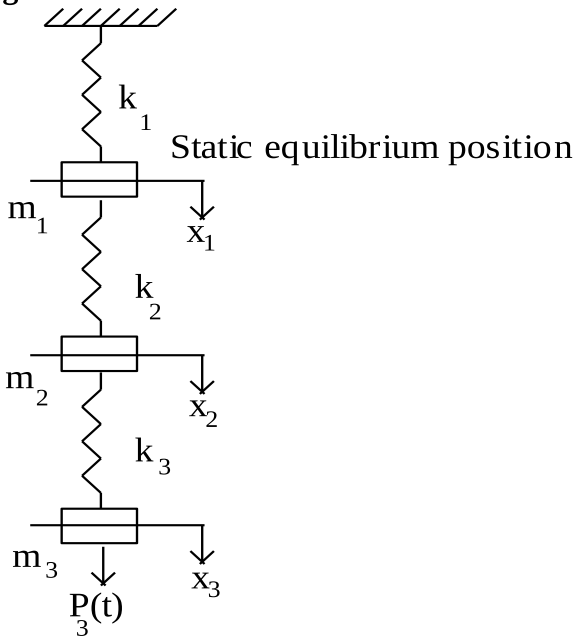
EIGENVALUES & IEGENVECTORS

Eigenvalue Problem:



$x_1 = x_1(t)$
$x_2 = x_2(t)$
$x_3 = x_3(t)$

$m\underset{\sim}{a} = \underset{\sim}{F}$ for each masses;

$a = \ddot{x} = \dfrac{d^2x}{dt^2}$ $\quad m_1\ddot{x}_1 = F_1$

$m_2\ddot{x}_2 = F_2$

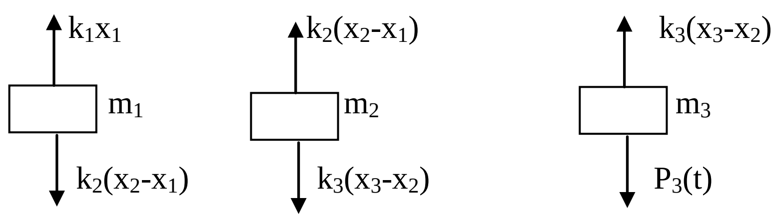
$m_3\ddot{x}_3 = F_3$

FBD of masses:

Elongation in spring 1 = x_1 $\quad F=kx$

Elongation in spring 2 = $x_2 - x_1$

Elongation in spring 3 = $x_3 - x_2$



$$m_1\ddot{x}_1 = k_2(x_2 - x_1) - k_1x_1 \quad \Rightarrow \quad m_1\ddot{x}_1 + (k_1 + k_2)x_1 - k_2x_2 = 0$$

$$m_2\ddot{x}_2 = k_3(x_3 - x_2) - k_2(x_2 - x_1) \quad \Rightarrow \quad m_2\ddot{x}_2 - k_2x_1 + (k_2 + k_3)x_2 - k_3x_3 = 0$$

$$m_3\ddot{x}_3 = P_3(t) - k_3(x_3 - x_2) \quad \Rightarrow \quad m_3\ddot{x}_3 - k_3x_2 + k_3x_3 = P_3(t)$$

$$\begin{bmatrix} m_1 & 0 & 0 \\ 0 & m_2 & 0 \\ 0 & 0 & m_3 \end{bmatrix}\begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \\ \ddot{x}_3 \end{bmatrix} + \begin{bmatrix} k_1 + k_2 & -k_2 & 0 \\ -k_2 & k_2 + k_3 & -k_3 \\ 0 & -k_3 & k_3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ P_3(t) \end{bmatrix}$$

$\underline{M}$	$\underline{\ddot{X}}$	$\underline{K}$	$\underline{X}$	$\underline{P}$
Mass Matrix	Acc. Matrix	Stifness Matrix	Displ. Matrix	Applied Force

$\underline{M}\ \underline{\ddot{X}} + \underline{K}\ \underline{X} = \underline{0}$ For free vibrations $\underline{P} = \underline{0}$

