
- Like the velocity component in the Euler system, our unknown v is constrained to lie in the interval $(-1/\epsilon, 1/\epsilon)$ limited by the (inverse of the) light speed parameter.
- Like the Euler system, by sending the light speed to infinity one recovers the classical (non-relativistic) model.

Proof. 1. We derive $w = T^0(v) = \frac{v}{\sqrt{1-\epsilon^2 v^2}} \in \mathbb{R}$ with respect to v

$$T^0_v(v) = \frac{1}{(1-\epsilon^2 v^2)^{3/2}} > 0, \qquad v \in (-1/\epsilon, 1/\epsilon),$$

so that $T^0(v)$ is increasing and one-to-one from $(-1/\epsilon, 1/\epsilon)$ onto $\mathbb{R}$.

2. Applying a change of variable $w := \frac{v}{\sqrt{1-\epsilon^2 v^2}}$, one obtains $v = \pm \frac{w}{\sqrt{1+\epsilon^2 w^2}}$ and substituting in (4.8), it follows that

$$\partial_0 w + \partial_1 f_\epsilon(w) = 0,$$

where

$$f_\epsilon(w) = \frac{1}{\epsilon^2}\Big(\pm \sqrt{1+\epsilon^2 w^2} - 1\Big).$$

To see that the flux function is strictly convex we derive it twice with respect to w such that

$$f'_\epsilon(w) = \frac{w}{\sqrt{1+\epsilon^2 w^2}}, \qquad f''_\epsilon(w) = \frac{1}{(1+\epsilon^2 w^2)^{3/2}} > 0,$$

which is the desired result. On the other hand we have

$$\partial_v T^1(v) = \frac{v}{(1-\epsilon^2 v^2)^{3/2}}, \qquad \partial_v T^0(v) = \frac{1}{(1-\epsilon^2 v^2)^{3/2}},$$

so that

$$\frac{\partial_v T^1(v)}{\partial_v T^0(v)} = v = \frac{w}{\sqrt{1+\epsilon^2 w^2}} = f'_\epsilon(w).$$

Next since derivative $f''_\epsilon(w)$ of $f'_\epsilon(w)$ is strictly positive, we conclude that $f'_\epsilon(w)$ is strictly increasing in variable $T^0(v)$. Analogously, if the flux is strictly concave one obtains the desired results.

3. Recall that we can write $T^0(v)$ and $T^1(v)$ in the variable of u so that

$$T^0(v) = \frac{1}{\epsilon}\sinh(\epsilon u) = u + O(\epsilon^2 u^3), \qquad T^1(v) = \frac{1}{\epsilon^2}\Big(\cosh(\epsilon u) - 1\Big) = \frac{u^2}{2} + O(\epsilon^2 u^4).$$

Observe that if ϵ is sufficiently close to 0, one obtains the standard inviscid Burgers equation (4.15).

□

4.2.3 The non-relativistic case

Recall that the Galilean transformations $(x^0, x^1) \mapsto (\overline{x}^0, \overline{x}^1)$ is defined by

$$\overline{x}^0 = x^0, \quad \overline{x}^1 = x^1 - Vx^0.$$

Moreover, given a speed V (describing $\mathbb{R}$), the velocity component v in the coordinate system (x^0, x^1) is related to the component $\overline{v}$ in the coordinates $(\overline{x}^0, \overline{x}^1)$ given by

$$\overline{v} = v - V.$$

Note that one recovers the Galilean transformations by relativistic case with $\epsilon \to 0$.

The following theorem shows the invariance property of the given conservation law with the exact forms of flux functions. We will see that the Burgers equation in non-relativistic case satisfies this property.

Theorem 4.3 (A derivation of the (non-relativistic) Burgers equation)**.** *The conservation law*

$$\partial_0 T^0(v) + \partial_1 T^1(v) = 0, \tag{4.16}$$

is invariant under Galilean transformations if and only if the flux functions T^0 and T^1 are linear and quadratic, respectively. In particular, if $T^0(v) = v$, then after a suitable normalization one gets $T^1(v) = v^2/2$.

Proof. It follows by Galilean transformations that

$$\frac{\partial \overline{x}^1}{\partial x^1} = 1, \quad \frac{\partial \overline{x}^1}{\partial x^0} = -V \quad \frac{\partial \overline{x}^0}{\partial x^1} = 1, \quad \frac{\partial \overline{x}^0}{\partial x^0} = 1, \quad \frac{\partial \overline{v}}{\partial v} = 1.$$

By using the Chain rule

$$\partial_0 T^0 = \frac{\partial T^0}{\partial \overline{x}^0}\frac{\partial \overline{x}^0}{\partial x^0} + \frac{\partial T^0}{\partial \overline{x}^1}\frac{\partial \overline{x}^1}{\partial x^0} = \frac{\partial T^0}{\partial \overline{x}^0} - V\frac{\partial T^0}{\partial \overline{x}^1}, \quad \text{and} \quad \partial_1 T^1 = \frac{\partial T^1}{\partial \overline{x}^1},$$

and substituting in (4.16), it follows that

$$\partial_0 T^0(\overline{v} + V) + \partial_1\Big(T^1(\overline{v} + V) - VT^0(\overline{v} + V)\Big) = 0.$$

Thus it has the same structure as (4.16) if

$$\begin{aligned} &T^0(\overline{v} + V) = T^0(\overline{v}) + C_1, \\ &T^1(\overline{v} + V) - V\,T^0(\overline{v} + V) = T^1(\overline{v}) + C_2, \end{aligned} \tag{4.17}$$

where C_1 and C_2 are constants depending upon V. Next we derive the first equation of (4.17) with respect to $\overline{v}$

$$T^0_{\overline{v}}(\overline{v} + V) = T^0_{\overline{v}}(\overline{v}),$$

by which we conclude that $T^0_{\overline{v}}$ is periodic of period V. As a result, T^0 is a linear function for arbitrary V, which is of the form

$$T^0(v) = Cv + \tilde{C},$$

for some constants C and $\tilde{C}$.

On the other hand multiplying the first equation of (4.17) by V and then adding to the second one, we get

$$T^1(\overline{v} + V) = T^1(\overline{v}) + V\,T^0(\overline{v}) + VC_1 + C_2.$$

Next using the fact that T^0 is a linear function given as above, and deriving twice the given equation with respect to $\overline{v}$, it follows that

$$\begin{aligned} &T^1(\overline{v} + V) - T^1(\overline{v}) = CV\,\overline{v} + V\tilde{C} + VC_1 + C_2, \\ &T^1_{\overline{v}}(\overline{v} + V) - T^1_{\overline{v}}(\overline{v}) - CV = 0, \\ &T^1_{\overline{v}\,\overline{v}}(\overline{v} + V) - T^1_{\overline{v}\,\overline{v}}(\overline{v}) = 0. \end{aligned}$$

Therefore $T^1_{\overline{v}\,\overline{v}}$ is periodic of period V which implies that the first derivative $T^1_{\overline{v}}$ is linear for arbitrary V. Since the derivative of quadratic functions is linear, as a result we conclude that T^1 is quadratic, which is the desired result. In particular case, one obtains the Burgers equations with $T^0(v) = v$ and $T^1(v) = v^2/2$.

□

4.3 The effect of the geometry

4.3.1 General hyperbolic balance laws

For simplicity in the presentation, we assume that the spacetime and the conservation law under consideration admit symmetries that allow for a reduction to dimension 1 + 1. The generalization to arbitrary dimensions is straightforward. We assume that the manifold is described by a single chart and, after identification, we set $M = \mathbb{R}_+ \times \mathbb{R}$. In coordinates (x^0, x^1) with $\partial_\alpha := \partial/\partial x^\alpha$ (with $\alpha = 0, 1$), the hyperbolic balance laws under consideration reads

$$\partial_0(\omega\, T^0(v)) + \partial_1(\omega\, T^1(v)) = \omega\, S(v), \tag{4.18}$$

where $v : M \to \mathbb{R}$ is the unknown function and $T^\alpha = T^\alpha(v)$ and $S = S(v)$ are prescribed (flux and source) fields on M, while $\omega = \omega(x)$ is a positive weight-function. This equation is *hyperbolic in the direction* $\partial/\partial x^1$ provided

$$\partial_v T^0(v) > 0, \tag{4.19}$$

which we always assume from now on.

