METU
Department of Mathematics

Math 405
Combinatorics

Lecture Notes
Chapter II

Special Numbers

Ali Doğanaksoy

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1. BINOMIAL COEFFICIENTS

Binomial coefficients are one of the most common and useful family of real numbers which arise in many areas of mathematics, especially in combinatorics. A binomial coefficient has two indices, and the coefficient indexed by r, k is denoted usually by $\binom{r}{k}$ read as 'r choose k' . There are many ways to introduce the binomial coefficients. The one which explains the naming is that, a *binomial* is an algebraic expression that contains two terms, for example, $x + y$, where x and y are real numbers. Then the *binomial coefficient* $\binom{r}{k}$ is the coefficient of $x^k y^{r-k}$ when we expand $(x + y)^r$. Due to the simple and symmetric form of $(x + y)^r$, it is not surprising to face with this expression in almost all areas of mathematics. There is a combinatorial property of binomial coefficients which dominates all other properties: when the indices are nonnegative integers and $k \leq n$, the binomial coefficient $\binom{n}{k}$ is equal to the number of k element subsets of an n element set. This is the reason why we read $\binom{r}{k}$ as 'r choose k' even when r is not an integer and there is nothing to choose. Again for this reason, in many books this property is taken as definition of binomial coefficients.

Some special cases of the binomial theorem were known from ancient times. For example, Euclid mentioned the binomial theorem for $(x + y)^2$. In 6th century, $(x + y)^3$was considered in India.

In fact, by the 6th century, the Hindu mathematicians probably knew how to compute binomial coeffficients, and a clear statement of this rule can be found in the 12th century text by Bhaskara. The binomial theorem as such can be found in the work of 11th-century Arabian mathematician Al-Karaji, who described the triangular pattern of the binomial coefficients. He also provided a mathematical proof of both the binomial theorem and Pascal's triangle, using a primitive form of mathematical induction. The problem was also discussed by the Omar Khayyam, who was probably familiar with the formula to higher orders. The term "binomial coefficient" was introduced in 1544 by Michael Stifeland and the notation $\binom{n}{k}$ was introduced by Andreas von Ettingshausen in 1826. Some alternative notations are $C(n,\ k)$, ${}^{n}C_{k}$, ${}_{n}C_{k}$, C_k^n, C_n^k, $C_{n,k}$.

For a nonzero $r \in \mathbb{R}$ and a positive integer k, the ***binomial coefficient*** $\binom{r}{k}$ is the coefficient of x^k in the power series representation of $(1+x)^r$:

$$(1+x)^r = \sum_{k=0}^{\infty} \binom{r}{k} x^k$$

where $|x| < 1$.

Since the k-th derivative of $(1+x)^r$ is $r(r-1)\cdots(r-k+1)(1+x)^{r-k}$, we obtain $(1+x)^r = \sum_{k=0}^{\infty} \frac{d^{(k)}}{dx}(1+x)^r\Big|_{x=0} \frac{x^k}{k!} = \sum_{k=0}^{\infty} r(r-1)\cdots(r-k+1)\frac{x^k}{k!}$ and consequently

$$\binom{r}{k} = \frac{r^{\underline{k}}}{k!} = \frac{r(r-1)\cdots(r-k+1)}{k!}.$$

By convention we take $\binom{r}{0} = 1$ for any $r \in \mathbb{R}$ and $\binom{0}{k} = 0$ for any positive integer k.

Example 1. **a)** $\binom{r}{1} = r$,

b) $\binom{r}{2} = \frac{r(r-1)}{2}$,

c) $\binom{5}{3} = \frac{5\cdot4\cdot3}{6} = 10$,

d) $\binom{\frac{1}{2}}{k} = \frac{1}{k!}\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)\cdots\left(-\frac{2k-3}{2}\right) = \frac{(-1)^{k-1}}{2^k k!}\cdot\frac{(2k-2)!}{2\cdot4\cdots(2k-2)} = \frac{(-1)^{k-1}}{2^{2k-1}k!}\cdot\frac{(2k-2)!}{(k-1)!} = \frac{(-1)^{k-1}}{2^{2k-1}k}\cdot\binom{2k-2}{k-1}$,

e) $\binom{-3}{4} = \frac{(-3)(-4)(-5)(-6)}{4!} = 15$.

Binomial Coefficients and Number of Subsets

If n is a positive integer and $k \le n$, then the expression for $\binom{n}{k}$ can be written as

$$\binom{n}{k} = \frac{n(n-1)\cdots(n-k+1)}{k!}\cdot\frac{(n-k)!}{(n-k)!} = \frac{n!}{k!\,(n-k)!}.$$

It is seen that for $n \in \mathbb{Z}^+$ and $k \le n$, the binomial coefficient $\binom{n}{k}$ agrees with $C(n,k)$, the number of k-subsets of an n-set. For this reason, in some sources binomial coefficients are defined to be the number of k-subsets and the notation $\binom{n}{k}$ is used to mean $C(n,k)$.

$\binom{n}{k}$ and $C(n,k)$ are two functions which act with the same rule on different domains. In fact $C(n,k)$ is a restriction of the function $\binom{n}{k}$ to a smaller set. It follows then, a property or identity proved to be true for $\binom{n}{k}$ is necessarily true for $C(n,k)$. But a proof given for some property of $C(n,k)$ is required to be extended to $\binom{n}{k}$, by checking whether the property is valid for non-integer values of n.

Combinatorial Proofs and Polynomial Argument

In dealing with counting issues, there are two common classes of proofs: ***algebraic proofs*** and ***combinatorial proofs***. An algebraic proof of a property is achieved by transforming the expressions with the aid of substitutions and arithmetic operations. A combinatorial proof (or bijective proof or double counting) is achieved by showing that both sides of the equality count the same thing. In general, a combinatorial proof considers integer arguments whereas an algebraic proof respects all possible points in the domain.

A property of $C(n,k)$ which is proved to be true algebraically, holds for $\binom{n}{k}$ (n no more restricted to integers) as well. If a property is proved combinatorically to be true for $C(n,k)$, in order to extend it to $\binom{n}{k}$, we commonly use an important propery which is known as the ***polynomial argument***. This argument goes like that: if two polynomials of degree at most d agree at $d+1$ distinct points, then they agree everywhere. It follows that if an equality, whose both sides are polynomials, is proved to be true by a combinatorial proof, then by polynomial argument it holds everywhere.

For example, from the expressions $C(n,k)=\frac{n!}{k!(n-k)!}$ and $C(n,n-k)=\frac{n!}{(n-k)!k!}$ it follows that $C(n,k)=C(n,n-k)$. This is an algebraic proof which automatically covers the case $\binom{n}{k}=\binom{n}{n-k}$ as well. On the other hand, we can say that $C(n,k)$ counts all subsets with k elements and $C(n,n-k)$ counts complements of these subsets. Since each subset has a unique complement, $C(n,k)=C(n-k)$. This is a combinatorial proof and it guarantees the equality only to hold only for integer values of n, thus it does not directly implies that $\binom{n}{k}=\binom{n}{n-k}$. But, $C(n,k)$ are $C(n,n-k)$ are both polynomial expressions which agree on integers, then by polynomial argument they agree everywhere, thus $\binom{r}{k}=\binom{r}{r-k}$ for any $r\in\mathbb{R}$.

For integers n and k, if we set $C(n,k)=0$ for $k<0$ or $k>n$, then the binomial theorem $(a+b)^n=\sum_{k=0}^{n}\binom{n}{k}a^k b^{n-k}$ where $a,b\in\mathbb{R}$ can be written as

$$(a+b)^n=\sum_{k=0}^{\infty}C(n,k)\,a^k b^{n-k}.$$

LEMMA 1.1. For any real number r and a nonnegative integer k,

$$\binom{-r}{k} = (-1)^k \binom{r+k-1}{k}.$$

Proof. A straightforward computation yields

$$\begin{aligned}\binom{-r}{k} &= \frac{1}{k!}\cdot(-r)(-r-1)\cdots(-r-k+1)\\ &= (-1)^k\frac{1}{k!}(r+k-1)\cdots(r+1)r\end{aligned}$$

which gives the desired result. ∎

The property

$$\binom{r}{k} = (-1)^k \binom{-r+k-1}{k} \tag{1}$$

is known as ***upper negation.***

We observe an interesting relation between the binomial ceofficients and the number of k-subsets. For a set with n elements, recall that $\binom{n+k-1}{k}$ is the number of ways of choosing k elements of X if repetitions are allowed. Then we see that $(-1)^k\binom{-n}{k}$ and $\binom{n}{k}$ denote respectively, the number of ways of choosing k elements if repetitions are allowed and not allowed, respectively.

Using the identity $\binom{-r}{k} = (-1)^k\binom{r+k-1}{k}$, for any $r \in \mathbb{R}$, we obtain the power series representation of $\frac{1}{(1-x)^r}$ as

$$\begin{aligned}\frac{1}{(1-x)^r} &= (1-x)^{-r}\\ &= \sum_{k=0}^{\infty}\binom{-r}{k}(-x)^k\\ &= \sum_{k=0}^{\infty}\binom{r+k-1}{k}x^k.\end{aligned}$$

It follows that $\frac{x^n}{(1-x)^{n+1}} = \sum_{k=0}^{\infty}\binom{n+k}{k}x^{k+n}$. Since $\binom{n+k}{k} = \binom{n+k}{n}$ and $\binom{m}{n} = 0$ for integers $m < n$, we can write the power series representation of $\frac{x^n}{(1-x)^{n+1}}$ as

$$\frac{x^n}{(1-x)^{n+1}} = \sum_{k=0}^{\infty}\binom{k}{n}x^k. \tag{2}$$

Basic Properties

Let X be set with n elements. It has only one subset which contains no elements, namely the empty set and it has only one subset which has n elements, the set X itself. This means that $C(n,1) = C(n,n)$=1 and by polynomial argument

$$\binom{r}{0} = \binom{r}{r} = 1, \qquad r \in \mathbb{R}.$$

For each element x, X has a unique subset with one element (the subset which includes only x) and a unique subset with $n-1$ (the subset which excludes only x). Then $C(n,1) = C(n,n-1) = n$ and by polynomial argument

$$\binom{r}{1} = \binom{r}{r-1} = r, \qquad r \in \mathbb{R}.$$

$C(n,k)$ and $C(n,n-k)$ count respectively, the subset with k elements and subset $n-k$ elements. If we associate each subset with its complement, to each subset with k elements there corresponds a unique subset with $n-k$ elements. Then $C(n,k) = C(n,n-k)$ and by polynomial argument

$$\binom{r}{k} = \binom{r}{r-k}, \qquad r \in \mathbb{R}, k \in \mathbb{N}_0.$$

From the definition it follows that if n is a positive integer and $k > n$, then $\binom{n}{k} = 0$. That is

$$\binom{r}{k} = 0, \qquad r, k \in \mathbb{N}_0, \;\; k > r.$$

By convention we may define the binomial coefficient $\binom{r}{k}$ for negative integer k by setting $\binom{r}{k} = 0$:

$$\binom{r}{k} = 0, \qquad r \in \mathbb{R}, k \in \mathbb{N}^-.$$

The following theorem gathers the identities which are useful in computations which involve binomial coefficients.

THEOREM 1.2. *Let r be a real number, k and m are nonnegative integers. We have the following identities.*

$$\binom{r}{k} = \frac{r}{r-k}\binom{r-1}{k}, \tag{3}$$

$$\binom{r}{k} = \frac{r}{k}\binom{r-1}{k-1}, \tag{4}$$

$$\binom{r}{k} = \frac{r-k+1}{k}\binom{r}{k-1}, \tag{5}$$

$$\binom{r-1}{k} - \binom{r-1}{k-1} = \frac{r-2k}{r}\binom{r}{k}, \tag{6}$$

$$\binom{r-1}{k} + \binom{r-1}{k-1} = \binom{r}{k}, \tag{7}$$

$$\binom{r}{m}\binom{r-m}{k} = \binom{r}{k}\binom{r-k}{m}, \tag{8}$$

$$\binom{r}{m}\binom{m}{k} = \binom{r}{k}\binom{r-k}{m-k}. \tag{9}$$

Proof. All the identities can be obtained by simple algebraic manipulations. Identities (3), (4) and (5) are obtained by quite similar steps:

$$\binom{r}{k} = \frac{r!}{k!\,(r-k)!} = \frac{r}{r-k} \cdot \frac{(r-1)!}{k!\,(r-k-1)!} = \frac{r}{r-k}\binom{r-1}{k},$$

$$\binom{r}{k} = \frac{r!}{k!\,(r-k)!} = \frac{r}{k} \cdot \frac{(r-1)!}{(k-1)!\,(r-k)!} = \frac{r}{k}\binom{r-1}{k-1},$$

$$\binom{r}{k} = \frac{r!}{k!\,(r-k)!} = \frac{r-k+1}{k} \cdot \frac{r!}{(k-1)!\,(r-k+1)!} = \frac{r-k+1}{k}\binom{r}{k-1}.$$

The difference and sum of (3) and (4) give the identities (6) and (7):

$$\binom{r-1}{k} - \binom{r-1}{k-1} = \left(\frac{r-k}{r} - \frac{k}{r}\right)\binom{r}{k} = \frac{r-2k}{r}\binom{r}{k}$$

$$\binom{r-1}{k} + \binom{r-1}{k-1} = \binom{r}{k}.$$

(8) is obtained as follows:

$$\binom{r}{m}\binom{r-m}{k} = \frac{r!}{m!\,(r-m)!} \cdot \frac{(r-m)!}{k!\,(r-m-k)!}$$

$$= \frac{r!}{k!\,(r-k)!} \cdot \frac{(r-k)!}{m!\,(r-k-m!)}$$

$$= \binom{r}{k}\binom{r-k}{m}.$$

(9) can be proved as

$$\binom{r}{m}\binom{m}{k} = \frac{r!}{m!\,(r-m)!} \cdot \frac{m!}{k!\,(m-k)!}$$

$$= \frac{r!}{k!\,(r-k)!} \cdot \frac{(r-k)!}{(r-m)!\,(m-k)!}$$

$$= \binom{r}{k}\binom{r-k}{m-k}.$$

which is the desired result. ■

Identity (4) is known as ***absorption property***. Now we provide a combinatorial proof for this identity. First express (4) as

$$k\binom{r}{k} = r\binom{r-1}{k-1}.$$

The left hand side counts the number of pairs (x, S), where $S \subset \{1,2, \dots, r\}$ with k elements and $x \in S$. There are $\binom{r}{k}$ choices for S, and for each of these there are k choices for x. The right hand side counts the same thing in a different order. First choose x (there are r choices) and then choose the other $k-1$ elements among the remainig $r-1$ elements there are $\binom{r-1}{k-1}$ choices.