Assume the solution:

x= asin(ωt+φ) where a : amplitude

ω: angular frequency

φ : phase angle

$$\underline{X} = \underline{a}\sin(\omega t + \phi)$$
$$\underline{\dot{X}} = \underline{a}\,\omega\cos(\omega t + \phi)$$
$$\underline{\ddot{X}} = \underline{a}\,\omega^2 \sin(\omega t + \phi)$$

$$\underline{M} = \left[-\omega^2 \underline{a}\sin(\omega t + \phi)\right] + \underline{K}\left[\underline{a}\sin(\omega t + \phi)\right] = \underline{0}$$
$$\left[-\omega^2 \underline{M}\underline{a} + \underline{K}\underline{a}\right]\sin(\omega t + \phi) = \underline{0}$$
$$\sin(\omega t + \phi) = 0 \rightarrow \omega t + \phi = 2\pi \rightarrow \phi = 2\pi - \omega t$$
$$\underline{K}\underline{a} = \omega^2 \underline{M}\underline{a} \quad \text{let } \omega^2 = \lambda$$

$\boxed{\underline{K}\ \underline{a} = \lambda\ \underline{M}\ \underline{a}}$: General Eigenvalue Problem:

λ : Eigevalue

$\underline{a}$: Eigenvector

$\underline{K}\ \underline{a} = \lambda\ \underline{M}\ \underline{a}$ (*) GEVP

Premultiply (*) by $\underline{M}^{-1}$

$$\underline{M}^{-1}\underline{K}\underline{a} = \lambda\ \underline{a} \quad \text{let } \underline{A} = \underline{M}^{-1}\underline{K}$$

$\boxed{\underline{A}\underline{a} = \lambda\ \underline{a}}$ Normal eigenvalue problem (NEVP)

Solution of NEVP:

$$\underline{A}\ \underline{x} = \lambda\ \underline{I}\ \underline{x}$$

$(\underline{A} - \lambda\underline{I})\underline{x} = \underline{0} \rightarrow$ homogeneous equation

In order to have a non trivial solution, $\left|\underline{A} - \lambda\underline{I}\right| = 0$

Solution of GEVP:

$$\underline{A}\ \underline{x} = \lambda\ \underline{B}\ \underline{x}$$

$(\underline{A} - \lambda\underline{B})\underline{x} = \underline{0} \rightarrow$ homogeneous equation for x

In order to have a non trivial solution, $\left|\underline{A} - \lambda\underline{B}\right| = 0$

Steps to be followed in the solution of an eigenvalue problem.

1. To find the eigenvalues corresponding to $\underline{A}_{nxn}$

$$\det(\underline{A} - \lambda \underline{I}) = \underline{0} \text{ or } \det(\underline{A} - \lambda \underline{B}) = 0$$

where λ (i.e. $\lambda_1, \lambda_2, \ldots \lambda_n$) are eigenvalues

$$|\underline{A} - \lambda \underline{I}| = 0 \rightarrow P_n(\lambda): \text{ characteristic polynomial of } \underline{A}$$

$$P_n(\lambda) = 0 \text{ characteristic equation of } \underline{A}$$

$$\rightarrow \lambda_1, \lambda_2 \ldots\ldots \lambda_n$$

2.

$$\underline{A}\underline{x} = \lambda \underline{x} \qquad \underline{A}\underline{x}^{(1)} = \lambda_1 \underline{x}^{(1)}$$

$$\underline{A}\underline{x}^{(2)} = \lambda_2 \underline{x}^{(2)}$$

$$\bullet$$

$$\bullet$$

$$\underline{A}\underline{x}^{(n)} = \lambda_n \underline{x}^{(n)}$$

$$\boxed{\underline{A}\,\underline{x}^{(i)} = \lambda_i \underline{x}^{(i)}}$$

Example:

$$\underline{A} = \begin{bmatrix} 26 & -2 & 2 \\ 2 & 21 & 4 \\ 4 & 2 & 28 \end{bmatrix} \quad |\underline{A} - \lambda \underline{I}| = 0 \quad \lambda \underline{I} = \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix}$$

$$\begin{vmatrix} 26-\lambda & -2 & 2 \\ 2 & 21-\lambda & 4 \\ 4 & 2 & 28-\lambda \end{vmatrix} = 0 \Rightarrow \quad (\lambda^3 - 75\lambda^2 + 1850 - 15000) = 0$$

$$= (\lambda - 20)(\lambda - 25)(\lambda - 30) = 0$$

$$\left.\begin{array}{l} \lambda_1 = 20 \\ \lambda_2 = 25 \\ \lambda_3 = 30 \end{array}\right\} \lambda_1 \langle \lambda_2 \langle \lambda_3$$