Note that the above balance law can be rewritten in the form

$$\partial_0 v + \partial_1 f(v) = \widetilde{S}(v), \tag{4.20}$$

with

$$\begin{aligned} \partial_v f(v) &:= \frac{\partial_v T^1(v)}{\partial_v T^0(v)}, \qquad \Omega := \ln \omega, \\ \widetilde{S}(v) &:= \frac{1}{\partial_v T^0(v)} \Big(S(v) - \partial_0 \Omega\, T^0(v) - \partial_1 \Omega\, T^1(v) \Big). \end{aligned} \tag{4.21}$$

However, one should keep in mind that (4.18) is the *geometric form* of this equation: it should be (and will be) preferred for the numerical discretization. For instance, when $\widetilde{S} \equiv 0$, then (4.18) is a genuine *conservation law,* while (4.20) does not retain the conservation form.

4.3.2 Derivation of a covariant scalar model

We consider the general conservation law

$$\nabla_\alpha T^\alpha(v) = 0 \quad \text{in } M, \tag{4.22}$$

where $v : M \to \mathbb{R}$ is an unknown scalar filed, and T^α is a prescribed vector field depending on v as a parameter.

We impose that T^α is a unit spacelike vector field, that is

$$g_{\alpha\beta} T^\alpha(v) T^\beta(v) = 1. \tag{4.23}$$

For instance, if (M, g) is the (1+1)-Minkovski spacetime in standard coordinates $(x^0, x^1) = (t, x)$, then

$$g = -\epsilon^{-2} dt^2 + dx^2$$

and

$$-\epsilon^{-2} T^0(v)^2 + T^1(v)^2 = 1.$$

It follows that

$$T^1(v) = \pm\Big(\epsilon^{-2} T^0(v)^2 + 1\Big)^{1/2} = \pm\sqrt{\epsilon^{-2} T^0(v)^2 + 1}.$$

Next by setting $z := T^0(v)$ we find the evolution equation

$$\partial_t z + \partial_x\Big(C \pm \sqrt{\epsilon^{-2} z^2 + 1}\Big) = 0$$

where the constant C was added for convenience. Observe that if we apply a change of variable $z = \epsilon^2 w$ to the above equation, then after a normalization, one can recover

$$\partial_t w + \partial_x\Big(\frac{-1 \pm \sqrt{\epsilon^2 w^2 + 1}}{\epsilon^2}\Big) = 0,$$

which is equivalent to the proposed equation (4.14).

4.3.3 Stationary solutions

We first consider (4.14) with a weight function ω such that

$$\begin{aligned} &\partial_0(\omega\, w) + \partial_1(\omega\, f_\epsilon(w)) = 0, \\ &f_\epsilon(w) = \frac{1}{\epsilon^2}\Big(-1 \pm \sqrt{1 + \epsilon^2 w^2}\Big), \end{aligned} \tag{4.24}$$

where $w \in \mathbb{R}$ and the weight function $\omega = \omega(x^1)$ does not depends on time. Then considering the following item

$$\partial_1(\omega\, f_\epsilon(w)) = 0$$

and recalling that f_ϵ is strictly convex, (or strictly concave), we find

$$\omega\, f_\epsilon(w) = c \quad \text{and} \quad f_\epsilon^{-1}(\frac{c}{\omega}) = w,$$

where c is constant. It follows that

$$f^{-1}_{\epsilon\pm}\Big(\frac{c}{\omega(x^1)}\Big) = \pm\sqrt{\frac{c^2}{\omega^2(x^1)}\epsilon^2 + 2\frac{c}{\omega(x^1)}} = \pm\frac{c}{\omega(x^1)}\sqrt{\epsilon^2 + 2\frac{\omega(x^1)}{c}},$$

and we denote

$$\tilde{v}_\epsilon(x^1) := f^{-1}_{\epsilon\pm}\Big(\frac{c}{\omega(x^1)}\Big). \tag{4.25}$$

This result lets us to conclude the following proposition.

Proposition 4.4 (Stationary solutions)**.** *The stationary solutions for the model* (4.24) *are described by*

$$\tilde{v}_\epsilon(x^1) = \pm\frac{c}{\omega(x^1)}\sqrt{\epsilon^2 + 2\frac{\omega(x^1)}{c}},$$

where $\epsilon > 0$, $\omega(x^1)$ *is given and* c *is constant.*

4.3.4 Relativistic zero-pressure Euler Equations

Recall the relativistic Euler equation

$$\begin{aligned}
&\partial_0\Big(\frac{p+\rho c^2}{c^2}\frac{v^2}{c^2-v^2} + \rho\Big) + \partial_1\Big((p+\rho c^2)\frac{v}{c^2-v^2}\Big) = 0,\\
&\partial_0\Big((p+\rho c^2)\frac{v}{c^2-v^2}\Big) + \partial_1\Big((p+\rho c^2)\frac{v^2}{c^2-v^2} + p\Big) = 0,
\end{aligned}$$

where p, ρ, u and c denote the pressure, density, velocity and light speed, respectively. The relativistic zero-pressure Euler equations is then recovered by substituting $p = 0$ so that

$$\begin{aligned}
&\partial_0\Big(\frac{\rho}{c^2-v^2}\Big) + \partial_1\Big(\frac{\rho\, v}{c^2-v^2}\Big) = 0,\\
&\partial_0\Big(\frac{\rho\, v}{c^2-v^2}\Big) + \partial_1\Big(\frac{\rho\, v^2}{c^2-v^2}\Big) = 0.
\end{aligned} \tag{4.26}$$

We suppose now that $c = 1/\epsilon$ and ρ as a constant so that (4.26) is rewritten by

$$\begin{aligned}
&\partial_0\Big(\frac{1}{1-\epsilon^2 v^2}\Big) + \partial_1\Big(\frac{v}{1-\epsilon^2 v^2}\Big) = 0,\\
&\partial_0\Big(\frac{v}{1-\epsilon^2 v^2}\Big) + \partial_1\Big(\frac{v^2}{1-\epsilon^2 v^2}\Big) = 0.
\end{aligned} \tag{4.27}$$

Next applying a change of variable

$$z = \frac{v}{1-\epsilon^2 v^2} \quad \text{such that} \quad v = v_\pm = \frac{-1 \pm \sqrt{1+4\epsilon^2 z^2}}{2\epsilon^2 z}$$

and substituting these in the second equation of (4.27), we get

$$\partial_0 z + \partial_1 \left(\frac{-1 \pm \sqrt{1 + 4\epsilon^2 z^2}}{2\,\epsilon^2} \right) = 0. \tag{4.28}$$

We recall now the proposed relativistic Burger equation (4.14)

$$\partial_0 w + \partial_1 \left(\frac{-1 \pm \sqrt{1 + \epsilon^2 w^2}}{\epsilon^2} \right) = 0.$$

One can observe that these two equations (4.14) and (4.28) are equivalent up to a constant factor.

4.4 Well-balanced finite volume approximation

4.4.1 Geometric formulation

We assume that the (1 + 1)-dimensional curved spacetime (M, ω) is globally hyperbolic, in the sense that there exists a foliation of M by spacelike, compact, oriented hypersurfaces H_t, $(t \in \mathbb{R})$ such that

$$M = \bigcup_{t \in \mathbb{R}} H_t,$$

where each slice has the topology of $\mathbb{R}$. We assume that M is foliated by these slices.

Let $\mathcal{T}^h = \bigcup_{K \in \mathcal{T}^h} K$ be a triangulation of the manifold M, which is made of (compact) spacetime elements K and satisfies the following conditions:

- The boundary ∂K of an element K is piecewise smooth $\partial K = \bigcup_{e \subset \partial K} e$ and contains exactly two spacelike faces, denoted by e_K^+ and e_K^-, and "timelike" elements
$$e^0 \in \partial^0 K := \partial K \setminus \left\{ e_K^+, e_K^- \right\}.$$
- The intersection $K \cap K'$ of two distinct elements K, K' is a common face of K, K'.
- $|K|$ and $|e_K^+|, |e_K^-|, |e^0|$ represent the measures of K and e_K^+, e_K^-, e^0, respectively.