A combinatorial proof of identity (9) is as follows. In a group of n students, just m students are wearing hats and exactly k of these students have ribbons wrapped around their hats. Now consider the identitiy

$$\binom{r}{m}\binom{m}{k} = \binom{r}{k}\binom{r-k}{m-k}.$$

The left hand side counts the ways of first choosing m students to wear the hats and then choosing k of these, to wrap ribbons around their hats. Right hand side counts the ways of first choosing k students to wear hats with ribbons and then choose $m-k$ students out of remaining $r-k$ students to wear hats without ribbons.

Pascal's Identity and Pascal's Triangle

Identity (7) is particularly important and is known as ***Pascal's[1] identity.*** In this section we examine this identity in more details and we give a combinatorial proof for it.

Pascal's Identity is a useful theorem of combinatorics which is often used to simplify complicated expressions involving binomial coefficients. For a selection of $k \geq 1$ objects from the set $\{1,2,\ldots,n\}$ there are two choices. The selection contains n or does not contain it. A selection not containing n has to be composed of the remaining $n-1$ elements thus, it can be composed in $C(n-1,k)$ different ways. If the selection contains n, then we have to choose the remaining $k-1$ elements out of $n-1$ elements and for this choice we have $C(n-1,k-1)$ possibilities. Then we conclude $C(n,k) = C(n-1,k) + C(n-1,k-1)$, and by polynomial argument

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}.$$

Naturally, the identity makes sense for $1 \leq k < n$. It is seen that Pascal's identity is also the ***basic recursion*** for the binomial coefficients. Starting with the initial condition $\binom{0}{0} = 0$, this identity enables us to compute all binomial coefficients recursively. Pascal's triangle[2] is a way of listing binomial coefficients as below:

$$\begin{array}{c}
\binom{0}{0} \\
\binom{1}{0} \quad \binom{1}{1} \\
\binom{2}{0} \quad \binom{2}{1} \quad \binom{2}{2} \\
\binom{3}{0} \quad \binom{3}{1} \quad \binom{3}{2} \quad \binom{3}{3} \\
\binom{4}{0} \quad \binom{4}{1} \quad \binom{4}{2} \quad \binom{4}{3} \quad \binom{4}{4} \\
\binom{5}{0} \quad \binom{5}{1} \quad \binom{5}{2} \quad \binom{5}{3} \quad \binom{5}{4} \quad \binom{5}{5} \\
\binom{6}{0} \quad \binom{6}{1} \quad \binom{6}{2} \quad \binom{6}{3} \quad \binom{6}{4} \quad \binom{6}{5} \quad \binom{6}{6} \\
\binom{7}{0} \quad \binom{7}{1} \quad \binom{7}{2} \quad \binom{7}{3} \quad \binom{7}{4} \quad \binom{7}{5} \quad \binom{7}{6} \quad \binom{7}{7} \\
\binom{8}{0} \quad \binom{8}{1} \quad \binom{8}{2} \quad \binom{8}{3} \quad \binom{8}{4} \quad \binom{8}{5} \quad \binom{8}{6} \quad \binom{8}{7} \quad \binom{8}{8}
\end{array}$$

[1] *Blaise Pascal (1623-1662), French mathematician, physicist, writer and philosopher.*

[2] *Pascal's identity and triangle were known long before Pascal by Chinese scholar Jia Xian, six hundred years before Pascal. Work of Jia Xian has passed us by Yang Hui (1238-1298). In Iran the triangle is known as the Hayyam triangle, named after Ömer Hayyam (1048-1131).*

Pascal's identity implies that each term of the array (except the rightmost and leftmost terms at each row) is equal to the sum of two terms in the row above which lie above-left and above-right. A simple construction of the triangle proceeds in the following manner. In row 0, the top-most row, the entry is 1 (that is, $\binom{0}{0}$). Then, to construct the elements of the following rows, the elemets at leftmost or rightmost columns are always 1 ($\binom{n}{0}$ and $\binom{n}{0}$). For the remaining terms, add the number above-left with the number above-the right of the given position. First 8 rows of the Pascal's triangle are shown below.

1

1 1

1 2 1

1 3 3 1

1 4 6 4 1

1 5 10 10 5 1

1 6 15 20 15 6 1

1 7 21 35 **35** 21 7 1

1 8 28 56 70 56 28 8 1

Sums of Row, Column and Diagonal Entries

Row n of Pascal's triangle consists of the terms $\binom{n}{k}$, $k = 0, \dots, n$. Consequently, the sum of all terms in a row is the number of all subsets of a set with n terms, that is $\sum_{k=0}^{n} C(n,k) = 2^n$. Since the right hand side is not a polynomial we can not directly write $\sum_{k=0}^{r} \binom{r}{k} = 2^r$. But, using the power series expansion $(1+x)^r = \sum_{k=0}^{\infty} \binom{r}{k} x^k$ of $(1+x)^r$ with $x = 1$ we get

$$\sum_{k=0}^{\infty} \binom{r}{k} = 2^r. \tag{10}$$

In Pascal's triangle we consider some particular sequences

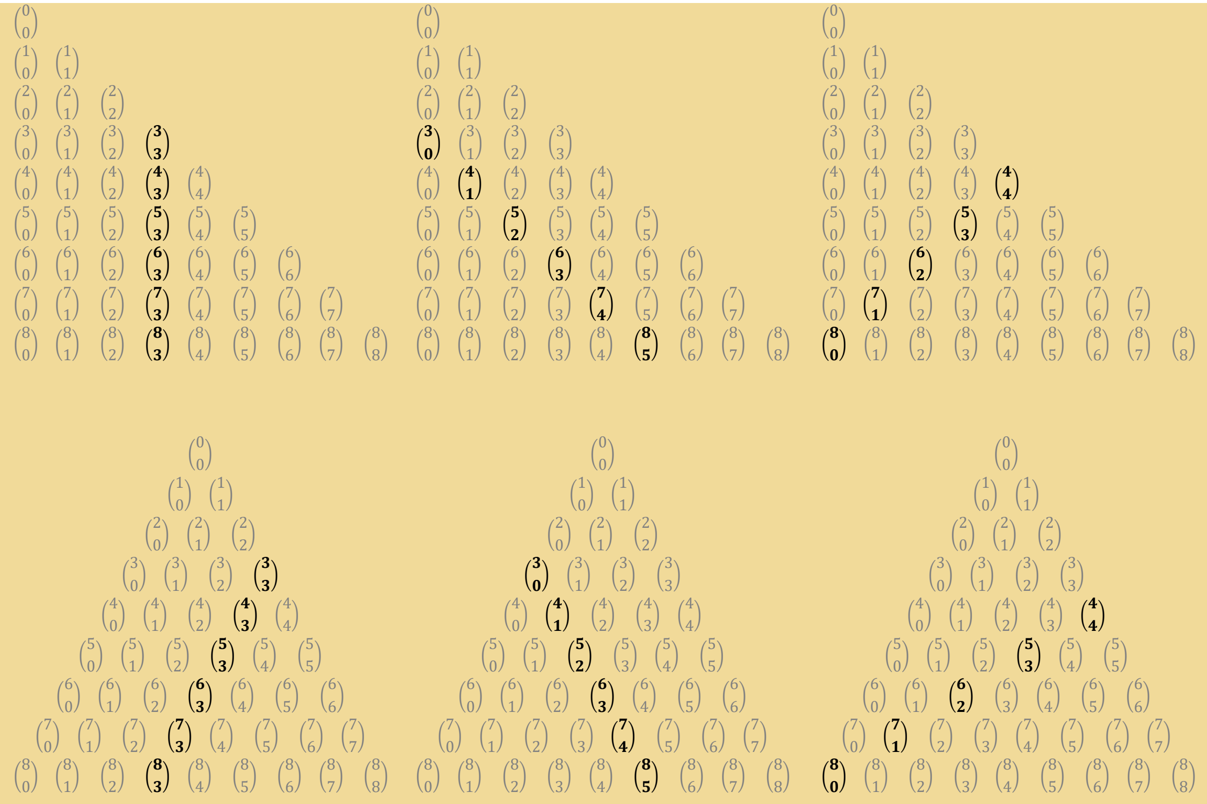
k-th column: $\binom{k}{k}, \binom{k+1}{k}, \binom{k+2}{k}, \dots$

k-th NW-SE diagonal: $\binom{k}{0}, \binom{k+1}{1}, \binom{k+2}{2}, \dots$

n-th NE-SW diagonal: $\binom{n}{0}, \dots, \binom{n-k}{k}, \dots, \binom{m}{t}$ (the last term is $\binom{n/2}{n/2}$ if n is even; $\binom{(n+1)/2}{(n-1)/2}$ otherwise).

Observe that the identity $\binom{k}{i}=\binom{k}{k-i}$ implies that the k-th column and k-th NW diagonal are composed of equal terms.

When we arrange Pascal's triangle in the following manner, it is more clear to understand why the sequence $\binom{k}{k},\binom{k+1}{k},\binom{k+2}{k},\ldots$ is called the k-th column.



3rd column   3rd NW-SE diagonal   8th NE-SW diagonal

The following theorem gives the sum of first k terms of the n −th column.

THEOREM 1.3 (*Upper Summation Formula*). *For any nonnegative integers n and k, the sum of first n terms of the k-th column is* $\binom{n+1}{k+1}$:

$$\sum_{i=0}^{n}\binom{i}{k}=\binom{n+1}{k+1}. \tag{11}$$

Proof. Since $\binom{0}{k}+\binom{1}{k}+\cdots+\binom{k-1}{k}=0$, we have to compute the sum $\binom{k}{k}+\binom{k+1}{k}+\binom{k+2}{k}+\cdots+\binom{n-1}{k}+\binom{n}{k}$. Replace $\binom{k}{k}$ with $\binom{k+1}{k+1}$. Then, by Pascal's identity, the sum of first two terms is $\binom{k+2}{k+1}$. Now the sum of first two terms of the resulting sequence is $\binom{k+3}{k+1}$. Continuing in this manner we reach the sum $\binom{n+1}{k+1}$.

An illustration of the proof is:

$$\underbrace{\underbrace{\underbrace{\underbrace{\underbrace{\underbrace{\binom{k+1}{k+1}+\binom{k+1}{k}}_{\binom{k+2}{k+1}}+\binom{k+2}{k}}_{\binom{k+3}{k+1}}+\binom{k+3}{k}}_{\binom{k+4}{k+1}}+\cdots}_{\ddots}+\binom{k+n-1}{k}}_{\binom{n}{k+1}}+\binom{k+n}{k}}_{\binom{n+1}{k+1}}.$$

Eventually $\binom{n+1}{k+1}$ is obtained. ■

Combinatorial Proof. The number of ways of distributing m candies to $k+1$ children is $\binom{m+k}{k}$. Now consider the problem of distributing *at most* m candies to $k+1$ children. This means that we distribute 0 or 1 or 2 … or m candies. Then the number of ways of distributing candies is $\binom{k}{k}+\binom{k+1}{k}+\cdots+\binom{m+k}{k}=\sum_{i=k}^{m+k}\binom{i}{k}$. On the other hand, when we distribute some of the candies, say $t \leq m$ candies, to $k+1$ children, there are $m-t$ candies left. Then we can consider a '$k+2$'nd child who is taking the remaining candies. So the number of ways of distributing *some* of m candies to $k+1$ children is same with the number of distributing *all* of m candies to $k+2$ children which is given by $\binom{\boldsymbol{m+k+1}}{\boldsymbol{k+1}}$. Then we conclude

$$\sum_{i=k}^{m+k}\binom{i}{k}=\binom{m+k+1}{k+1}$$

which gives the equality (11) after the substitution $n=m+k$. ■

COROLLARY 1.4 (*Parallel Summation Formula*). *For nonnegative integers n and k, the sum of first n terms of the k-th NW-SE diagonal is* $\binom{n+1}{k+1}$:

$$\sum_{i=0}^{n}\binom{k+i}{i}=\binom{n+1}{k+1}. \tag{12}$$

Proof. Since $\binom{k+i}{i}=\binom{k+i}{k}$, result follows directly from the previous theorem. ■

Denote the sum of n-th NE-SW diagonal entries by S_n

For n=0 we have $S_0=\binom{0}{0}=1$

For $n=1$ $\quad S_1=\binom{1}{0}=1$

For $n = 2$ $\quad S_2 = \binom{2}{0} + \binom{1}{0} = 2$

For $n = 3$ $\quad S_3 = \binom{3}{0} + \binom{2}{1} = 3$

For $n = 4$ $\quad S_4 = \binom{4}{0} + \binom{3}{1} + \binom{2}{2} = 5$

First terms of the sequence $\{S_n\}$ are 1,1,2,3,5, ... This resembles the Fibonacci sequence $\{\mathfrak{f}_n\} = 0, 1, 1, 2, 3, 5, \ldots$ each of whose terms, starting from the third, is the sum of preceding two terms.