$$\underline{A}\underline{x}^{(1)} = 20\ \underline{x}^{(1)}$$

$$\underline{A}\underline{x}^{(2)} = 25\ \underline{x}^{(2)}$$

$$\underline{A}\underline{x}^{(3)} = 30\ \underline{x}^{(3)}$$

$$\begin{bmatrix} 26 & -2 & 2 \\ 2 & 21 & 4 \\ 4 & 2 & 28 \end{bmatrix} \begin{bmatrix} x_1^{(1)} \\ x_2^{(1)} \\ x_3^{(1)} \end{bmatrix} = 20 \begin{bmatrix} x_1^{(1)} \\ x_2^{(1)} \\ x_3^{(1)} \end{bmatrix}$$

$$\begin{bmatrix} 26-20 & -2 & 2 \\ 2 & 21-20 & 4 \\ 4 & 2 & 28-20 \end{bmatrix} \begin{bmatrix} x_1^{(1)} \\ x_2^{(1)} \\ x_3^{(1)} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{aligned} &6x_1 - 2x_2 + 2x_3 = 0 \\ &\left.\begin{aligned} 2x_1 + x_2 + 4x_3 = 0 \\ 4x_1 + 2x_2 + 8x_3 = 0 \end{aligned}\right\} \textit{dependent} \end{aligned}$$

$$\text{Take an arbitrary } x_3 = \alpha \qquad \left.\begin{aligned} &6x_1 - 2x_2 = -2\alpha \\ &2x_1 + x_2 = -4\alpha \\ &x_2 = -2\alpha \quad \text{and} \quad x_1 = -\alpha \end{aligned}\right\} \rightarrow \underline{x}^{(1)} = \begin{bmatrix} -\alpha \\ -2\alpha \\ \alpha \end{bmatrix}$$

$$\text{or By GEM: } \begin{bmatrix} 6 & -2 & 2 \\ 2 & 1 & 4 \\ 4 & 2 & 8 \end{bmatrix} \approx \begin{bmatrix} 6 & -2 & 2 \\ 0 & 5/3 & 10/3 \\ 0 & 10/3 & 20/3 \end{bmatrix} \approx \begin{bmatrix} 6 & -2 & 2 \\ 0 & 5/3 & 10/3 \\ 0 & 0 & 0 \end{bmatrix} \rightarrow \underline{x}^{(1)} = \begin{bmatrix} -\alpha \\ -2\alpha \\ \alpha \end{bmatrix}$$

Example GEVP

$$\underline{C} = \begin{bmatrix} 0 & -1 \\ -1 & 2 \end{bmatrix} \quad \underline{B} = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \quad \underline{C}\ \underline{X} = \lambda\ \underline{B}\ \underline{X}$$

$$(\underline{C} - \lambda\ \underline{B})\underline{X} = \underline{0} \rightarrow \det(\underline{C} - \lambda\ \underline{B}) = 0$$

$$\lambda\ \underline{B} = \begin{bmatrix} 2\lambda & \lambda \\ \lambda & 2\lambda \end{bmatrix} \quad \underline{C} - \lambda\ \underline{B} = \begin{bmatrix} (-2\lambda) & (-1-\lambda) \\ (-1-\lambda) & (2-2\lambda) \end{bmatrix}$$

$$\begin{aligned} \det(\underline{C} - \lambda\ \underline{B}) &= -2\lambda(2-2\lambda) - (-1-\lambda)(-1-\lambda) \\ &= -4\lambda + 4\lambda^2 - 1 - 2\lambda - \lambda^2 = 3\lambda^2 - 6\lambda - 1 \end{aligned}$$

λ_1 =-0.1547

λ_2 =2.1547

$$\underline{C}\ \underline{X}^{(1)} = \lambda_1 \underline{B}\,\underline{X}^{(1)}$$

$$\begin{bmatrix} 0 & -1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} x_1^{(1)} \\ x_2^{(1)} \end{bmatrix} = -0.15 \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} x_1^{(1)} \\ x_2^{(1)} \end{bmatrix}$$

$$\left.\begin{aligned} -x_2^{(1)} = -0.30\,x_1^{(1)} - 0.15\,x_2^{(1)} \qquad & :0.30\,x_1 - 0.85\,x_2 = 0 \\ x_1^{(1)} + 2\,x_2^{(1)} = -0.15\,x_1^{(1)} - 0.30\,x_2^{(1)} \qquad & :-0.85\,x_1 + 2.30\,x_2 = 0 \end{aligned}\right\} \textit{dependent}$$

let $x_1 = \alpha$

$$x_2 = -\frac{0.85}{02.30}\alpha = -0.37\alpha \qquad \underline{x}^{(1)} = \begin{bmatrix} \alpha \\ -0.37\alpha \end{bmatrix}$$