We introduce the finite volume method by formally averaging the balanced law (4.18) over each element $K \in \mathcal{T}^h$ of the triangulation, by integrating in space and time

$$\int_K (\omega S) dV_M = \int_K \operatorname{div}^{\omega}(T(v))\, dV_M,$$

which is equal to

$$\int_{\partial^0 K} (\omega S) dV_{\partial^0 K} = \int_{e_K^+} (\omega T^0) dV_e - \int_{e_K^-} (\omega T^0) dV_e + \sum_{e^0 \in \partial^0 K} \int_{e^0} (\omega T^1) dV_{e^0},$$

and after arranging the terms

$$\int_{e_K^+} (\omega T^0) dV_e = \int_{e_K^-} (\omega T^0) dV_e - \sum_{e^0 \in \partial^0 K} \int_{e^0} (\omega T^1) dV_{e^0} + \int_{\partial^0 K} (\omega S) dV_{\partial^0 K}.$$

Given the averaged values v_K^- along e_K^- and $v_{K_{e^0}}^-$ along $e^0 \in \partial^0 K$, we need an approximation v_K^+ of the average value of the solution along e_K^+. To this end, we introduce the approximations

$$\int_{e_K^-} T^0 dV_e \simeq |e_K^-| \overline{T}_{e_K^-}(v_K^-), \qquad \int_{e^0} T^1 dV_{e^0} \simeq |e^0| \, q_{K,e^0}(v_K^-, v_{K_{e^0}}^-),$$

and

$$\int_{\partial^0 K} S \, dV_{\partial^0 K} \simeq |\partial^0 K| \, \overline{S}_{\partial^0 K},$$

where

$$\overline{T}_e^0(v) := \frac{1}{|e|} \int_e T^0(v) dV_e, \qquad \overline{S}_e(v) := \frac{1}{|e|} \int_e S(v) dV_e,$$

and to each element K, and each element $e^0 \in \partial^0 K$ we associate a locally Lipschitz numerical flux function $q_{K,e^0} : \mathbb{R}^2 \to \mathbb{R}$ satisfying certain assumptions (consistency, conservation and monotonicity properties). Moreover, the averaged values of ω are given by

$$\int_{e_K^+} \omega dV_e = \omega_{e_K^+}, \quad \int_{e_K^-} \omega dV_e = \omega_{e_K^-}, \quad \int_{e^0} \omega dV_{e^0} = \omega_{e^0}, \quad \int_{\partial^0 K} \omega dV_{\partial^0 K} = \omega_{\partial^0 K}.$$

Hence, in view of the above approximation formulas we may write the finite volume method of interest, as a discrete approximation of

$$\omega_{e_K^+} |e_K^+| \overline{T}_{e_K^+}^0(v_K^+) = \omega_{e_K^-} |e_K^-| \overline{T}_{e_K^-}^0(v_K^-) - \sum_{e^0 \in \partial^0 K} |e^0| \omega_{e^0} q_{K,e^0}(v_K^-, v_{K_{e^0}}^-) + \omega_{\partial^0 K} |\partial^0 K| \, \overline{S}_{\partial^0 K}(v_K^-). \tag{4.29}$$

Moreover, for the stability of this scheme, we impose the following CFL condition

$$\frac{\Delta x^0}{\Delta x^1} \sup_v \left| \frac{(T^1)'(v)}{(T^0)'(v)} \right| < 1. \tag{4.30}$$

4.4.2 Formulation in local coordinates

We choose local coordinate on M. Then on this coordinate we choose a triangulation cartesian such that we divide space into equally spaced cells $J_j = (x^1_{j-1/2}, x^1_{j+1/2})$ of size Δx^1, centered at x^1_j and consider each element $K \in M$ so that it is composed of space-time cells, i.e.

$$K_{i,j} := I_i \times J_j = (x^0_i, x^0_{i+1}) \times (x^1_{j-1/2}, x^1_{j+1/2}).$$

Then the finite volume approximation in local coordinates will be of the form

$$\omega_{i,j}\,\overline{T}_{i+1,j} = \omega_{i,j}\,\overline{T}_{i,j} - \lambda\Big(\omega_{i,j+1/2}\,q_{i,j+1/2} - \omega_{i,j-1/2}\,q_{i,j-1/2}\Big) + \omega_{K_{i,j}}|\Delta x^0|\,\overline{S}_{i,j} \tag{4.31}$$

where $\lambda := \Delta x^0/\Delta x^1$, $\overline{T}_{i,j} := \overline{T}(v_{i,j}) = \overline{T}(v^i_j)$ and $\overline{S}_{i,j} := \overline{S}(v_{i,j}) = \overline{S}(v^i_j)$.

4.4.3 Numerical experiments

We finalize this study with numerical results. We consider the relativistic version of the scalar Burgers equation (4.8)–(4.9). From the proof of the main theorem we recall that the flux functions can be rewritten after a change of variable $u = \frac{1}{2\epsilon}\ln\left(\frac{1+\epsilon v}{1-\epsilon v}\right)$ by

$$\begin{aligned} T^0(v) &= T^0\Big(\phi(u)\Big) = \frac{1}{\epsilon}\sinh(\epsilon u), \\ T^1(v) &= T^1\Big(\phi(u)\Big) = \frac{1}{\epsilon^2}\Big(\cosh(\epsilon u) - 1\Big), \end{aligned}$$

where $u \in \mathbb{R}$. In the following numerical experiments, we will observe the convergence of the proposed scheme (4.31) for this model. We suppose that the source term vanishes, i.e. $S = 0$.

Moreover we will be interested in the choice of the smooth function

$$\omega(x^1) = x^1(x^1 - 2m),$$

where m represents a "mass" parameter.

Remarks for numerical tests

Let $u_0 = 0.9\ \sin(x)$ be the initial value. Note that $x = x^1$. Since $\omega = x(x-2m)$, we will examine our scheme with different values of mass m, and, in particular, we will be interested in behavior of the scheme for the interval $x > 2m$. One can observe by the following numerical experiments that the figures are similar in the mentioned interval for different values of m. It can be also observed that the points $x = 2m$ are the critical points containing shock waves. We first fix the interval $(0, 4\pi)$. We divide it in equal subintervals ΔX so that if we denote by J the total number of points in this interval, then each subinterval will be $\Delta X = 4\pi/\,J$. In the following tests we fix J = 100. Moreover, $N, \Delta T$ and CFL denote the number of iterations, the time difference between two iterations and the Courant-Friedrichs-Lewy number, respectively.

In the first three experiments (see Fig 4.1, 4.2, 4.3), we fix $\epsilon = 0.9$ for interval $(0, 4\pi)$ and we observe the scheme for different values of mass, ($m = 0, 1, 2$). Then in Fig 4.4–4.5, we fix $m = 1$ and we observe the difference between two figures by changing ϵ in the interval $(2m, 8\pi)$, that is, we are interested in the case that $x > 2m$. For instance, for $m = 1$ we have the interval $(2, 8\pi)$. We repeat this procedure for $m = 2, 3, 4$ (see Fig 4.6–4.10). One observes that after some number of iterations in the interval $x > 2m$, if we fix an ϵ, we have almost the same figures for different values of m, which shows the convergence of the proposed scheme.