THEOREM 1.5. *The sum of n-th NE-SW diagonal entries is the Fibonacci number* $\mathfrak{f}_{n+1}$.

Proof. From the computations above we have $S_0 = \mathfrak{f}_1, S_1 = \mathfrak{f}_2$. Now it is sufficient to show that the terms of the sequence $\{S_n\}$ satisfy the recursion $S_n = S_{n-1} + S_{n-2}$ for any integer $n \geq 3$. We have

$$\begin{aligned} S_n &= \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n-k}{k} \\ &= 1 + \sum_{k=1}^{\lfloor n/2 \rfloor} \binom{n-k-1}{k} + \sum_{k=1}^{\lfloor n/2 \rfloor} \binom{n-k-1}{k-1} \\ &= \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n-k-1}{k} + \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n-k-2}{k} \\ &= S_{n-1} + S_{n-2} \end{aligned}$$

which proves the assertion. ■

Finite Sums Involving Binomial Coefficients

In the previous section we have shown that the sum of the n th row terms of Pascal's triangle is 2^n, namely $\sum_{k=0}^{n} \binom{n}{k} = 2^n$. Unfortunately, we do not have any closed form to express the partial sum

$$\sum_{k=0}^{m} \binom{n}{k}.$$

Now consider the alternating partial sum $\sum_{k=0}^{n} (-1)^k \binom{n}{k}$. Let $n > 0$, then by substituting $x = 1$ in $(1-x)^n = \sum_{k=0}^{n} (-1)^k \binom{n}{k} x^k$, we immediately have

$$\sum_{k=0}^{n} (-1)^k \binom{n}{k} = 0. \tag{13}$$

Since $(-1)^k = 1$ for even k and $(-1)^k = -1$ for odd k, if $n > 0$, then (13) can be written as $\sum_{k \text{ even}}^{n} \binom{n}{k} - \sum_{k \text{ odd}}^{n} \binom{n}{k} = 0$.

Then $\sum_{k \text{ even}}^{n} \binom{n}{k} = \sum_{k \text{ odd}}^{n} \binom{n}{k} = 2^{n-1}$ or

$$\sum_{k=0}^{n} \binom{n}{2k} = 2^{n-1}$$

and

$$\sum_{k=0}^{n} \binom{n}{2k+1} = 2^{n-1}$$

which means that, for any nonempty set, the number of subsets with an odd number of elements is equal to the number of subsets with an even number of elements.

Moreover, for the partial sums we have:

$$\sum_{k=0}^{m} (-1)^k \binom{n}{k} = (-1)^m \binom{n-1}{m}.$$

The equality for partial sums can be obtained as follows:

$$\begin{aligned}\sum_{k=0}^{m} (-1)^k \binom{n}{k} &= 1 + \sum_{k=1}^{m} (-1)^k \binom{n-1}{k} + \sum_{k=1}^{m} (-1)^k \binom{n-1}{k-1} \\ &= \sum_{k=0}^{m-1} (-1)^k \binom{n-1}{k} + (-1)^m \binom{n-1}{m} - \sum_{k=0}^{m-1} (-1)^k \binom{n-1}{k} \\ &= (-1)^m \binom{n-1}{m}.\end{aligned}$$

The binomial formula $(1+x)^n = \sum_{k=0}^{n} \binom{n}{k} x^k$ is very useful to obtain interesting identities. For example, differentiating both sides we have $n(1+x)^{n-1} = \sum_{k=0}^{n} k \binom{n}{k} x^{k-1}$. By substituting $x = 1$ we get

$$\sum_{k=0}^{n} k \binom{n}{k} = n2^{n-1}. \tag{14}$$

Now we provide a combinatorial proof of (14). In a class with n students we wish to choose a committee and a chairperson for that committee. Right hand side counts the number of ways of choosing the chairman (n ways) and then among the remaining $n-1$ students, choosing other members of the committee consisting of any number of members, probably empty (2^{n-1} ways).

Left hand side counts the number of ways of first choosing some committee of k members ($\binom{n}{k}$ ways) then among choosing a chairperson among them (k ways). As k may take any value, $k = 0, \dots, n$, by summation we obtain the equality.

It should be noted that, a set with n elements has $\binom{n}{k}$ subsets ech of which contains k elements. Then the total number of elements in these subsets is $k\binom{n}{k}$. This observation leads us to an interpretation of (14). The sum on the left hand side is the total number of all elements in all subsets of the set. On the other hand, each particular element of the set appears in exactly 2^{n-1} subsets. It follows that the total number of all elements in all subsets is $n2^{n-1}$ which is the right hand side.

Since the set has 2^n subsets, the average number of elements of a subset is $\frac{n}{2}$.

Differentiating both sides of the binomial formula once more we get

$$n(n-1)(1+x)^{n-2} = \sum_{k=0}^{n} k(k-1)\binom{n}{k}x^{k-2}$$

which gives the identity $\sum_{k=0}^{n}(k^2-k)\binom{n}{k} = n(n-1)2^{n-2}$ for $x = 1$. Then we have

$$\sum_{k=0}^{n} k^2\binom{n}{k} = n(n+1)2^{n-2}. \tag{15}$$

Now for arbitrary $r, s \in \mathbb{R}$ we write

$$\begin{aligned}
\sum_{m=0}^{\infty}\binom{r}{m}x^m &= (1+x)^r \\
&= (1+x)^s(1+x)^{r-s} \\
&= \left(\sum_{n=0}^{\infty}\binom{s}{n}x^n\right)\left(\sum_{j=0}^{\infty}\binom{r-s}{j}x^j\right) \\
&= \sum_{n=0}^{\infty}\sum_{j=0}^{\infty}\binom{s}{n}\binom{r-s}{j}x^{n+j} \\
&= \sum_{m=0}^{\infty}\left(\sum_{k=0}^{m}\binom{s}{k}\binom{r-s}{m-k}\right)x^m.
\end{aligned}$$

Comparing the coefficients of x^m we obtain the identity which is called ***Vandermonde's convolution***:

$$\sum_{k=0}^{m}\binom{s}{k}\binom{r-s}{m-k} = \binom{r}{m}. \tag{16}$$

For a combinatorial proof of this equality, notice that the right hand side counts the number of ways to choose a group of m students from a class of r students. Assume that there are $s \geq m$ girls and $r - s$ boys at the class. The left hand side counts the same thing according to cases depending on the number of girls i on the committee, which can range from 0 to m. Since in such a case, there are $\binom{s}{k}$ ways to select the girls and $\binom{r-s}{m-k}$ ways to select the boys, the number of such committees is $\binom{s}{k}\binom{r-s}{m-k}$. Now the result follows.

When $s = 1$ Vandermonde's convolution gives Pascal's identity in the form

$$\binom{r}{k} = \binom{r-1}{k} + \binom{r-1}{k-1}.$$

For $s = 2$ we obtain the identity

$$\binom{r}{k} = \binom{r-2}{k} + 2\binom{r-2}{k-1} + \binom{r-2}{k-2}.$$

In Vandernomnde's convolution, by taking $r = 2n$ and $s = m = n$, for a positive integer n we obtain

$$\sum_{k=0}^{n} \binom{n}{k}^2 = \binom{2n}{n}. \tag{17}$$

Now let n, k be positive integers. Using (2) we write

$$\begin{aligned}
\sum_{i=0}^{\infty} \binom{i}{k+1} x^i &= \frac{x^{k+1}}{(1-x)^{k+2}} \\
&= x\left(\frac{x^j}{(1-x)^{j+1}}\right)\left(\frac{x^{k-j}}{(1-x)^{k-j+1}}\right) \\
&= x\left(\sum_{a=0}^{\infty} \binom{a}{j} x^a\right)\left(\sum_{b=0}^{\infty} \binom{b}{k-j} x^b\right) \\
&= \sum_{a=0}^{\infty}\sum_{b=0}^{\infty} \binom{a}{j}\binom{b}{k-j} x^{a+b+1} \\
&= \sum_{i=0}^{\infty}\left(\sum_{m=0}^{n} \binom{m}{j}\binom{i-m}{k-j}\right) x^{i+1}.
\end{aligned}$$

By comparing the coefficients of x^{n+1} we obtain

$$\sum_{m=0}^{n} \binom{m}{j}\binom{n-m}{k-j} = \binom{n+1}{k+1}. \tag{18}$$

Note that, taking $j = k$ in (18) gives (11) in the form $\sum_{m=0}^{n}\binom{m}{k} = \binom{n+1}{k+1}$.

To obtain a generalization of (14), we apply the binomial formula to $(2+x)^n$ in two different ways:

$$
\begin{aligned}
(2+x)^n &= \big(1+(1+x)\big)^n \\
&= \sum_{k=0}^{n} \binom{n}{k} (1+x)^k \\
&= \sum_{k=0}^{n} \binom{n}{k} \sum_{i=0}^{k} \binom{k}{i} x^i \\
&= \sum_{k=0}^{n} \sum_{i=0}^{k} \binom{n}{k} \binom{k}{i} x^i \\
&= \sum_{i=0}^{n} \sum_{k=i}^{n} \binom{n}{k} \binom{k}{i} x^i
\end{aligned}
$$

and

$$(2+x)^n = \sum_{i=0}^{n} \binom{n}{i} 2^{n-i} x^i.$$

By comparing the coefficients of x^i in these expressions, we have

$$\sum_{k=i}^{n} \binom{n}{k} \binom{k}{i} = \binom{n}{i} 2^{n-i} \tag{19}$$

for any nonnegative integers n and i. By substitutions $i = n - m$, and $k = n - i$ the identity (19) gives $\sum_{i=0}^{n} \binom{n}{i} \binom{n-i}{n-m} = \binom{n}{m} 2^m$ which can be rearranged to write

$$\sum_{k=0}^{n} \binom{n}{k} \binom{n-k}{m-k} = \binom{n}{m} 2^m. \tag{19'}$$

To have a combinatorial proof of this equality, we generalize the idea which is followed for the identity (14). Now our model is to choose a committee and i distinguished members of this committee. Rest of the proof is similar to that of (14).

BINOMIAL INVERSION FORMULA

In this section we develop a technique, known as binomial inversion which has many interesting applications .

LEMMA 1.6. *If n and $m \leq n$ are positive integers, then*

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}\binom{k}{m} = (-1)^m\delta_m^n.$$

Proof.

$$\begin{aligned} x^n &= \big(1+(x-1)\big)^n \\ &= \sum_{k=0}^{n}\binom{n}{k}(x-1)^k \\ &= \sum_{k=0}^{n}\binom{n}{k}\sum_{m=0}^{k}\binom{k}{m}(-1)^{k-m}x^m \\ &= \sum_{k=0}^{n}\sum_{m=0}^{k}(-1)^{k-m}\binom{n}{k}\binom{k}{m}x^m \\ &= \sum_{m=0}^{n}\left(\sum_{k=m}^{n}(-1)^{k-m}\binom{n}{k}\binom{k}{m}\right)x^m. \end{aligned}$$

Comparing the coefficients of x^m claim follows. ■

The ***transform*** of a given a sequence $\{f_n\}$, under ***binomial inversion*** is the sequence $\{g_n\}$ with

$$g_n = \sum_{k=0}^{n}(-1)^k\binom{n}{k}f_k.$$

THEOREM 1.7 (*Binomial Inversion Formula).* *The transform under binomial inversion is an involution.That is, if $\{f_n\}$ and $\{g_n\}$ are sequences such that*

$$g_n = \sum_{k=0}^{n}(-1)^k\binom{n}{k}f_k,$$

then

$$f_n = \sum_{k=0}^{n}(-1)^k\binom{n}{k}g_k.$$

Proof. We simply compute the transform of $\{g_n\}$:

$$\begin{aligned}\sum_{k=0}^{n}(-1)^k\binom{n}{k}g_k &= \sum_{k=0}^{n}(-1)^k\binom{n}{k}\sum_{i=0}^{k}(-1)^i\binom{k}{i}f_i\\ &= \sum_{k=0}^{n}(-1)^k\sum_{i=0}^{k}(-1)^i\binom{n}{k}\binom{k}{i}f_i\\ &= \sum_{i=0}^{n}(-1)^i f_i\sum_{k=i}^{n}(-1)^k\binom{n}{k}\binom{k}{i}\\ &= \sum_{i=0}^{n}f_i\delta_i^n\end{aligned}$$

The last term we have obtained is f_n. ■

If $\{g_n\}$ is the transform of $\{f_n\}$, we call these sequences a binomial inversion pair. An alternative version of the theorem is as follows.