$\alpha^2 + 0.137\,\alpha^2 = 1 \qquad \alpha = \pm 0.94$

$\underline{x}^{(1)} = \pm 0.94 \begin{bmatrix} 1 \\ -0.37 \end{bmatrix} \rightarrow$ normalized eigenvector. (Normalization wrt length of a vector.)

$$\underline{C}\ \underline{X}^{(2)} = \lambda_2 \underline{B}\,\underline{X}^{(2)}$$

$$\begin{bmatrix} 0 & -1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} x_1^{(2)} \\ x_2^{(2)} \end{bmatrix} = 2.15 \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} x_1^{(2)} \\ x_2^{(2)} \end{bmatrix}$$

$$-x_2^{(2)} = 4.30x_1^{(2)} + 2.15x_2^{(2)}$$
$$-x_1^{(2)} + 2x_2^{(2)} = 2.15x_1^{(2)} + 4.30x_2^{(2)}$$

$$\left.\begin{aligned} 4.30\,x_1 + 3.15\,x_2 = 0 \\ 3.15\,x_1 + 2.30\,x_2 = 0 \end{aligned}\right\} \textit{dependent}$$

let $x_1 = \alpha$

$$x_2 = -\frac{3.15}{4.30}\alpha = -0.73\alpha \qquad \underline{x}^{(2)} = \begin{bmatrix} \alpha \\ -0.73\alpha \end{bmatrix}$$

$\alpha^2 + 0.53\alpha^2 = 1 \qquad \alpha = \pm 0.81$

$\underline{x}^{(2)} = \pm 0.81 \begin{bmatrix} 1 \\ -0.73 \end{bmatrix} \rightarrow$ normalized eigenvector. (Normalization wrt length of a vector.)

Properties of eigenvalues and eigenvectors:

Def 1: A matrix $\underline{A}$ is said to be positive definite if $\underline{x}^T \underline{A}\, \underline{x} > \underline{0}$ for an arbitrary vector $\underline{x}$.

Remark : if components of a vector can be written as: $x_1^2 + x_2^2 + x_3^2 = 1 \Rightarrow$ positive definite in quadratic forms. Unit vectors are always positive definite quadratic forms.

Def 2: Eigenvectors are said to be orthogonal if they satisfy $\underline{x}^{(i)T} \underline{x}^{(j)} = \underline{0}$ for $i \neq j$.

Def 3: Eigenvectors are said to be orthonormal (mutually perpendicular and unit vectors); if they satisfy: $\underline{x}^{(i)T} \underline{x}^{(j)} = \delta_{ij}$

Def 4: Eigenvectors are said to be orthogonal wrt. a matrix $\underline{C}$, if they satisfy:

$\underline{x}^{(i)T} \underline{C}\, \underline{x}^{(j)} = \underline{0} \quad i \neq j$

Theorem Given a square matrix $\underline{A}$ which is:

- Real
- Symmetric
- Positive definite

Then:

i) The eigenvalues of $\underline{A}$ are real

ii) The eigenvectors are orthogonal

iii) The eigenvalues of $\underline{A}$ are positive

Diagonal form of a matrix:

Def : Modal Matrix $\underline{Q} = \left[\underline{x}^{(1)} \quad \underline{x}^{(2)} \ldots\ldots \quad \underline{x}^{(n)}\right]$ $\quad x^{(i)}$: are eigenvectors

If $\underline{D}$ is the digonal form of $\underline{A}$, then,

$$\underline{D} = \underline{Q}^{-1}\underline{A}\underline{Q} = \begin{bmatrix} \lambda_1 & 0 & 0 & 0 \\ 0 & \lambda_2 & 0 & 0 \\ . & & & \\ & . & . & . \\ . & & & \\ 0 & 0 & 0 & \lambda_n \end{bmatrix}$$

Example :

$$\underline{A}=\begin{bmatrix}10 & -6 & 0\\ -6 & 10 & 0\\ 0 & 0 & 1\end{bmatrix}\quad \begin{matrix}\lambda_1=1\\ \lambda_2=4\\ \lambda_3=16\end{matrix}\quad \underline{x}^{(1)}=\begin{bmatrix}0\\0\\1\end{bmatrix}\quad \underline{x}^{(2)}=\begin{bmatrix}1/\sqrt{2}\\1/\sqrt{2}\\0\end{bmatrix}\quad \underline{x}^{(3)}=\begin{bmatrix}1/\sqrt{2}\\-1/\sqrt{2}\\0\end{bmatrix}$$