Figure 4.1: $m = 0, \epsilon = 0.9$

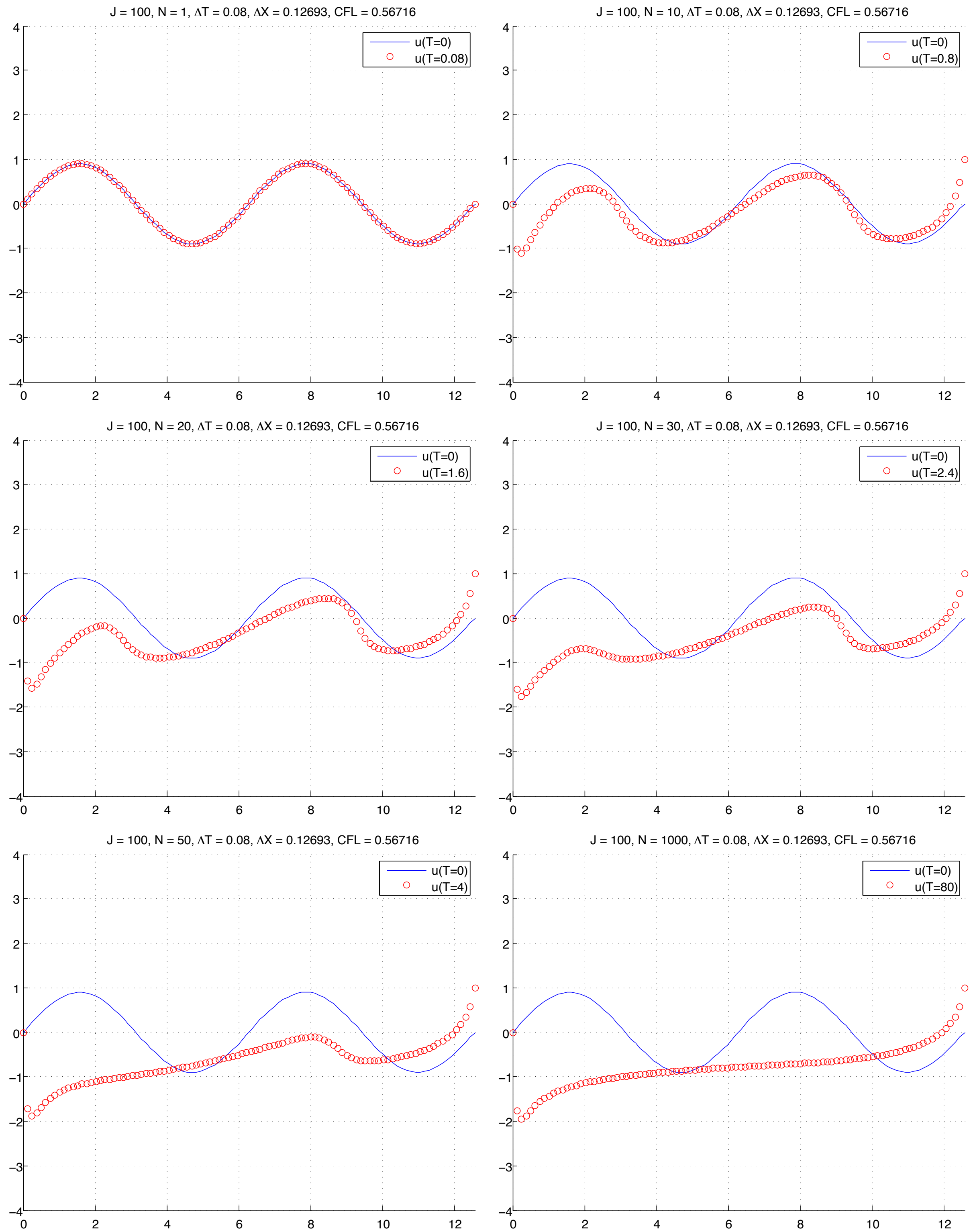


Figure 4.2: $m = 1, \epsilon = 0.9$

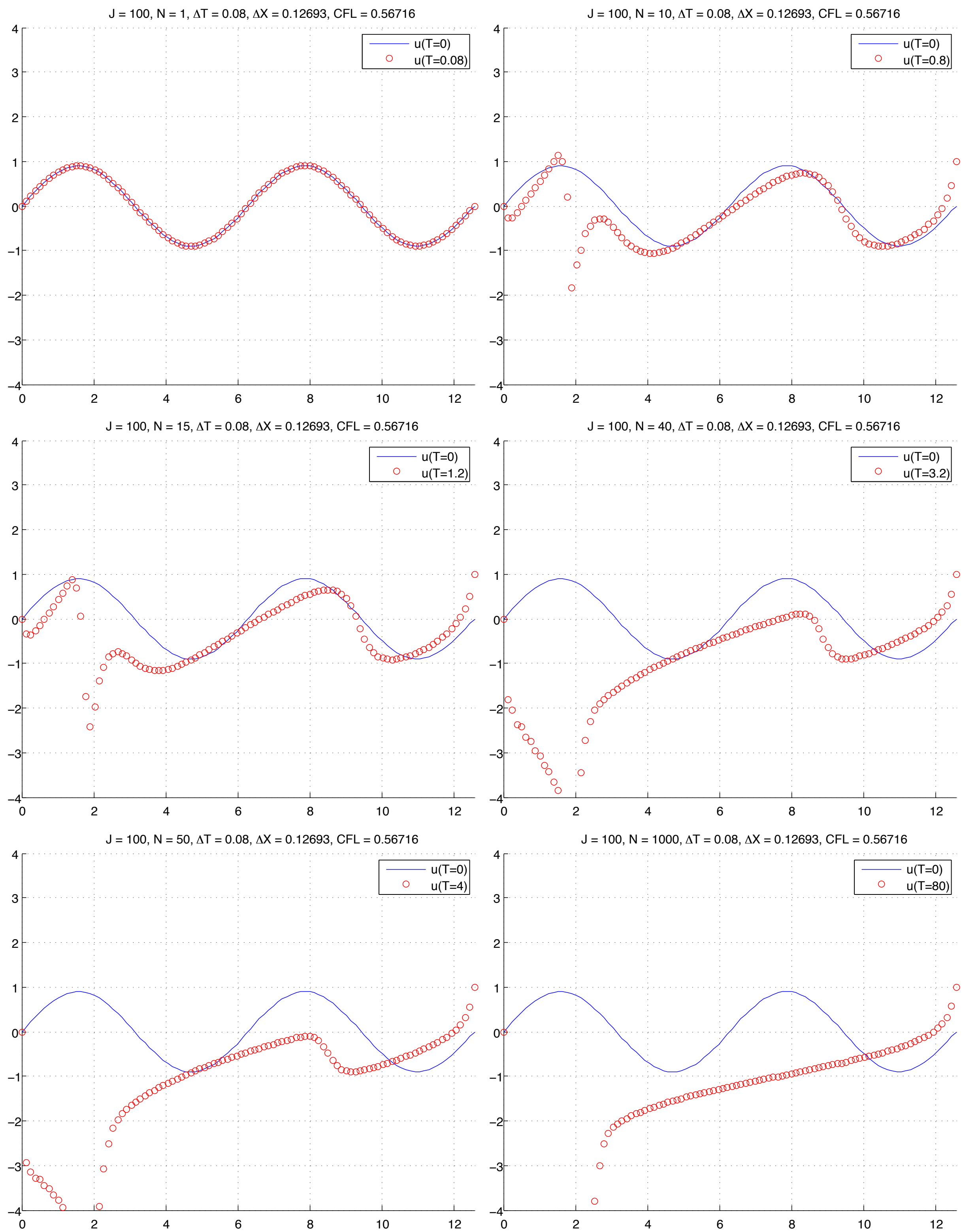


Figure 4.3: $m = 2$, $\epsilon = 0.9$

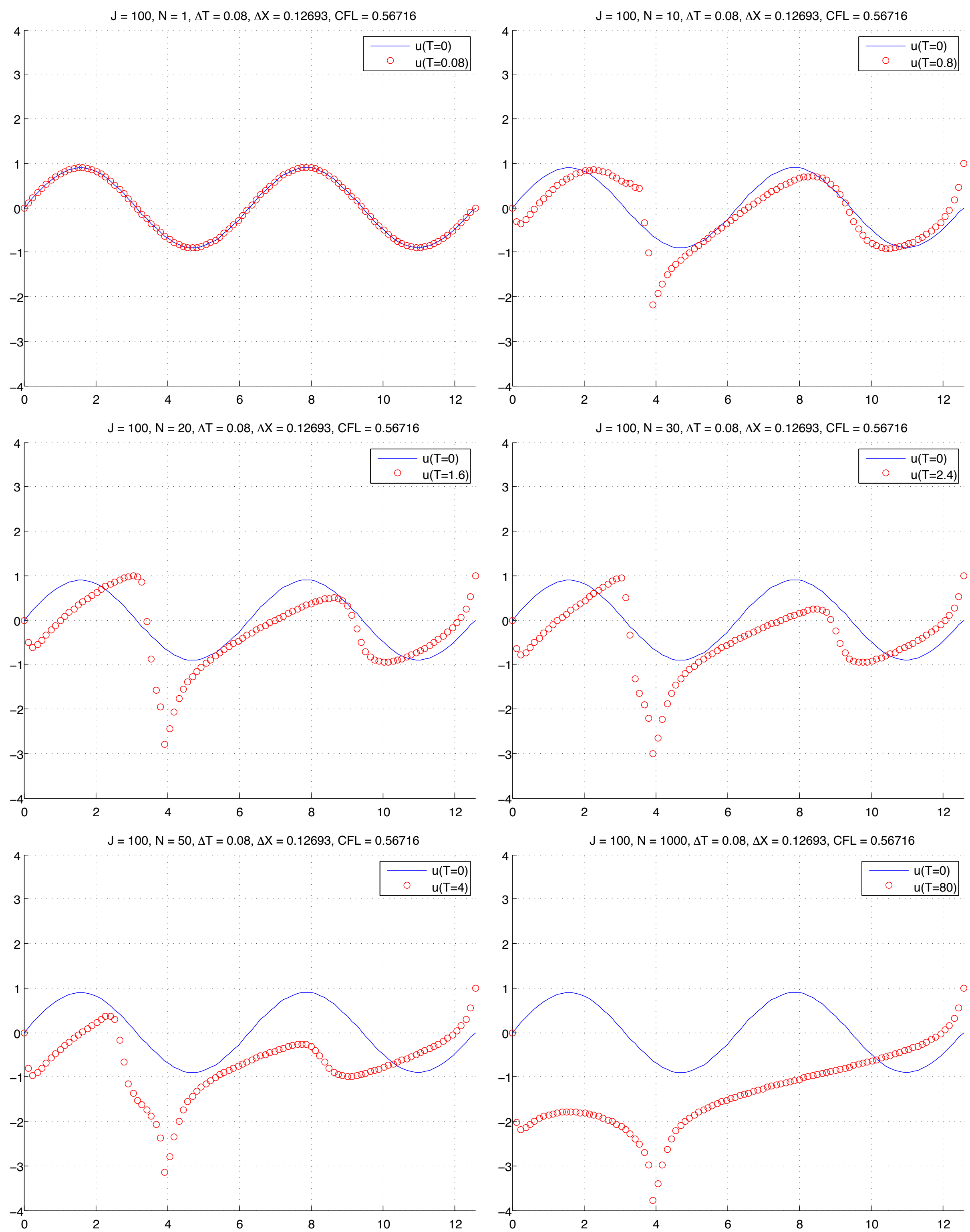


Figure 4.4: $m = 1$, $\epsilon = 0.9$, Interval: $x > 2$

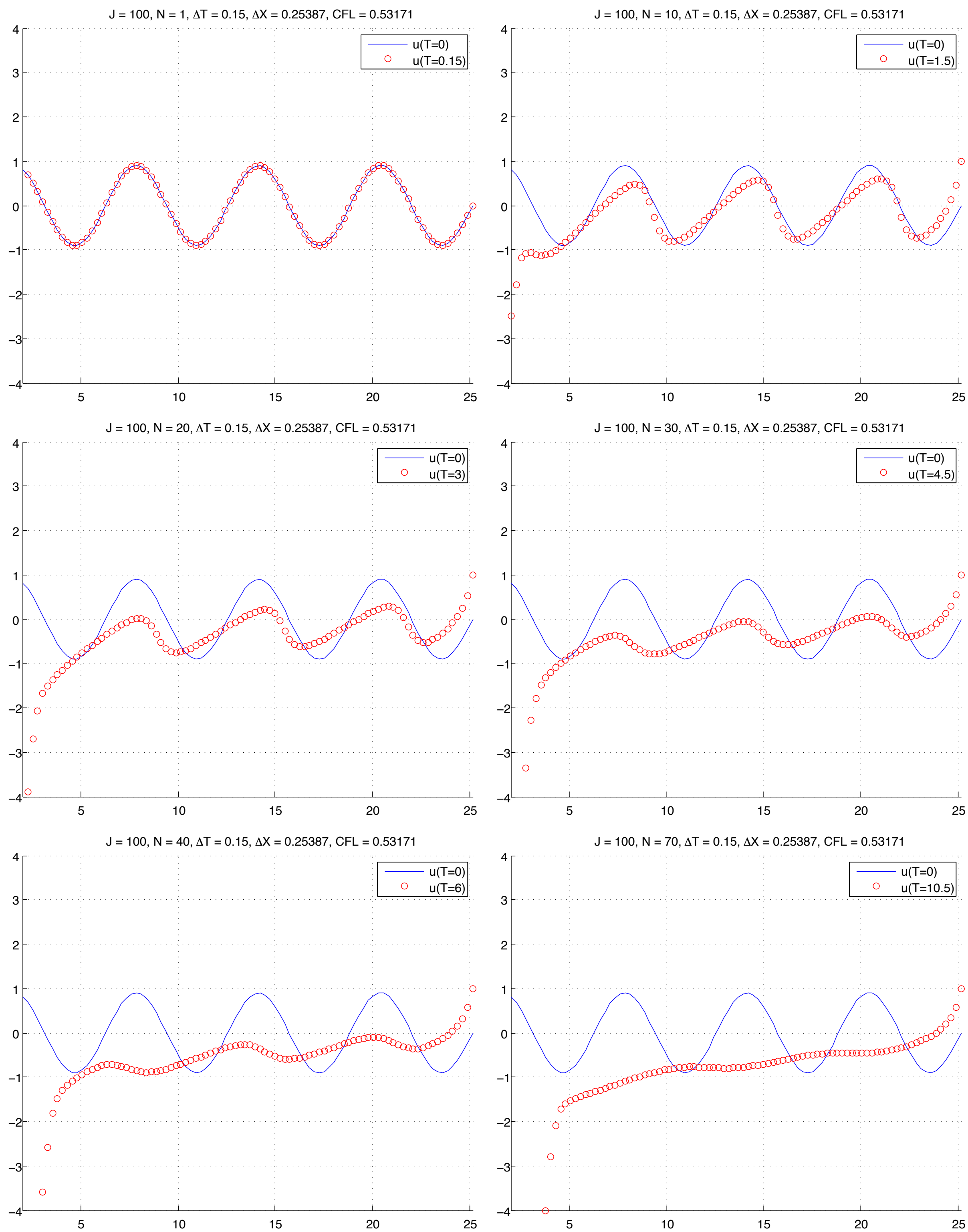


Figure 4.5: $m = 1$, $\epsilon = 0.5$, Interval: $x > 2$

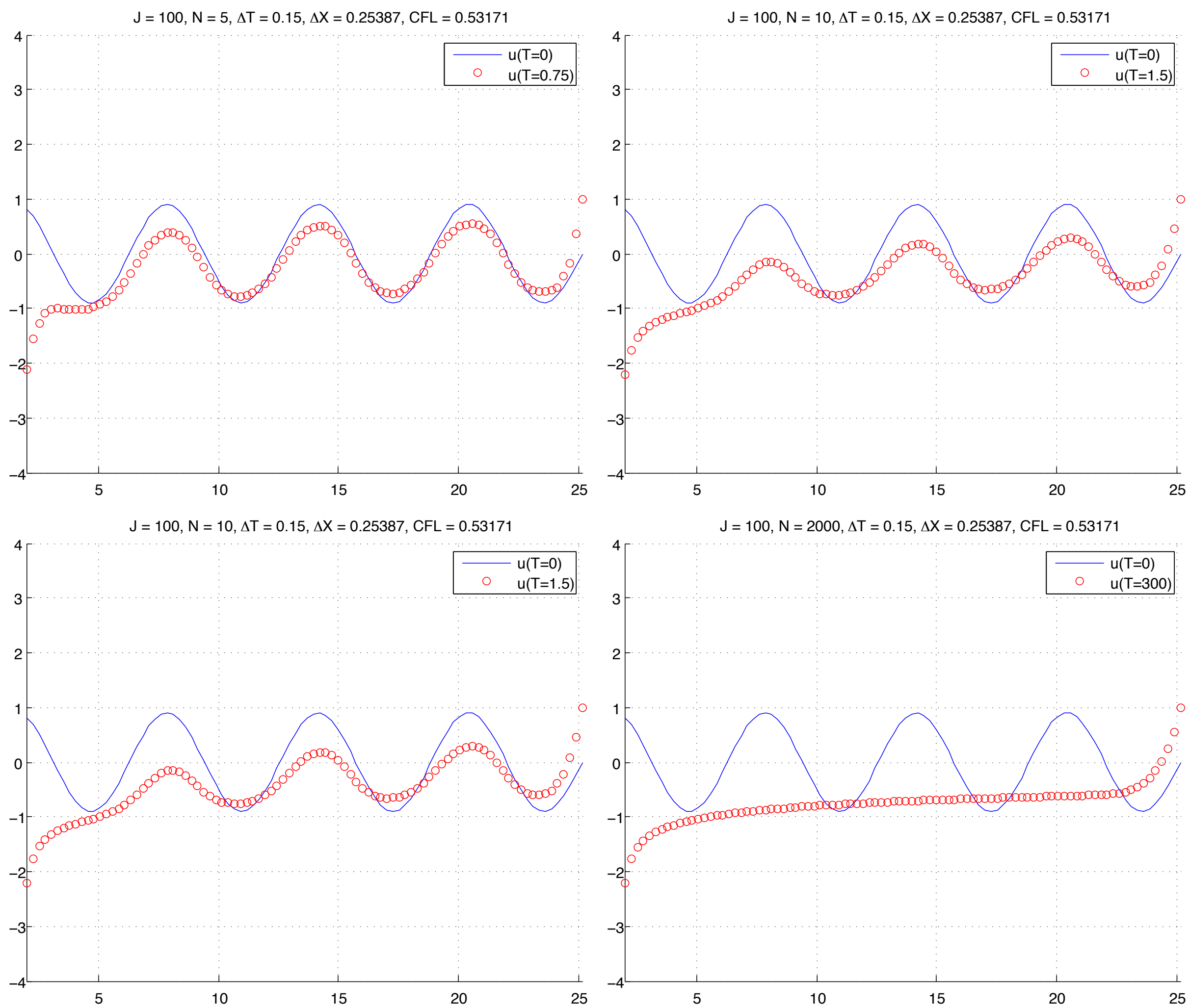


Figure 4.6: $m = 2$, $\epsilon = 0.9$, Interval: $x > 4$

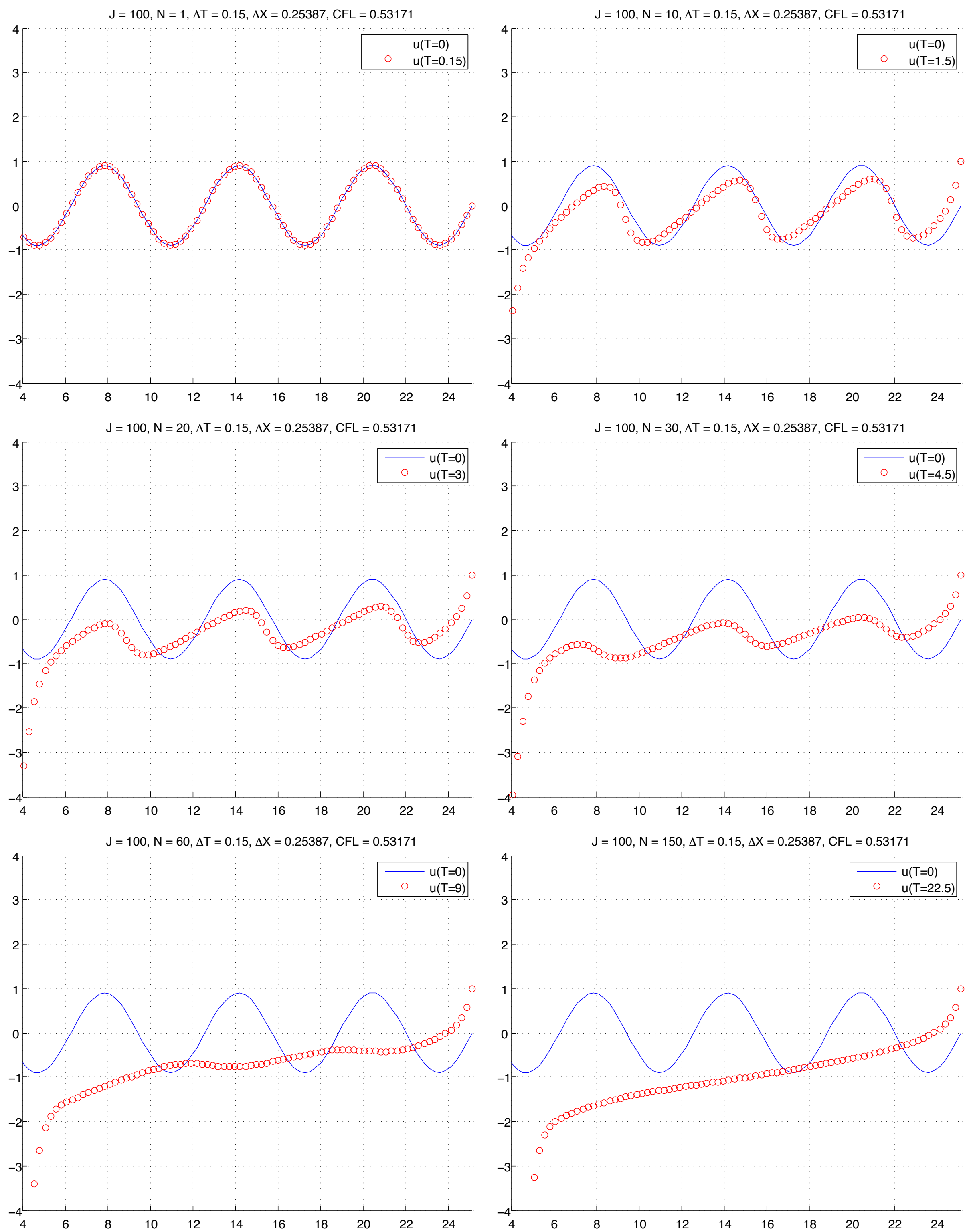


Figure 4.7: $m = 2$, $\epsilon = 0.5$, Interval: $x > 4$

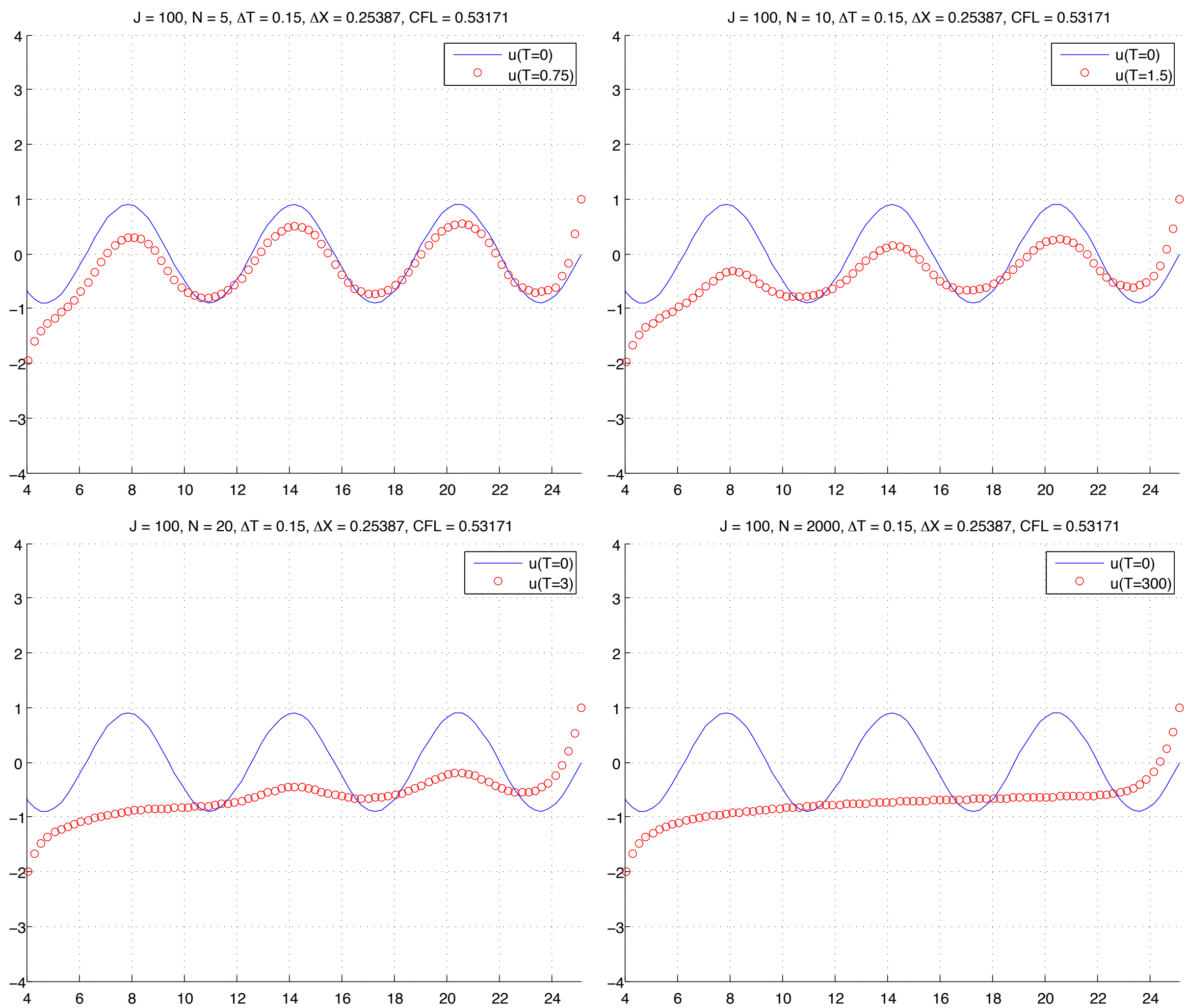


Figure 4.8: $m = 3$, $\epsilon = 0.9$, Interval: $x > 6$