THEOREM 1.8. *If $\{f_n\}$ and $\{g_n\}$ are sequences such that*

$$g_n = \sum_{k=0}^{n}\binom{n}{k}f_k,$$

then

$$f_n = \sum_{k=0}^{n}(-1)^{n-k}\binom{n}{k}g_k\,. \quad ■$$

Example 2. If we let $\{f_n\}$ to be the constant sequence $f_n = 1$ for $n = 0,1,\ldots$, then

$$\sum_{k=0}^{n}\binom{n}{k}f_k = \sum_{k=0}^{n}\binom{n}{k} = 2^n = g_n.$$

By binomial inversion we have

$$\sum_{k=0}^{n}(-1)^{n-k}2^k\binom{n}{k} = \sum_{k=0}^{n}(-1)^{n-k}\binom{n}{k}g_k = f_n$$

which gives

$$\sum_{k=0}^{n}(-2)^k\binom{n}{k} = (-1)^n.$$

Example 3. Let n and $i \leq n$ be positive integers. If $f_n = \binom{n}{i}$, then $\sum_{k=0}^{n}\binom{n}{k}f_k = \sum_{k=0}^{n}\binom{n}{k}\binom{k}{i} = \binom{n}{i}2^{n-i} = g_n$ and by binomial inversion

$$\begin{aligned}\binom{n}{i} &= f_n \\ &= \sum_{k=0}^{n}(-1)^{n-k}\binom{n}{k}\binom{k}{i}2^{k-i} \\ &= \sum_{k=0}^{n}(-1)^{n-k}\binom{n}{k}\binom{k}{i}2^{k-i} \\ &= (-1)^n 2^{-i}\sum_{k=0}^{n}(-2)^k\binom{n}{k}\binom{k}{i}.\end{aligned}$$

So we obtain

$$\sum_{k=0}^{n}(-2)^k\binom{n}{k}\binom{k}{i} = (-1)^n 2^{n-i}\binom{n}{i}.$$

Generating Functions

THEOREM 1.9. *For a fixed real number r, generating function of the sequence $\left\{\binom{r}{k}\right\}$ is*

$$\sum_{k=0}^{\infty}\binom{r}{k}x^k = (1+x)^r.$$

Proof. Assertion is just rephrasing the definition of binomial coefficients. ■

THEOREM 1.10. *For a fixed nonnegative k, generating function of the sequence $\left\{\binom{n}{k}\right\}$ is*

$$\sum_{n=0}^{\infty}\binom{n}{k}x^n = \frac{x^k}{(1-x)^{k+1}}.$$

Proof. Assertion follows from equation (2). ■

THEOREM 1.11. *A bivariate generating function of the binomial coefficients is*

$$\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\binom{n}{k}x^k y^n = \frac{1}{1-y-xy}.$$

Proof. Write the power series representation of $\frac{1}{1-y-xy}$:

$$\frac{1}{1-y(1+x)} = \sum_{n=0}^{\infty} y^n (1+x)^n$$
$$= \sum_{n=0}^{\infty} y^n \sum_{k=0}^{\infty} \binom{n}{k} x^k.$$

Then, claim follows. ■

THEOREM 1.12. *A bivariate generating function of the binomial coefficients is*

$$\sum_{n=0}^{\infty}\sum_{k=0}^{\infty} \binom{n+k}{k} x^k y^n = \frac{1}{1-x-y}.$$

Proof. Just write the power series representation of $\frac{1}{1-x-y}$:

$$\frac{1}{1-(x+y)} = \sum_{n=0}^{\infty} (x+y)^n$$
$$= \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} \binom{n}{k} x^k y^{n-k}$$
$$= \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} \binom{n+k}{k} x^k y^n.$$

This completes the proof. ■

THEOREM 1.13. *The exponential bivariate generating function of the binomial coefficients is*

$$\sum_{n=0}^{\infty}\sum_{k=0}^{\infty} \frac{1}{(n+k)!} \binom{n+k}{k} x^k y^n = e^{x+y}.$$

Proof. Writing the power series representation of e^{x+y} leads the proof directly. ■

MULTINOMIAL COEFFICIENTS

Binomial coefficients are generalized to ***multinomial coefficients*** as follows. For a non-negative integer n and non-negative integers $k_1, k_2, \ldots, k_r$ such that $k_1 + k_2 + \cdots + k_r = n$, multinomial coefficient $\binom{n}{k_1, k_2, \ldots, k_r}$ is

$$\binom{n}{k_1, k_2, \ldots, k_r} = \frac{n!}{k_1! \cdots k_r!}.$$

Note that the multinomial coefficient $\binom{n}{k_1, k_2}$ is the binomial coefficient $\binom{n}{k_1}$. It is also easy to verify that

$$\binom{n}{k_1, k_2, \dots, k_r} = \binom{n}{k_1}\binom{n-k_1}{k_2}\binom{n-k_1-k_2}{k_3}\cdots\binom{n-k_1-\cdots-k_{r-2}}{k_{r-1}}.$$

Multinomial coefficient $\binom{n}{k_1,k_2,\dots,k_r}$ is coefficient of $x_1^{k_1}x_2^{k_2}\cdots x_r^{k_r}$ in $(x_1+\cdots+x_r)^n$ in

$$(x_1+x_2+\cdots+x_r)^n = \sum_{\substack{k_1,k_2,\dots,k_r=0 \\ k_1+\cdots+k_r=n}}^{\infty} \binom{n}{k_1, k_2, \dots, k_r} x_1^{k_1}x_2^{k_2}\cdots x_r^{k_r}.$$

Some Other Properties

- Binomial coefficient $\binom{n}{k}$, where $0 < k < n$, is divisible by a prime p if and only if n is a power of p.
- Let $k, m > 1$ and n be nonnegative integers. At least one of $\binom{n}{k}, \binom{n+1}{k}, \binom{n+2}{k}, \dots, \binom{n+k}{k}$ is not divisible by m.
- Let k and n be positive integers and p be a prime. If $\binom{n}{k-1} \nmid p$ and $\binom{n}{k} \nmid p$, then $\binom{n+1}{k} \nmid p$, except the case, when $n+1$ is divisible by p.
- **Lucas Theorem**. Write the positive integers n and k in a prime base p:

$$n = n_d p^d + n_{d-1}p^{d-1} + \cdots + n_1 p + n_0,$$
$$k = k_d p^d + k_{d-1}p^{d-1} + \cdots + k_1 p + k_0.$$

  Then

$$\binom{n}{k} \equiv \binom{n_d}{k_d}\binom{n_{d-1}}{k_{d-1}}\cdots\binom{n_1}{k_1}\binom{n_0}{k_0} \pmod{p}.$$

- If n is a positive integer and p is a prime such that $(p-1)|n$, then

$$\binom{n}{p-1} + \binom{n}{2(p-1)} + \binom{n}{3(p-1)} \cdots + \binom{n}{n} \equiv 1 \pmod{p}$$

- If $p \ge 5$ is a prime, then

$$\binom{2p}{p} \equiv 2 \pmod{p^3},$$
$$\binom{p^2}{p} \equiv p \pmod{p^5},$$
$$\binom{p^3}{p^2} \equiv \binom{p^2}{p} \pmod{p^8}.$$

Table of Binomial Coefficients

$n \backslash k$	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17
0	1																	
1	1	1																
2	1	2	1															
3	1	3	3	1														
4	1	4	6	4	1													
5	1	5	10	10	5	1												
6	1	6	15	20	15	6	1											
7	1	7	21	35	35	21	7	1										
8	1	8	28	56	70	56	28	8	1									
9	1	9	36	84	126	126	84	36	9	1								
10	1	10	45	120	210	252	210	120	45	10	1							
11	1	11	55	165	330	462	462	330	165	55	11	1						
12	1	12	66	220	495	792	924	792	495	220	66	12	1					
13	1	13	78	286	715	1287	1716	1716	1287	715	286	78	13	1				
14	1	14	91	364	1001	2002	3003	3432	3003	2002	1001	364	91	14	1			
15	1	15	105	455	1365	3003	5005	6435	6435	5005	3003	1365	455	105	15	1		
16	1	16	120	560	1820	4368	8008	11440	12870	11440	8008	4368	1820	560	120	16	1	
17	1	17	136	680	2380	6188	12376	19448	24310	24310	19448	12376	6188	2380	680	136	17	1

Table of binomial coefficients $\binom{n}{k}$ for $0 \leq n \leq 17$.

Exercises

1. Let n, m be positive integers. Prove the following.

a) $\binom{n}{m} < \dfrac{n^n}{m^m (n-m)^{n-m}}$,

b) $\displaystyle\sum_{k=0}^{n} (-1)^{k+n} \binom{n}{k} k^n = n!$,

c) $\displaystyle\sum_{k=1}^{n} (-1)^{n+1} \frac{1}{k} \binom{n}{k} = H_k$,

d) $\displaystyle\sum_{k=0}^{n} k \binom{n}{k}^2 = \frac{n}{2} \binom{2n}{n}$,

e) $\displaystyle\sum_{k=0}^{n} k^2 \binom{n}{k}^2 = n^2 \binom{2n-2}{n-1}$,

f) $\displaystyle\sum_{k=0}^{n} \binom{2k}{k} \binom{2(n-k)}{n-k} = 4^n$,

g) $\displaystyle\sum_{k=0}^{n-1} \binom{n-1}{k} \frac{(k+1)!}{n^k} = n!$

2. Let n, m be positive integers. Provide combinatorial proofs for the following identities.

a) $\binom{2n}{2} = 2\binom{n}{2} + n^2$,

b) $\binom{m+n}{2} = \binom{m}{2} + \binom{n}{2} + mn$,

c) $\displaystyle\sum_{k=1}^{n} k^3 \binom{n}{k} = n^2 (n+3) 2^{n-3}$,

d) $\displaystyle\sum_{k=0}^{n} 2^k \binom{n}{k} = 3^n$.

3. Evaluate

a) $\sum_{k=0}^{2n}(-1)^k k^n \binom{2n}{k}$,

b) $\sum_{k=0}^{n}(-1)^k \binom{n}{k}\left(1-\frac{k}{n}\right)$,

c) $\sum_{k=0}^{n}(-1)^k \binom{n}{k}^3$,

d) $\sum_{k=0}^{n}\binom{n}{k} 2^{k-n}$,

e) $\sum_{k=0}^{n}\binom{2k}{k} 4^{-k}$,

4. Let A be a set with n elements. Show that

$$\sum_{X,Y\subset A}|X\cup Y| = \frac{3n}{4},$$

$$\sum_{X,Y\subset A}|X\cap Y| = \frac{n}{4}.$$

5. Let A be a set with n elements and X, Y are two arbitrary subsets with $|A| = |B| = k$ elements. Compute the expected values of $|X\cap Y|$ and $|X\cup Y|$.

6. Show that for any nonnegative integer n, the equation $\binom{a}{1}+\binom{b}{2}+\binom{c}{3}=n$ has a unique solution.

7. Show that if $8|n$, then the number of subsets of $\{1,\dots,n\}$ whose number of elements is divisible by 4 is $2^{n-2}+2^{n/2-1}$. Discuss the case when $8\nmid n$.

8. Let p be a prime and n, k nonnegative integers. Show that

a) $\binom{p-1}{k} \equiv (-1)^k \pmod{p}$

b) $\binom{p}{k} \equiv 0 \pmod{p}$ $k = 1, 2, \dots, p-1$.

c) $\binom{2n}{n} \equiv (-4)^n \binom{(p-1)/2}{n} \pmod{p}$ where $n \le \frac{p-1}{2}$.

9. Show that the coefficient of x^k in $(1+x+x^2+x^3)^n$ is $\sum_{i=0}^{k}\binom{n}{i}\binom{n}{k-2i}$.

10. In this question we focus on parities of binomial coefficients in Pascal's triangle.

a) Prove that the number of odd binomial coefficients in n-th row of Pascal triangle is equal to 2^r, where r is the number of 1's in the binary expansion of n.

b) Prove that if $n = 2^k - 1$ for some k, then $\binom{n}{m}$ is odd for all $m = 0,1,\dots,n$.

c) Let n be odd. Show that the set $\left\{\binom{n}{1},\binom{n}{2},\dots,\binom{n}{m}\right\}$, where $m = (n-1)/2$, contains an odd number of odd integers.

d) Prove that in the first 10^6 rows of Pascal's triangle, the percentage of odd coefficients is less than 1%.

11. Let k be a fixed positive integer. Show that $\frac{d}{dx}\binom{x}{k} = \binom{x}{k}\sum_{i=0}^{k-1}\frac{1}{x-i}$.

12. Let k be a fixed positive integer. Find the local maximum value of the function $f(x) = \binom{x}{k}$.

List of Basic Identities Involving Binomial Coefficients

In the following, n, m and h are nonnegative integers, r and s are real numbers.