$$\underline{Q}=\begin{bmatrix}0 & 1/\sqrt{2} & 1/\sqrt{2}\\ 0 & 1/\sqrt{2} & -1/\sqrt{2}\\ 1 & 0 & 0\end{bmatrix}\quad \underline{Q}^{-1}=\begin{bmatrix}0 & 0 & 1\\ 1/\sqrt{2} & 1/\sqrt{2} & 0\\ 1/\sqrt{2} & -1/\sqrt{2} & 0\end{bmatrix}$$

$$\underline{D}=\begin{bmatrix}0 & 0 & 1\\ 1/\sqrt{2} & 1/\sqrt{2} & 0\\ 1/\sqrt{2} & -1/\sqrt{2} & 0\end{bmatrix}\begin{bmatrix}10 & -6 & 0\\ -6 & 10 & 0\\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix}0 & 1/\sqrt{2} & 1/\sqrt{2}\\ 0 & 1/\sqrt{2} & -1/\sqrt{2}\\ 1 & 0 & 0\end{bmatrix}=\begin{bmatrix}1 & 0 & 0\\ 0 & 4 & 0\\ 0 & 0 & 16\end{bmatrix}$$

Theorem : If $\underline{A}$ is a symmetric matrix, then the modal matrix $\underline{Q}$ is orthogonal.

$$\underline{Q}^{-1}=\underline{Q}^{T}$$

Square root of a matrix:

let $\underline{D}$ be a diagonal matrix:

$$\underline{D}=\begin{bmatrix}d_{11} & 0 & 0 & \dots & 0\\ 0 & d_{22} & & & \\ 0 & & d_{33} & & \\ . & & & . & \\ . & & & . & \\ 0 & & & & d_{nn}\end{bmatrix}$$

$$\underline{D}^2=\underline{D}\underline{D}=\begin{bmatrix}d_{11} & 0 & 0 & \dots & 0\\ 0 & d_{22} & & & \\ 0 & & d_{33} & & \\ . & & & . & \\ . & & & . & \\ 0 & & & & d_{nn}\end{bmatrix}\begin{bmatrix}d_{11} & 0 & 0 & \dots & 0\\ 0 & d_{22} & & & \\ 0 & & d_{33} & & \\ . & & & . & \\ . & & & . & \\ 0 & & & & d_{nn}\end{bmatrix}=\begin{bmatrix}d_{11}^2 & 0 & 0 & \dots & 0\\ 0 & d_{22}^2 & & & \\ 0 & & d_{33}^2 & & \\ . & & & . & \\ . & & & . & \\ 0 & & & & d_{nn}^2\end{bmatrix}$$

then, $\sqrt{D} = \begin{bmatrix} \sqrt{d_{11}} & 0 & 0 & \cdots & 0 \\ 0 & \sqrt{d_{22}} & & & \\ 0 & & \sqrt{d_{33}} & & \\ . & & & . & \\ . & & & . & \\ 0 & & & & \sqrt{d_{nn}} \end{bmatrix}$

for any matrix $\underline{A}$, $\underline{A}^{1/2} = \underline{Q}\ \underline{D}^{1/2}\underline{Q}^{-1}$ $\quad \underline{A}^{1/2}\ \underline{A}^{1/2} = \underline{A}$

In general $\underline{A}^{1/n} = \underline{Q}\ \underline{D}^{1/n}\underline{Q}^{-1}$ or $\underline{A}^{n} = \underline{Q}\ \underline{D}^{n}\underline{Q}^{-1}$

Example:

$$\underline{A} = \begin{bmatrix} 10 & -6 & 0 \\ -6 & 10 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad \underline{Q} = \begin{bmatrix} 0 & 1/\sqrt{2} & 1/\sqrt{2} \\ 0 & 1/\sqrt{2} & -1/\sqrt{2} \\ 1 & 0 & 0 \end{bmatrix}$$

$$\underline{Q}^{-1} = \underline{Q}^{T} \qquad \underline{D} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 16 \end{bmatrix}$$

$$\underline{A}^{1/2} = \underline{Q}\ \underline{D}^{1/2}\underline{Q}^{-1}$$

$$\underline{A}^{1/2} = \begin{bmatrix} 0 & 1/\sqrt{2} & 1/\sqrt{2} \\ 0 & 1/\sqrt{2} & -1/\sqrt{2} \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 4 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \\ 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 1/\sqrt{2} & -1/\sqrt{2} & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 3 & -1 & 0 \\ -1 & 3 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

<u>Note :</u> trace of $\underline{A} \equiv tr(\underline{A}) = \sum_{i=1}^{n} a_{ij} = a_{11} + a_{22} + \ldots\ldots\ldots\ldots + a_{nn}$