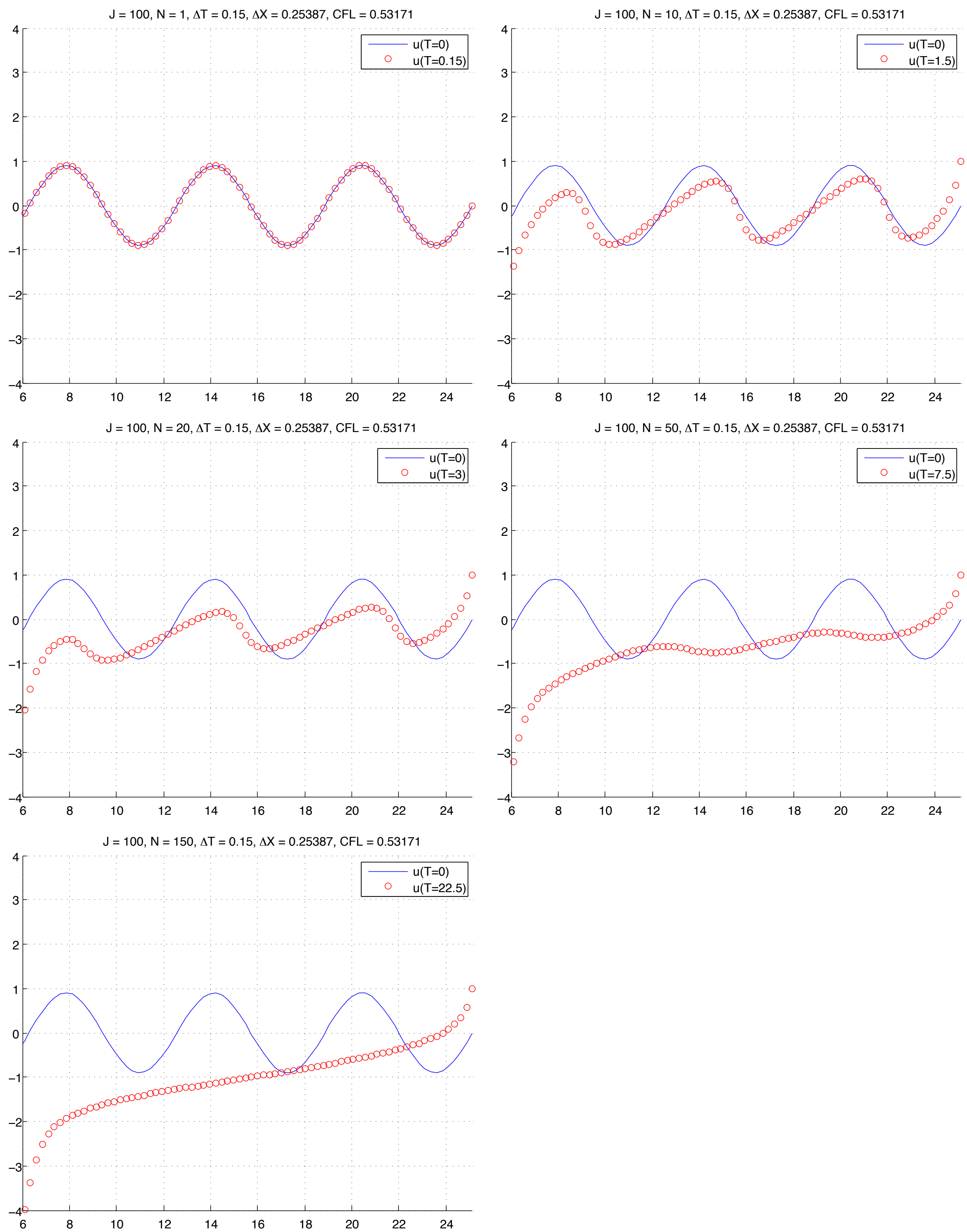


Figure 4.9: $m = 3$, $\epsilon = 0.5$, Interval: $x > 6$

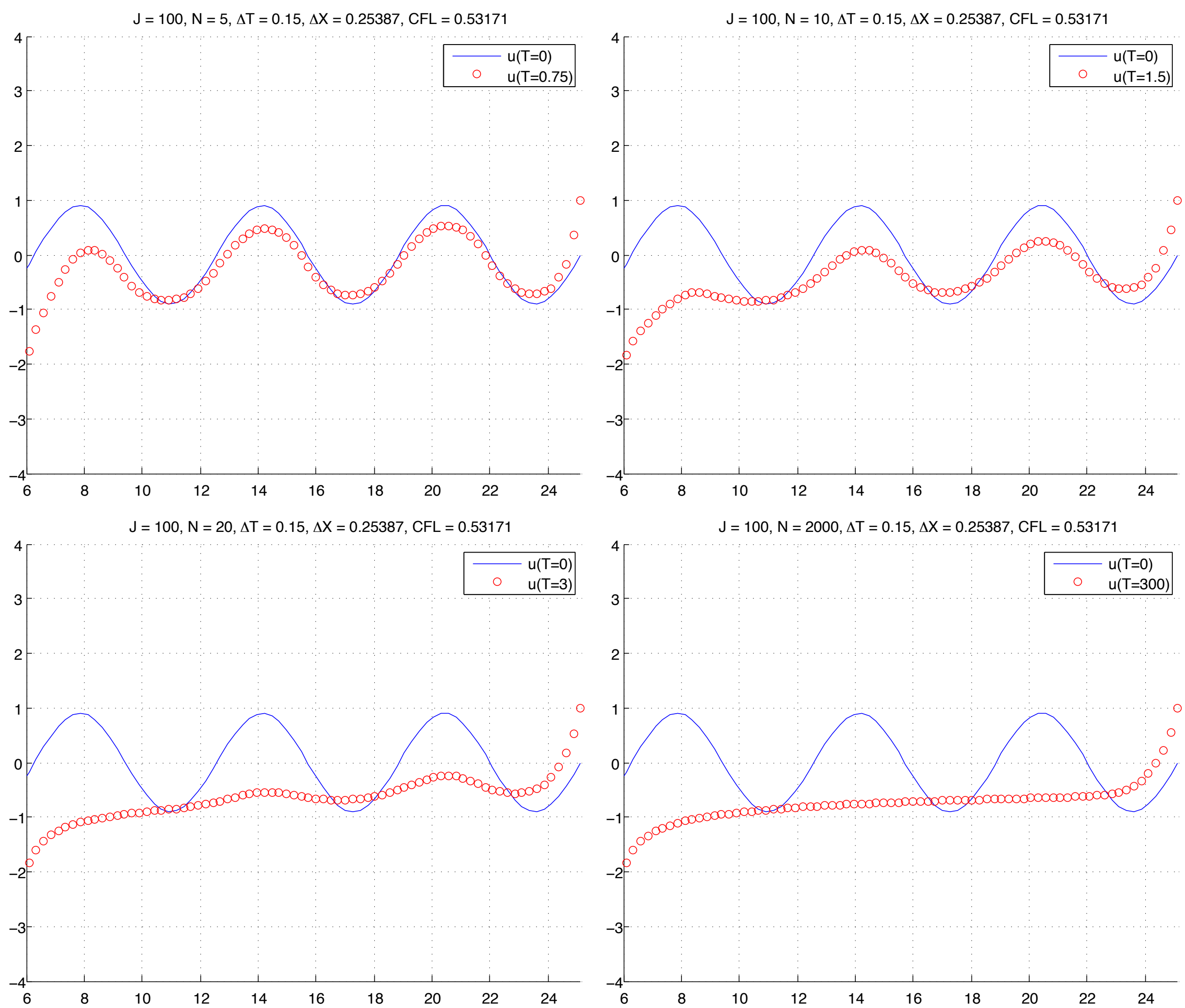
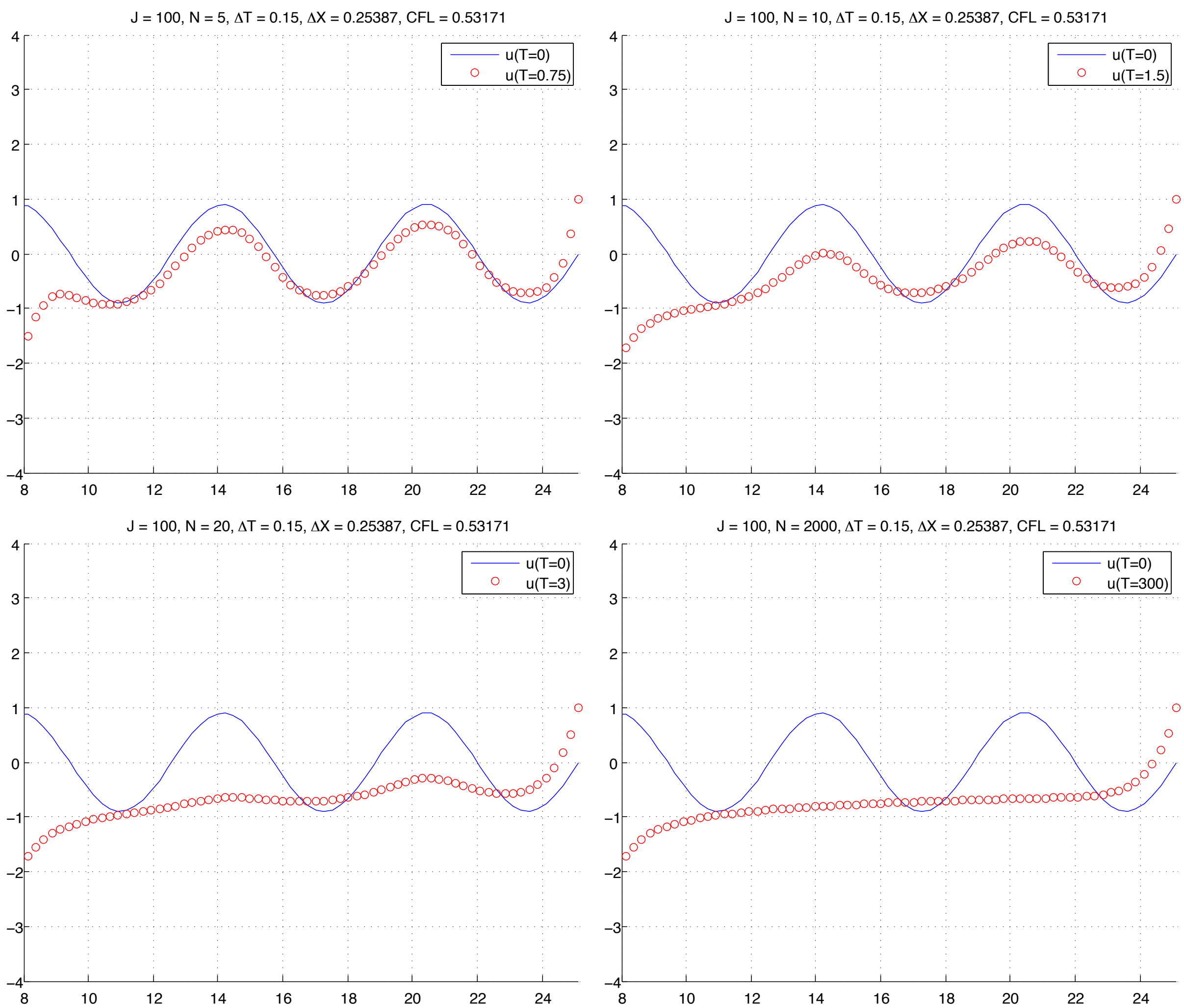


Figure 4.10: $m = 4$, $\epsilon = 0.5$, Interval: $x > 8$



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Résumé

La première partie de ce travail de thèse est consacrée à l'étude de la méthode de volumes finis pour les lois de conservation hyperboliques sur une variété. Nous étudions tout d'abord une première approche qui nécessite l'existence d'une métrique lorentzienne. Notre résultat principal établit la convergence de schémas de volumes finis du premier ordre pour une large classe de maillages. Ensuite, nous proposons une nouvelle approche