Basic Definition

$$\binom{r}{m} = \frac{r(r-1)\cdots(r-k+1)}{k!}$$

$$\binom{n}{m} = \frac{n!}{m!\,(n-m)!}$$ *Factorial Expansion*

Special Values

$$\binom{0}{m} = 0$$

$$\binom{r}{0} = \binom{n}{r} = 1$$

$$\binom{r}{1} = \binom{r}{r-1} = r$$

$$\binom{r}{2} = \binom{r}{r-2} = \frac{r(r-1)}{2}$$

Symmetry

$$\binom{r}{m} = \binom{r}{n-m}$$

Factoring

$$\binom{r}{m} = \frac{r}{r-m}\binom{r-1}{m}$$

$$\binom{r}{m} = \frac{r}{m}\binom{r-1}{m-1}$$ *Absorption* *m≠0*

$$\binom{r}{m} = \frac{r-m+1}{m}\binom{r}{m-1}$$ *m≠0*

Basic Identities

$$\binom{r}{m} = (-1)^m\binom{m-r-1}{m}$$ *Upper Negation*

$$\binom{r-1}{m} - \binom{r-1}{m-1} = \frac{r-2m}{r}\binom{r}{m}$$ *$r \neq 0$*

$$\binom{r-1}{m} + \binom{r-1}{m-1} = \binom{r}{m}$$ *Basic Recursion Pascal's Identity*

$$\binom{r}{m}\binom{r-m}{h} = \binom{r}{h}\binom{r-h}{m}$$

$$\binom{r}{m}\binom{m}{h} = \binom{r}{h}\binom{r-h}{m-h}$$ *Trinomial Revision*

Generating Functions

$$\sum_{k=0}^{\infty}\binom{r}{k}x^k = (1+x)^r$$

$$\sum_{n=0}^{\infty}\binom{n}{k}x^n = \frac{x^k}{(1-x)^{k+1}}$$

$$\sum_{n,k=0}^{\infty}\binom{n}{k}x^k y^n = \frac{1}{1-y-xy}$$

$$\sum_{n,k=0}^{\infty}\binom{n+k}{k}x^k y^n = \frac{1}{1-x-y}$$

$$\sum_{n,k=0}^{\infty}\frac{1}{(n+k)!}\binom{n+k}{k}x^k y^n = e^{x+y}$$

$$\sum_{k=0}^{\infty}(-1)^k\binom{-n}{k}x^k = \frac{1}{(1-x)^r}$$

Row-Column-Diagonal Sums

$$\sum_{k=0}^{n}\binom{n}{k} = 2^n$$ *Lower Summation*

$$\sum_{k=0}^{m}(-1)^k\binom{n}{k} = (-1)^m\binom{n-1}{m}$$ *Alternating Lower Summation*

$$\sum_{i=0}^{n}\binom{i}{k} = \binom{n+1}{k+1}$$ *Upper Summation*

$$\sum_{i=0}^{n}\binom{k+i}{i} = \binom{n+1}{k+1}$$ *Parallel Summation*

$$\sum_{k=0}^{\lfloor n/2\rfloor}\binom{n-k}{k} = \mathfrak{f}_{n+1}$$ *Column Sums*

Finite Sums

$$\sum_{k=0}^{n}\binom{n}{2k} = 2^{n-1}$$ *Number of Even Subsets*

$$\sum_{k=0}^{n}\binom{n}{2k+1} = 2^{n-1}$$ *Number of Odd Subsets*

$$\sum_{k=0}^{n}k\binom{n}{k} = n2^{n-1}$$ *Sum of Subset Cardinalities*

$$\sum_{k=0}^{n}k^2\binom{n}{k} = n(n+1)2^{n-2}$$

Sums of Products

$$\sum_{k=0}^{m}\binom{s}{k}\binom{r-s}{m-k} = \binom{r}{m}$$ *Vandermonde Convolution*

$$\sum_{i=0}^{n}\binom{n}{i}^2 = \binom{2n}{n}$$ *Square Summation*

$$\sum_{m=0}^{n}\binom{m}{j}\binom{n-m}{k-j} = \binom{n+1}{k+1}$$

$$\sum_{k=0}^{n}\binom{n}{k}\binom{n-k}{m-k} = \binom{n}{m}2^m$$

$$\sum_{k=i}^{n}\binom{n}{k}\binom{k}{m} = \binom{n}{i}2^{n-m}$$ *Subset Subset Sum*

Binomial Inversion

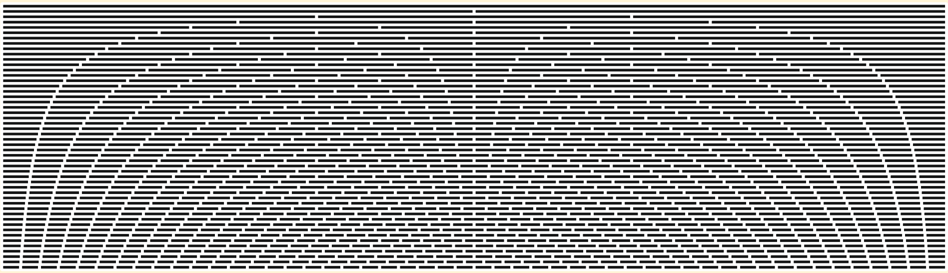
$$g_n = \sum_{k=0}^{n}(-1)^k\binom{n}{k}f_k \Leftrightarrow f_n = \sum_{k=0}^{n}(-1)^k\binom{n}{k}g_k$$

or

$$g_n = \sum_{k=0}^{n}\binom{n}{k}f_k \Leftrightarrow f_n = \sum_{k=0}^{n}(-1)^{n-k}\binom{n}{k}g_k$$

2. HARMONIC NUMBERS

The sum of reciprocals of first n positive integers is n-th harmonic number. Harmonic numbers are important in many brances of mathematics, especially in number theory. Harmonic numbers are called harmonic series as well. They are closely related to the Riemann zeta function, and appear in the expressions of various special functions.



For each positive integer n, the n th ***harmonic number*** $\boldsymbol{H_n}$ is defined as

$$H_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}$$

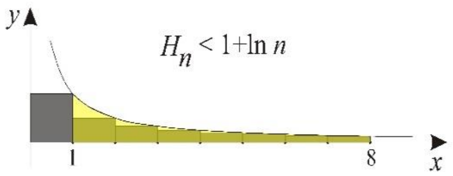
and by convention $H_0 = 0$.

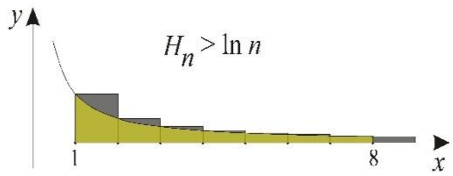
Harmonic Numbers And Logarithmic Function

As $H_n = \sum_{k=1}^{n} \frac{1}{k}$ and the logarithmic function computes $\ln(n) = \int_1^n \frac{1}{x} dx$ harmonic numbers can be visualized as the discrete version of the logarithmic function. In fact, the similarity is quite beyond just being visual. For example, we may compare the derivative $\frac{d}{dx} \ln x = \frac{1}{x}$ of the logarithmic function and the finite difference $\Delta H_k = \frac{1}{k+1}$ of the harmonic number.

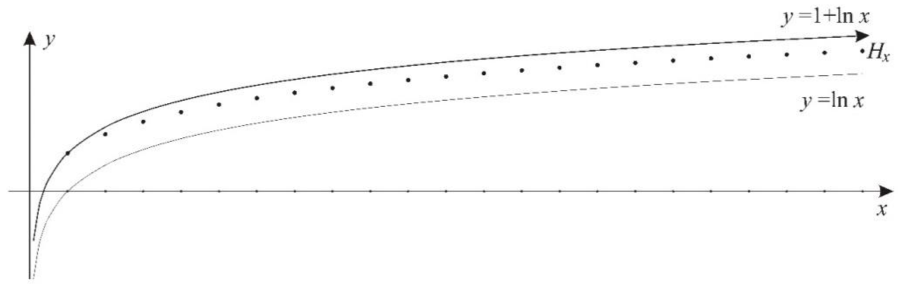
It is easy to observe that for $n > 1$,

$$\ln n < H_n < 1 + \ln n.$$





Following figure illustrates the graphs of $y = \ln x$, $y = 1 + \ln x$ and $y = H_x$.



It follows that a rough approximation for H_n is

$$H_n \approx \ln n.$$

The observation $H_n > \ln n$ implies divergence of the sequence $\{H_n\}$. Below theorem proves this fact with a different approach.

THEOREM 2.1. *The sequence $\{H_n\}$ is divergent.*

Proof. Write

$$H_{2^k} = 1 + \frac{1}{2} + \overbrace{\underbrace{\frac{1}{3} + \frac{1}{4}}_{\substack{\text{2 terms} \\ \text{smallest } 1/4}} + \underbrace{\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}}_{\substack{\text{4 terms} \\ \text{smallest } 1/8}} + \cdots + \underbrace{\frac{1}{2^{k-1}+1} + \cdots + \frac{1}{2^k}}_{\substack{2^{k-1} \text{ terms} \\ \text{smallest } 1/2^k}}}^{k-1 \text{ groups, sum of each } > 1/2}.$$

It follows that for $k > 1$,

$$H_{2^k} > 1 + \frac{k}{2}$$

which implies that $\{H_n\}$ is divergent. ■

THEOREM 2.2. *For any integer* $n > 1$,

$$1 + \frac{n}{2} < H_{2^n} < n.$$

Proof. First inequality is already obtained in the proof of previous theorem. For any positive integer n, we have $H_n \leq 1 + \ln n$. In particular for any $k > 1$, $H_{2^k} < 1 + k \ln 2 < k$. ■

THEOREM 2.3. H_n *is an integer only when* $n = 1$.

Proof. Let $n > 1$ and b be the least common multiple of the integers $1, \dots, n$. We can write $b = 2^k m$ for some odd integer m and $k > 1$ such that $2^k \leq n$. Then $bH_n = \frac{b}{1} + \frac{b}{2} + \cdots + \frac{b}{2^k} + \cdots + \frac{b}{n}$ and exactly one term on the right hand side is odd. Therefore, the sum on right hand side is odd. Since b is even, H_n cannot be an integer. ■

Generating Function Of $\{H_n\}$

THEOREM 2.4. *Generating function of* $\{H_n\}$ *is*

$$-\frac{\ln(1-x)}{1-x} = \sum_{k=1}^{\infty} H_n x^n.$$

Proof. By integrating both sides of $\frac{1}{1-x} = \sum_{k=0}^{\infty} x^k$, we see that $-\ln(1-x) = \sum_{k=1}^{\infty} \frac{x^k}{k}$ is the generating function of the sequence $1, \frac{1}{2}, \frac{1}{3}, \dots$ of reciprocals of natural numbers. Consequently, $-\ln(1-x) \cdot \frac{1}{1-x}$ is the generating function of the partial sums of reciprocals of natural numbers, namely the harmonic numbers. ■

Generalized Harmonic Numbers - A Better Approximation For H_n

For any integer $r > 1$, the ***generalized harmonic number*** $H_n^{(r)}$ is defined as follows

$$H_n^{(r)} = \sum_{k=1}^{n} \frac{1}{k^r}.$$

Since the series $\sum_{k=0}^{\infty} 1/k^r$ is convergent when $r > 1$, we can define

$$\zeta(r) = \lim_{n\to\infty} \sum_{k=r}^{n} \frac{1}{k^r} = H_{\infty}^{(r)}.$$

The function $z \mapsto \zeta(z)$ is called ***Riemann[3] zeta function***. Euler has proven that if n is a positive integer, then

$$\zeta(2n) = \frac{p}{q}\pi^{2n}$$

for some positive integers p and q. No such representation in terms of π is known for the values ζ(2n+1).

Some values of ζ function are

$$\zeta(2) = \frac{\pi^2}{6} \qquad \zeta(3) = 1.20205\cdots$$

$$\zeta(4) = \frac{\pi^4}{90} \qquad \zeta(5) = 1.03692\cdots$$

$$\zeta(6) = \frac{\pi^6}{945} \qquad \zeta(7) = 1.00834\cdots$$

Derivative of the function $F(x) = \sum_{i=1}^{\infty} \frac{x^{-i}}{i} = \sum_{i=1}^{\infty} \frac{1}{i}\left(\frac{1}{x}\right)^i$ is

$$F'(x) = \sum_{i=1}^{\infty} -\frac{1}{x^2}\left(\frac{1}{x}\right)^{i-1} = -\frac{1}{x^2}\sum_{i=0}^{\infty}\left(\frac{1}{x}\right)^i$$

$$= -\frac{1}{x^2}\left(\frac{1}{1-\frac{1}{x}}\right)$$

$$= -\frac{1}{x(x-1)}$$

$$= \frac{1}{x} - \frac{1}{(x-1)}.$$

[3] *Georg Friedrich Bernhard Riemann (1826-1866), German mathematician.*

It follows that $\ln x - \ln(x-1) = \int \left(\frac{1}{x} - \frac{1}{(x-1)}\right) dt = \int F'(x)dx = F(x) = \sum_{i=0}^{\infty} \frac{1}{i}\left(\frac{1}{x}\right)^i$. Then we have $\ln \frac{k}{k-1} = \frac{1}{k} + \frac{1}{2k^2} + \frac{1}{3k^3} + \cdots$ which is convergent for $k > 1$. Write the equations for $k = 2,3,\cdots,n$:

$$k = 2 \qquad \ln 2 - \ln 1 = \frac{1}{2} + \frac{1}{2 \cdot 2^2} + \cdots$$

$$k = 3 \qquad \ln 3 - \ln 2 = \frac{1}{3} + \frac{1}{2 \cdot 3^2} + \cdots$$

$$\vdots \qquad \vdots$$

$$k = n \qquad \ln n - \ln(n-1) = \frac{1}{n} + \frac{1}{2 \cdot n^2} + \cdots$$

Term by term summation gives

$$\begin{aligned} \ln n - \ln 1 &= H_n - 1 + \frac{1}{2}\left(H_n^{(2)} - 1\right) + \frac{1}{3}\left(H_n^{(3)} - 1\right) + \cdots \\ &= H_n - 1 + \sum_{k=2}^{\infty} \frac{1}{k}\left(H_n^{(k)} - 1\right) \end{aligned}$$

or

$$H_n - \ln n = 1 - \sum_{k=2}^{\infty} \frac{1}{k}\left(H_n^{(k)} - 1\right).$$

The limit $\lim_{n\to\infty}\left[1 - \sum_{k=2}^{\infty} \frac{1}{k}\left(H_n^{(k)} - 1\right)\right] = 1 - \sum_{k=2}^{\infty} \frac{1}{k}(\zeta(k) - 1)$ exists and its value γ is known as Euler[4] Mascheroni[5] constant[6], that is

$$1 - \sum_{k=2}^{\infty} \frac{1}{k}(\zeta(k) - 1) = \gamma.$$

Now we have

$$\lim_{n\to\infty}(H_n - \ln n) = 1 - \frac{1}{2}(\zeta(2) - 1) - \frac{1}{3}(\zeta(3) - 1) - \cdots = \gamma.$$

As a precise approximation for H_n we can write

$$H_n \approx \ln n + \gamma.$$

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[4] *Leonhard Euler (1707-1703), Swiss mathematician. One of the greatest three mathematicians of all times.*
[5] *Lorenzo Mascheroni (1750-1800), Italian mathematician.*
[6] $\gamma = 0.57721\ 56649\ 01532\ 86060\ 65120\ 90082\ 40243\ 10421\ 59335\ 93992 \cdots$

In fact

$$H_n = \ln n + \gamma + \frac{1}{2n} - \frac{1}{12n^2} + \frac{\varepsilon_n}{120n^4} \qquad 0 < \varepsilon_n < 1.$$

Two useful bounds for H_n are given by the following inequalities:

$$\frac{1}{2(n+1)} < \ H_n - \ln n - \gamma < \frac{1}{2n},$$

$$\frac{1}{24(n+1)^2} < H_n - \ln\left(n + \frac{1}{2}\right) - \gamma < \frac{1}{24n^2}.$$

Finite Sums Involving Harmonic Numbers

In this section we compute the first nine of the following finite sums:

$$\sum_{k=1}^{n} H_k = (n+1)H_n - n$$

$$\sum_{k=1}^{n} H_k^{(r)} = (n+1)H_n^{(r)} - H_n^{(r-1)}$$

$$\sum_{k=1}^{n} kH_k = \frac{n(n+1)}{2}\left(H_{n+1} - \frac{1}{2}\right)$$

$$\sum_{k=1}^{n}\sum_{k=i}^{k} H_i = \binom{n+2}{2}(H_{n+2} - H_2)$$

$$\sum_{k=1}^{n} \frac{H_k}{k} = \frac{1}{2}\left((H_n)^2 + H_n^{(2)}\right)$$

$$\sum_{k=1}^{n} (-1)^{k-1}\frac{1}{k}\binom{n-1}{k-1} = \frac{1}{n}$$

$$\sum_{k=1}^{n} (-1)^{k-1}\binom{n}{k}\frac{1}{k} = H_n$$

$$\sum_{k=1}^{n} (-1)^{k-1}\binom{n}{k} H_k = \frac{1}{n}$$

$$\sum_{k=m}^{n} \binom{k}{m} H_k = \binom{n+1}{m+1}\left(H_{n+1} - \frac{1}{m+1}\right)$$

$$\sum_{k=0}^{n} \binom{n}{k} H_k = 2^n\left(H_n - \sum_{k=1}^{n}\frac{1}{k2^k}\right)$$

$$\sum_{k=1}^{n} \binom{n}{k}^2 H_k = \binom{2n}{n}(2H_n - H_{2n})$$

PROPOSITION 2.5. *Partial sums of harmonic numbers and generalized harmonic numbers are as follows:*

$$\sum_{k=1}^{n} H_k = (n+1)H_n - n,$$
$$\sum_{k=1}^{n} H_k^{(r)} = (n+1)H_n^{(r)} - H_n^{(r-1)}.$$

Proof. For the partial sums of harmonic numbers

$$\sum_{k=1}^{n} H_k = \sum_{k=1}^{n}\sum_{i=1}^{k}\frac{1}{i} = \sum_{i=1}^{n}\sum_{k=i}^{n}\frac{1}{i}$$
$$= \sum_{k=1}^{n}\frac{n-i+1}{i}$$
$$= (n+1)\sum_{k=1}^{n}\frac{1}{i} - n$$
$$= (n+1)H_n - n.$$

For the partial sums of generalized harmonic numbers we have

$$\sum_{k=1}^{n} H_k^{(r)} = \sum_{k=1}^{n}\sum_{i=1}^{k}\frac{1}{i^r}$$
$$= \sum_{i=1}^{n}\frac{1}{i^r}(n+1-i)$$
$$= (n+1)H_n^{(r)} - H_n^{(r-1)}.$$

This completes the proof. ■

We could obtain the first equality of the above proposition using the generating functions. Let $\{S_n\}$ be the sequence of partial sums of harmonic numbers, namely $\{S_n\} = H_1 + \cdots + H_n$. Since generating function of $\{H_n\}$ is $-\frac{\ln(1-x)}{1-x}$, generating function of S_n is $-\frac{\ln(1-x)}{1-x}\cdot\frac{1}{1-x}$. That is

$$-\frac{\ln(1-x)}{(1-x)^2} = \sum_{n=1}^{\infty} S_n x^n = \sum_{n=1}^{\infty}\left(\sum_{k=1}^{n} H_k\right)x^n.$$

On the other hand $-\ln(1-x) = \sum_{i=1}^{\infty}\frac{x^i}{i}$ and $\frac{1}{(1-x)^2} = \sum_{j=1}^{\infty} j\, x^{j-1}$ so we get

$$-\frac{\ln(1-x)}{(1-x)^2} = \sum_{i=1}^{\infty} \frac{x^i}{i} \cdot \sum_{j=1}^{\infty} j\, x^{j-1} = \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} \frac{j}{i} x^{i+j-1}$$

$$= \sum_{n=1}^{\infty} \left(\sum_{k=1}^{n} \left(\frac{n+1-k}{k} \right) \right) x^n$$

$$= \sum_{n=1}^{\infty} \big((n+1)H_n - n \big)\, x^n.$$

It follows that $S_n = (n+1)H_n - n$, in other words

$$\sum_{k=0}^{n} H_k = (n+1)H_n - n.$$

Now define $\{\xi_n\}$ to be the sequence of partial sums of $\{S_n\}$, that is $\xi_n = S_1 + S_2 + \cdots + S_n$. Generating function of $\{S_n\}$ is $-\frac{\ln(1-x)}{(1-x)^2}$, so generating function of $\{\xi_n\}$ is $-\frac{\ln(1-x)}{(1-x)^3}$:

$$-\frac{\ln(1-x)}{(1-x)^3} = \sum_{n=1}^{\infty} \xi_n\, x^n = \sum_{n=1}^{\infty} \left(\sum_{i=1}^{n} S_i \right) x^n$$

$$= \sum_{n=1}^{\infty} \left(\sum_{k=1}^{n} \big((k+1)H_k - k \big) \right) x^n$$

$$= \sum_{n=1}^{\infty} \left((n+1)H_n - \frac{n(n+3)}{2} + \sum_{k=1}^{n} kH_k \right) x^n.$$

which implies that

$$\xi_n = (n+1)H_n - \frac{n(n+3)}{2} + \sum_{k=1}^{n} kH_k.$$

On the other hand

$$-\frac{\ln(1-x)}{(1-x)^3} = \sum_{i=1}^{\infty} \frac{x^i}{i} \cdot \sum_{j=0}^{\infty} \binom{j+2}{2} x^j$$

$$= \sum_{i=1}^{\infty} \frac{x^i}{i} \cdot \sum_{j=1}^{\infty} \binom{j+1}{2} x^{j-1}$$

$$= \frac{1}{2} \sum_{i=1}^{\infty} \frac{x^i}{i} \cdot \sum_{j=1}^{\infty} j(j+1)\, x^{j-1}$$

$$= \frac{1}{2} \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \frac{j(j+1)}{i} x^{i+j-1}$$

$$= \frac{1}{2}\sum_{n=1}^{\infty}\left(\sum_{k=1}^{n}\left(\frac{(n+1-k)(n+2-k)}{k}\right)\right)x^n$$

$$= \frac{1}{2}\sum_{n=1}^{\infty}\left((n+1)(n+2)H_n - n(2n+3) + \frac{n(n+1)}{2}\right)x^n$$

$$= \frac{1}{2}\sum_{n=1}^{\infty}\left((n+1)(n+2)H_n - \frac{3n^2+5n}{2}\right)x^n$$

so that

$$\xi_n = \frac{1}{2}\left((n+1)(n+2)H_n - \frac{3n^2+5n}{2}\right).$$

Now we have

$$(n+1)H_n - \frac{n(n+3)}{2} + \sum_{i=1}^{n} kH_k = \frac{1}{2}\left((n+1)(n+2)H_n - \frac{3n^2+5n}{2}\right)$$

which can be rearranged to write

$$\sum_{k=1}^{n} kH_k = \frac{1}{2}\left(\big((n+1)(n+2) - 2(n+1)\big)H_n + n(n+3) - \frac{3n^2+5n}{2}\right)$$

$$= \frac{1}{2}\left(n(n+1)H_n - \frac{n^2}{2} + \frac{n}{2}\right) = \frac{1}{2}\left(n(n+1)H_{n+1} - \frac{n^2}{2} - \frac{n}{2}\right)$$

which simplifies into

$$\sum_{k=1}^{n} kH_k = \frac{n(n+1)}{2}\left(H_{n+1} - \frac{1}{2}\right).$$

PROPOSITION 2.6. *For any positive integer n,*

$$\sum_{k=1}^{n} kH_k = \frac{n(n+1)}{2}\left(H_{n+1} - \frac{1}{2}\right),$$

$$\sum_{k=1}^{n}\sum_{k=i}^{k} H_i = \binom{n+2}{2}(H_{n+2} - H_2).$$

Proof. In the paragraph preceding the statement of the proposition, the first equality has been obtained already. The second equality follows easily from the first one:

$$\sum_{k=1}^{n}\sum_{k=i}^{k} H_i = \sum_{k=1}^{n}\big((k+1)H_k - k\big)$$

$$= \sum_{k=1}^{n}(kH_k + H_k - 1))$$

$$= \frac{n(n+1)}{2}\left(H_{n+1} - \frac{1}{2}\right) + (n+1)H_n - n - \frac{n(n+1)}{2}$$

$$= \frac{(n+1)(n+2)}{2}\left(H_{n+2} - \frac{3}{2}\right).$$

By writing $\frac{(n+1)(n+2)}{2} = \binom{n+2}{2}$ and $\frac{3}{2} = H_2$ we obtain the desired equality. ■

PROPOSITION 2.7. For any positive integer n,

$$\sum_{k=1}^{n}\frac{H_k}{k} = \frac{1}{2}\left((H_n)^2 + H_n^{(2)}\right).$$

Proof. We write

$$\sum_{k=1}^{n}\frac{H_k}{k} = \sum_{k=1}^{n}\sum_{i=1}^{k}\frac{1}{k}\cdot\frac{1}{i} = \sum_{i=1}^{n}\frac{1}{i}\sum_{k=i}^{n}\frac{1}{k} = \sum_{i=1}^{n}\frac{1}{i}(H_n - H_{i-1})$$

$$= (H_n)^2 - \sum_{i=1}^{n}\frac{1}{i}H_{i-1} = (H_n)^2 - \sum_{i=1}^{n}\frac{1}{i}\left(H_i - \frac{1}{i}\right)$$

$$= (H_n)^2 - \sum_{i=1}^{n}\frac{1}{i}H_i + \sum_{i=1}^{n}\frac{1}{i^2}$$

which results in $2\sum_{k=1}^{n}\frac{H_k}{k} = (H_n)^2 + H_n^{(2)}$. ■

PROPOSITION 2.8. *For any positive integer n,*

$$\sum_{k=1}^{n}(-1)^{k-1}\frac{1}{k}\binom{n-1}{k-1} = \frac{1}{n},$$

$$\sum_{k=1}^{n}(-1)^{k-1}\frac{1}{k}\binom{n}{k} = H_n,$$

$$\sum_{k=1}^{n}(-1)^{k-1}\binom{n}{k}H_k = \frac{1}{n}.$$

Proof. We compute the sum on the right hand side of the first equality:

$$\sum_{k=1}^{n}(-1)^{k-1}\frac{1}{k}\binom{n-1}{k-1} = \frac{1}{n}\sum_{k=1}^{n}(-1)^{k-1}\binom{n}{k}$$

$$= \frac{1}{n}\left(1-\sum_{k=0}^{n}(-1)^{k}\binom{n}{k}\right)$$

$$= \frac{1}{n}(1-(1-1)^{n})$$

$$= \frac{1}{n}.$$

Similarly, for the second one we have:

$$\sum_{k=1}^{n}(-1)^{k-1}\frac{1}{k}\binom{n}{k} = \sum_{k=0}^{n-1}(-1)^{k}\frac{1}{k+1}\binom{n}{k+1}$$

$$= \sum_{k=0}^{n-1}(-1)^{k}\frac{1}{k+1}\sum_{i=k}^{n-1}\binom{i}{k}$$

$$= \sum_{i=0}^{n-1}\sum_{k=0}^{i}(-1)^{k}\frac{1}{k+1}\binom{i}{k}$$

$$= \sum_{i=0}^{n-1}\sum_{k=1}^{i+1}(-1)^{k-1}\frac{1}{k}\binom{i}{k-1}$$

$$= \sum_{i=0}^{n-1}\frac{1}{1+i}$$

$$= H_n.$$

Since $\sum_{k=0}^{n}(-1)^{k}\binom{n}{k} = 0$, we can write $-\sum_{k=i}^{n}(-1)^{k}\binom{n}{k} = \sum_{k=0}^{i-1}(-1)^{k}\binom{n}{k}$. We use this equality to prove the last inequality:

$$\sum_{k=1}^{n}(-1)^{k-1}\binom{n}{k}H_k = \sum_{k=1}^{n}\sum_{i=1}^{k}(-1)^{k-1}\binom{n}{k}\frac{1}{i}$$

$$= \sum_{i=1}^{n}\frac{1}{i}\sum_{k=i}^{n}(-1)^{k-1}\binom{n}{k}$$

$$= \sum_{i=1}^{n} \frac{1}{i} \sum_{k=0}^{i-1} (-1)^k \binom{n}{k}$$

$$= \sum_{i=1}^{n} \frac{1}{i} (-1)^{i-1} \binom{n-1}{i-1}$$

$$= \frac{1}{n} \sum_{i=1}^{n} (-1)^{i-1} \binom{n}{i}$$

$$= \frac{1}{n}. \quad \blacksquare$$

Note that, the equality $\sum_{i=0}^{t}(-1)^i\binom{n}{i} = (-1)^i\binom{n-1}{i}$, which holds for all integers $0 < n$ and $t < n$, is used in the proof.