basée sur des champs de formes différentielles. Dans ce travail, nous introduisons une nouvelle version de la méthode de volumes finis, qui requiert uniquement la structure de n-forme sur une variété de dimension $(n + 1)$.

La seconde partie porte sur les estimations d'erreur pour la méthode de volumes finis et sur la mise en oeuvre d'un modèle de fluides. Nous considérons tout d'abord les lois de conservation hyperboliques posées sur une variété riemannienne et nous établissons une estimation d'erreur en norme L^1 pour une classe de schémas de volumes finis pour l'approximation des solutions entropiques du problème de Cauchy. Nous étudions ensuite les équations hyperboliques posées sur un espace-temps courbe. En imposant que le flux vérifie une propriété naturelle d'invariance de Lorentz, nous identifions une loi de conservation unique à une normalisation près, qui peut être vue comme une version relativiste de l'équation classique de Burgers.

Abstract

The first part of this thesis is devoted to the study of finite volume methods for conservation laws on manifolds. We study first an approach based on a metric on Lorentzian manifolds. Our main result establishes the convergence of monotone and first-order finite volume schemes for a large class of (space and time) triangulations. Next, we consider another approach based on differential forms. We establish a new version of the finite volume methods which only requires the knowledge of family of n-volume form on an $(n + 1)$-manifold.

The second part is concerned with error estimates for finite volume methods and the implementation of a model of relativistic compressible fluids. We consider first nonlinear hyperbolic conservation laws posed on a Riemannian manifold, and we establish an L^1-error estimate for a class of finite volume schemes allowing for the approximation of entropy solutions to the initial value problem. Next, we consider the hyperbolic balance laws posed on a curved spacetime endowed with a volume form, and, after imposing a natural Lorentz invariance property we identify a unique balance law which can be viewed as a relativistic version of Burgers equation.