A one line proof of the last equality is obtained once we notice that, from the second equality, $\{H_n\}$ is transform of $\left\{\frac{1}{n}\right\}$ under binomial inversion. Thus the third equality follows immediately.

PROPOSITION 2.9. *For any positive integers n and m*

$$\sum_{k=m}^{n} \binom{k}{m} H_k = \binom{n+1}{m+1}\left(H_{n+1} - \frac{1}{m+1}\right).$$

Proof.

$$\sum_{k=m}^{n} \binom{k}{m} H_k = \sum_{k=m}^{n} \sum_{i=1}^{k} \frac{1}{k} \binom{k}{m}$$

$$= \sum_{k=m}^{n} \sum_{i=1}^{k} \frac{1}{i} \binom{k}{m}$$

$$= \sum_{k=m}^{n} \sum_{i=1}^{m} \frac{1}{i} \binom{k}{m} + \sum_{k=m+1}^{n} \sum_{i=m+1}^{k} \frac{1}{i} \binom{k}{m}$$

$$= \left(\sum_{k=m}^{n} \binom{k}{m}\right)\left(\sum_{i=1}^{m} \frac{1}{i}\right) + \sum_{i=m+1}^{n} \frac{1}{i} \sum_{k=i}^{n} \binom{k}{m}$$

$$= \binom{n+1}{m+1} H_m + \sum_{i=m+1}^{n} \frac{1}{i}\left(\binom{n+1}{m+1} - \binom{i}{m+1}\right)$$

$$= \binom{n+1}{m+1} H_m + \binom{n+1}{m+1}(H_n - H_m) - \sum_{i=m+1}^{n} \frac{1}{i} \binom{i}{m+1}$$

$$= \binom{n+1}{m+1} H_n - \frac{1}{m+1} \sum_{i=m+1}^{n} \binom{i-1}{m}$$

$$= \binom{n+1}{m+1} H_n - \frac{1}{m+1} \binom{n}{m+1}$$

$$= \binom{n+1}{m+1} H_{n+1} - \frac{1}{n+1} \binom{n+1}{m+1} - \frac{1}{m+1} \binom{n}{m+1}$$

$$= \binom{n+1}{m+1} H_{n+1} - \frac{1}{m+1} \binom{n}{m} - \frac{1}{m+1} \binom{n}{m+1}$$

$$= \binom{n+1}{m+1} H_{n+1} - \frac{1}{m+1} \left(\binom{n}{m} + \binom{n}{m+1} \right).$$

Now, using the identity $\binom{n}{m} + \binom{n}{m+1} = \binom{n+1}{m+1}$ we obtain the desired equality. ■

An Integral Representation And Extension To Real Numbers

A direct consequence of the identity $\frac{1-x^n}{1-x} = 1 + x + \cdots + x^{n-1}$ is

$$H_n = \int_0^1 \frac{1-x^n}{1-x} dx.$$

With the transform $u = 1 - x$ in the above integral we get

$$H_n = \int_0^1 \frac{1-(1-u)^n}{u} du$$

$$= \int_0^1 \left[\sum_{k=1}^{n} (-1)^{k-1} \binom{n}{k} u^{k-1} \right] du$$

$$= \sum_{k=1}^{n} (-1)^{k-1} \binom{n}{k} \int_0^1 u^{k-1} du$$

$$= \sum_{k=1}^{n} (-1)^{k-1} \frac{1}{k} \binom{n}{k}.$$

Then

$$H_n = \sum_{k=1}^{n} (-1)^{k-1} \frac{1}{k} \binom{n}{k}.$$

For any real number $0 < x < 1$ we define H_x by setting

$$H_x = \int_0^1 \frac{1 - t^x}{1 - t} dt.$$

Values of H_x for $x > 1$ or $x < 0$ can be computed from the recurence

$$H_x = H_{x-1} + \frac{1}{x}.$$

Some examples are

$$H_{\frac{1}{2}} = 2 - 2\ln 2,$$

$$H_{\frac{1}{3}} = 3 - \frac{\pi}{2\sqrt{3}} - \frac{3}{2}\ln 3,$$

$$H_{\frac{3}{4}} = \frac{4}{3} - 3\ln 2 + \frac{\pi}{2}.$$

IDENTITIES INVOLVING H_x

$$H_x = x \sum_{k=1}^{\infty} \frac{1}{k(x+k)},$$

$$H_{2x} = \frac{1}{2}\left(H_x + H_{x-\frac{1}{2}}\right) + \ln 2,$$

$$\int_0^1 H_x dx = \gamma,$$

$$\int_0^n H_x dx = n\gamma + \ln(n!),$$

$$\int_0^a H_x^{(2)} dx = a\frac{\pi^2}{6} - H_a,$$

$$\int_0^a H_x^{(3)} dx = a\zeta(3) - \frac{1}{2}H_a^{(2)}.$$

Some Infinite Sums Involving Harmonic Numbers

$$\sum_{k=1}^{\infty} \frac{1}{k(k+x)} = x \cdot H_x\,,$$

$$\sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k} \binom{n}{k} H_k = H_n^{(2)},$$

$$\sum_{k=1}^{\infty} \frac{H_k}{k^2} = 2\zeta(3),$$

$$\sum_{k=1}^{\infty} \frac{H_k}{k2^k} = \frac{1}{2}\zeta(2) = \frac{1}{12}\pi^2,$$

$$\sum_{k=1}^{\infty} \frac{H_k^2}{k^2} = \frac{17}{4}\zeta(4) = \frac{17}{360}\pi^4,$$

$$\sum_{k=1}^{\infty} \frac{H_k^2}{(k+1)^2} = \frac{11}{4}\zeta(4) = \frac{11}{360}\pi^4,$$

$$\sum_{k=1}^{\infty} \frac{H_k}{k^3} = \frac{5}{4}\zeta(4) = \frac{1}{72}\pi^4.$$

Table of Harmonic Numbers

n	H_n	n	H_n	n	H_n	n	H_n	n	H_n	n	H_n
1	1	21	3,64536	41	4,30293	61	4,69626	81	4,97782	100	5,18738
2	1,50000	22	3,69081	42	4,32674	62	4,71239	82	4,99002	200	5,87803
3	1,83333	23	3,73429	43	4,35000	63	4,72827	83	5,00207	300	6,28266
4	2,08333	24	3,77596	44	4,37273	64	4,74389	84	5,01397	400	6,56993
5	2,28333	25	3,81596	45	4,39495	65	4,75928	85	5,02574	500	6,79282
6	2,45000	26	3,85442	46	4,41669	66	4,77443	86	5,03737	600	6,97497
7	2,59286	27	3,89146	47	4,43796	67	4,78935	87	5,04886	700	7,12901
8	2,71786	28	3,92717	48	4,45880	68	4,80406	88	5,06022	800	7,26245
9	2,82897	29	3,96165	49	4,47921	69	4,81855	89	5,07146	900	7,38016
10	2,92897	30	3,99499	50	4,49921	70	4,83284	90	5,08257	1000	7,48547
11	3,01988	31	4,02725	51	4,51881	71	4,84692	91	5,09356	2000	8,17836
12	3,10321	32	4,05850	52	4,53804	72	4,86081	92	5,10443	3000	8,58375
13	3,18013	33	4,08880	53	4,55691	73	4,87451	93	5,11518	4000	8,87139
14	3,25156	34	4,11821	54	4,57543	74	4,88802	94	5,12582	5000	9,09450
15	3,31823	35	4,14678	55	4,59361	75	4,90136	95	5,13635	6000	9,27681
16	3,38073	36	4,17456	56	4,61147	76	4,91451	96	5,14676	7000	9,43095
17	3,43955	37	4,20159	57	4,62901	77	4,92750	97	5,15707	8000	9,56447
18	3,49511	38	4,22790	58	4,64625	78	4,94032	98	5,16728	9000	9,68225
19	3,54774	39	4,25354	59	4,66320	79	4,95298	99	5,17738	10000	9,78760
20	3,59774	40	4,27854	60	4,67987	80	4,96548	100	5,18738	100000	12,09015

Problems Involving Harmonic Numbers

There are many interesting problems whose solutions involve harmonic numbers. Here we give several examples of such questions. Exercises 9-10, at the end of this section are also examples of such problems.

Problem 1. Remove those terms in $1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\cdots$ such that its denominator in decimal expansion contains the digit 9, then prove that the sequence is bounded.

The integers without the digit 9 in the interval $[10^{m-1}, 10^m - 1]$ are m-digit numbers. The first digit from the left cannot be the digits 0 and 9, (8 choices), the other digits cannot contain 9, hence nine choices for each. Altogether there are $8 \cdot 9^{m-1}$ such integers. The sum of their reciprocals is less than $8\left(\frac{9}{10}\right)^{m-1}$. The sum of all such numbers is therefore less then

$$\sum_{m=1}^{\infty} 8\left(\frac{9}{10}\right)^{m-1} = 8 \cdot \frac{1}{1-\frac{9}{10}} = 80.$$

Problem 2. Assume that we start keeping snowfall records this year. We wish to find the expected number of records that will occur in the next 20 years.

The first year is necessarily a record. The second year will be a record if the snowfall in the second year is greater than that in the first year. By symmetry, this probability is $1/2$. In general let P_k be 1 if the k-th year is a record and 0 otherwise. To find $E(P_k)$, we need only find the probability that the k-th year is a record. But the record snowfall for the first k years is equally likely to fall in any one of these years, so $E(P_k) = 1/k$. Therefore, if S_n is the total number of records observed in the first n years, $(S_n) = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} = H_n$. Therefore, in ten years the expected number of records is $H_{20} \approx \ln 20 + \gamma = 3.98\cdots$.

Problem 3. A folk dance performance group of n dancers is to be splitted into two circular formations. In how many different ways can this be done?

Distinguish one of the dancers, say A and form the circles groups in two steps:

- First pick k dancers to join A and form a circle ($\binom{n-1}{k}k! = \frac{(n-1)!}{(n-k-1)!}$ ways),

- Next form the second circle with the remaining dancers ($(n-k-2)!$ ways).

Since for each $k = 0, \dots, n-2$, there are $\frac{(n-1)!}{(n-k-1)}$ ways, number of total ways is

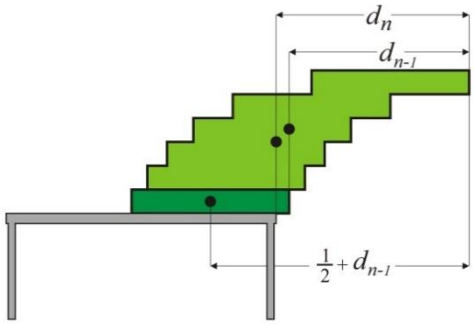
$$(n-1)!\left[\frac{1}{n-1} + \frac{1}{n-2} + \cdots + \frac{1}{2} + 1\right] = (n-1)!\,H_{n-1}.$$

The problem actually asks to compute $\left[{n \atop 2}\right]$. Thus, we have obtained the relation

$$\left[{n \atop 2}\right] = (n-1)!\,H_{n-1}.$$

between harmonic numbers and Stirling numbers (See Section 5 of this chapter).

Problem 4. We have n identical books of unit length. What is the maximum overhang that can be achieved when the books are stacked as a pile (of one book at each level) at the edge of a table?



Let d_n be the maximum offset distance of a stack of n books. When the largest offset is obtained, the center of mass of the n books must lie right above the table's edge and the center of mass of the $n-1$ top books must lie right above the edge of the book at the bottom. By computing the total moment of n books with respect to the right edge we obtain

$$nd_n = \left(d_{n-1} + \frac{1}{2}\right) + (n-1)d_{n-1}.$$

This relation can be solved for d_n to obtain the recurrence relation $d_n = d_{n-1} + \frac{1}{2n}$.

By iteration we get

$$d_2 = d_1 + \frac{1}{4}$$
$$d_3 = d_2 + \frac{1}{6} = d_1 + \frac{1}{4} + \frac{1}{6}$$
$$d_4 = d_3 + \frac{1}{8} = d_1 + \frac{1}{4} + \frac{1}{6} + \frac{1}{8}.$$

It is easy to see that

$$d_n = \frac{1}{2}\left(2d_1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}\right).$$

For only one book obviously $d_1 = \frac{1}{2}$. Then we conclude $d_n = \frac{1}{2}H_n$. Since the sequence $\{H_n\}$ is divergent, the maximum amount of overhang will become arbitrarily large as the number of books grows.

Problem 5. One end of an infinitely stretchable rubber band, initially 1-meter-long, is nailed to a wall. A bug is on the rubber band at the end attached to wall. At each second, the ant crawls 1 centimeter toward the other end and when the ant stops, the band is stretched by 1 meter. In the first second ant crawls 1 cm and after the band is stretched, it is 2 centimeters apart from the wall. Two seconds later it is 4.5 centimeters apart, so on. Will the bug ever reach the other end? If so, when?

Let L_n be the length of the band after n seconds, then $L_n = L_0 + nL_0 = (1+n)L_0$ where $L_0 = 100\ cm$ is the length of the band at the beginning. Let S_n be the distance (in terms of centimeters) of ant from the wall after n seconds.

At the beginning of n th second, length of band is $L_{n-1} = nL_0$ centimeters and the ant is S_{n-1} centimeters apart from the wall. Now, ant crawls to the point $S_{n-1}+1$ and the band is stretched to length $L_n = (1+n)L_0$. As a result, ant is now

$$S_n = \frac{1+n}{n}(S_{n-1}+1)$$

centimeters apart from the wall. We write first few terms of the sequence $\{S_t\}$

$$S_1 = \frac{2}{1}\cdot 1$$
$$S_2 = \frac{3}{2}\left(\frac{2}{1}+1\right) = 3\left(1+\frac{1}{2}\right)$$
$$S_3 = \frac{4}{3}\cdot\left(\frac{3}{3}+\frac{3}{2}+1\right) = 4\left(1+\frac{1}{2}+\frac{1}{3}\right)$$
$$S_4 = \frac{5}{4}\left(\frac{4}{4}+\frac{4}{2}+\frac{4}{3}+1\right) = 5\left(1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}\right)$$

Inductively it can be shown that $S_n = (n+1)H_n$. The ant reaches the other end of the band after n_0 seconds if $S_{n_0} \geq L_{n_0}$ or $(n_0+1)H_{n_0} \geq (1+n_0)L_0$ that is , $H_{n_0} \geq L_0$.

The sequence $\{H_n\}$ is divergent so, no matter how large L_0 is, for some n_0, eventually we will have $H_{n_0} \geq L_0$. Using the rough approximation $H_n \approx \ln n$, we can write $n_0 \approx e^{L_0}$. When L_0 is 100, it requires e^{100} seconds (more than 8×10^{35} years) for the ant to reach the other end.

Problem 6. The secretary problem is an optimal stopping problem that has been studied extensively in the fields of applied probability, statistics, and decision theory. It is also known as the marriage problem, the sultan's dowry problem, the fussy suitor problem, and the best choice problem. The problem can be stated as follows:

– There is a single position to fill,
– There are n applicants for the position,
– The applicants can be ranked from best to worst with no ties,
– The applicants are interviewed sequentially in a random order,
– After each interview, the applicant is accepted or rejected,
– The decision to accept or reject an applicant can be based only on the relative - ranks of the applicants interviewed so far,
– Rejected applicants cannot be recalled.

The object is to select the best applicant. The payoff is 1 for the best applicant and zero otherwise. We can go through some of the applicants and use that information in order to choose the best applicant out of the rest we have not seen so far. For example, we if look at k applicants and see who is the best applicant out of them, then we look through the $n-k$ applicants and choose the first one who is better than the best of the first k. Find the value of k for which choosing the best applicant is maximum.

If the best applicant is among the first k, probability to win is clearly 0. If the best applicant is at t th place $(t>k)$, he could chosen only if the best of the first $t-1$ applicants is among the first k. Probability of this event is

$$p=\frac{k}{t-1}$$

for $t=k+1,k+2,\dots,n$. Since the applicant can be at any position with probability $\frac{1}{n}$ overall probability to win is

$$P_k=\frac{k}{n}\left(\frac{1}{k}+\frac{1}{k+1}+\cdots+\frac{1}{n-1}\right)=\frac{k}{n}(H_{n-1}-H_{k-1}).$$

For the maximum value of P_k we must have $P_k\geq P_{k-1}\quad and \quad P_k\geq P_{k+1}$.

First inequality gives

$$\frac{k}{n}(H_{n-1}-H_{k-1})\geq\frac{k-1}{n}(H_{n-1}-H_{k-2})$$

$$kH_{n-1}-kH_{k-1}\geq kH_{n-1}-H_{n-1}-kH_{k-2}+H_{k-2}$$

$$H_{n-1}\geq k(H_{k-1}-H_{k-2})+H_{k-2}$$

that is $H_{n-1}=1+H_{k-1}$.

Second inequality gives

$$\frac{k}{n}(H_{n-1} - H_{k-1}) \geq \frac{k+1}{n}(H_{n-1} - H_k)$$

$$kH_{n-1} - kH_{k-1} \geq kH_{n-1} + H_{n-1} - kH_k - H_k$$

so $H_{n-1} \leq 1 + H_k$.

Combining two results we get $H_{k-1} \leq H_{n-1} - 1 \leq H_k$ or

$$H_{n-1} - 1 \leq H_k \leq H_{n-1} - 1 + \frac{1}{k}.$$

Then we conclude that k is the smallest integer for which $H_k \geq H_{n-1} - 1$.

For large values of n, the approximation $H_n \approx \ln n + \gamma$ yields

$$\ln k + \gamma \geq \ln(n-1) + \gamma - \ln e$$

which gives that $k = \left\lceil \frac{n-1}{e} \right\rceil$.

Exercises

1. Prove that the following equalities hold for any positive integer n:

a) $\sum_{k=1}^{n} \frac{H_k}{k+1} = \frac{1}{2}\left((H_{n+1})^2 + H_{n+1}^{(2)}\right)$,

b) $\sum_{k=1}^{n} \frac{H_k}{k+2} = \frac{1}{2}\left((H_{n+2})^2 + H_{n+3}^{(2)}\right) + \frac{1}{n+2} - 1$,

c) $\sum_{k=1}^{n} \frac{H_k}{k+3} = \frac{1}{2}\left((H_{n+3})^2 + H_{n+3}^{(2)}\right) + \frac{3}{2(n+3)} + \frac{1}{2(n+2)} - \frac{7}{4}$,

d) $\sum_{k=1}^{n} \frac{1}{k(n-k+1)} = \frac{2}{n+1} H_n$.

2. Evaluate

$$\sum_{k=1}^{\infty} \frac{H_k}{2^k}.$$

3. Define the sequence W_n as follows: $W_n = 1$ and for $n > 1$

$$W_n = \begin{cases} W_{n-1} & \text{if} \quad \text{decimal representation of } n \text{ contains at least one even digit} \\ W_{n-1} + \dfrac{1}{n} & \text{if} \quad \text{otherwise} \end{cases}.$$

Show that $W_n < 7$ for any positive integer n.

4. Show that

a) $n(n+1)^{\frac{1}{n}} < n + H_n$ for $n > 1$,

b) $(n-1)n^{-1/(n-1)} < n - H_n$ for $n > 2$.

5. Let m and n be positive integers with $m > n$. Prove that

$$\sum_{k=0}^{2n} \frac{1}{m-n+k} > \frac{2n+1}{m},$$

and deduce that, if $r > 1$ is an integer and $s = \frac{3^r-1}{2}$, then

$$H_r > s.$$

6. Let $\{a_n\}$ be a sequence of distinct positive integers. Prove that for any positive integer n,

$$H_n \leq \sum_{k=1}^{n} \frac{a_k}{k^2}.$$

7. For each positive integer k, let $H_n(k)$ be the smallest harmonic number larger than k and define the k-th excess as $E(k) = H_n(k) - k$.

a) Determine whether the seuqence of excesses is convergent or not.

b) Observe that

$$E(1) = 0.5, \qquad E(3) = 0.0833\cdots, \qquad E(3) = 0.0199\cdots, \qquad E(4) = 0.0272\cdots.$$

It is seen that the sequence of excesses is not monotonically decreasing. Find the smallest $n > 4$ such that $E(n) < E(n+1)$.

8. In single-lane traffic, with no overtaking, each slow car is followed by a bunch of cars wishing to go faster, but unable to do so. If n cars set out, expectedly how many bunches will form?

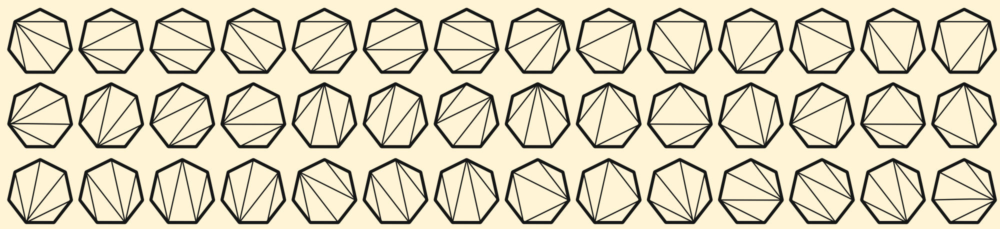
9. Suppose you have a thousand wooden beams and want to find their minimum breaking strain. You build a simple machine which applies a gradually increasing force F to a beam which is supported at its ends in a horizontal position. By increasing the force F until the beam breaks, you can find the breaking strain of each beam. Suppose we denote the breaking strain of the n th beam by F_n. A test to destruction carried out in this way has one major disadvantage. At the end you know the precise value of F_n for every n but have destroyed all the beams in the process. However, we did not actually want the precise value of F_n for every n, only the minimum value of F_n for $1 < n < 1000$. Develop a procedure to find that minimum value by breaking, expectedly, no more than 10 beams.

10. Your aim is to cross 1000 kilometers of desert using a jeep which can carry at most 80 liters of fuel at any time and can travel 5 kilometers of distance on 1 liter of fuel assuming that the jeep's fuel consumption is constant. At the beginning you are at a base where there is an unlimited supply of fuel. We assume that you can deposit fuel in containers at any point along the route for later use. Thus, for example, you can travel 100 kilometers into the desert, drop off 40 liters, and have just enough to get back to the starting point and refill. Now you can make a second trip. When you reach to the drop-off point, you have 60 liters left. You refill from the deposit and go another 100 miles into the desert. There you drop off 40 liters, get back to the deposit point and refill, getting just enough fuel to get back to the base. You now have no fuel at the 100 kilometer mark, but 40 liters at 200 kilometers into the desert. Can you get across the desert, and, if so, how many trips would it take?

3. CATALAN NUMBERS

The Catalan sequence was described in the 18th century by Leonhard Euler, who was interested in the number of different ways of dividing a polygon into triangles. The sequence is named after Eugène Charles Catalan, who discovered the connection to parenthesized expressions during his exploration of the Towers of Hanoi puzzle. The counting trick for Dyck words was found by D. André in 1887.

In 1988, it came to light that the Catalan number sequence had been used in China by the Mongolian mathematician Mingantu by 1730. That is when he started to write his book Ge Yuan Mi Lu Jie Fa, which was completed by his student Chen Jixin in 1774 but published sixty years later. P.J. Larcombe (1999) sketched some of the features of the work of Mingantu, including the stimulus of Pierre Jartoux, who brought three infinite series to China early in the 1700s.



DYCK SEQUENCES

Consider the following binary sequence s which consists of six 0's and six 1's: u_n: 0 1 0 1 0 0 0 1 1 0 1 1. An important property of this sequence is that, starting from the first term, 1's are never on the majority throughout the sequence. In the below table, k th column of third and fourth rows show the number of 0's and 1's appearing in the first k terms of the sequence. The tie in the last column is a direct consequence of the fact that the sequence is balanced. We have another tie on the fourth column and except these ties, at each column, fourth row entry is less than the third row entry.

n	1	2	3	4	5	6	7	8	9	10	11	12
u_n	**0**	**1**	**0**	**1**	**0**	**0**	**0**	**1**	**1**	**0**	**1**	**1**
# 0's	1	1	2	2	3	4	5	5	5	6	6	6
# 1's	0	1	1	2	2	2	2	3	4	4	5	6

The fourth row of the table is in fact the sequence of partial sums $\{S_k\}$ of $\{u_k\}$. Then, above property is equivalent to say that $2S_k \leq k$ for any $k = 1, \dots, 12$.

A finite binary sequence $u_1, \dots, u_n$ is called a ***Dyck*[7] *sequence*** if $2(u_1 + \cdots + u_k) \leq k$ for $k = 1, \dots, n$.

Call a binary sequence a (p, q) sequence where p and q are the numbers of 0's and 1's, respectively. We wish to find the number $C_{p,q}$ of (p, q) Dyck sequences. If $p \leq q$, then $C_{p,q} = 0$, so we naturally assume that $p \geq q$.

THEOREM 3.1. *For any integers $p \geq q \geq 0$, the number of (p, q) Dyck sequences is given by*

$$\frac{p+1-q}{p+1}\binom{p+q}{p}.$$

Proof. Let S be a (p, q) sequence which is not a Dyck sequence. Let t be the smallest index such that among the first t terms, the number of 1's outweighs the number of 0's. If we flip $(0 \leftrightarrow 1)$ the first t terms of the sequence we obtain a $(p+1, q-1)$ sequence.

Conversely, given an arbitrary $(p+1, q-1)$ sequence. Since $p+1 > q-1$, there is a smallest index t such that among the first t terms, the number of 0's outweighs the number of 1's. If we flip $(0 \leftrightarrow 1)$ the first t terms of the sequence we obtain a (p, q) sequence which is not a Dyck sequence.

[7] *Walther Franz Anton von Dyck (1856-1934), German mathematician.*

Combining the above arguments, we see that the set of $(p+1, q-1)$ sequences and the set (p, q) non-Dyck sequences are in 1-1 correspondence. It follows that the number of non-Dyck sequences is $\binom{p+q}{p+1}$. Since the number of all (p, q) sequences is $\binom{p+q}{p}$ we conclude that the number of Dyck sequences is $\binom{p+q}{p} - \binom{p+q}{p+1} = \binom{p+q}{p} - \frac{q}{p+1}\binom{p+q}{p} = \frac{p+1-q}{p+1}\binom{p+q}{p}$. ■

CATALAN NUMBERS

For each positive integer n, the Catalan[8] number $\mathcal{C}_n$ is the number of balanced $(p = q = n)$ Dyck sequences:

$$\mathcal{C}_n = \frac{1}{n+1}\binom{2n}{n}.$$

For $n = 0$, by convention, $\mathcal{C}_0 = 1$.

First few terms of the sequence $\{\mathcal{C}_n\}$ are $1, 1, 2, 5, 14, 42, 132, 429, 1430, 4862, \cdots$.

LEMMA 3.2. *For $n \geq 0$, Catalan numbers satisfy the recursion*

$$\mathcal{C}_{n+1} = \frac{2(2n+1)}{n+2}\,\mathcal{C}_n.$$

Proof. It follows from the definition that

$$\begin{aligned}\mathcal{C}_{n+1} &= \frac{1}{n+2}\binom{2n+2}{n+1} \\ &= \frac{1}{n+2}\cdot\frac{(2n+2)(2n+1)}{(n+1)^2}\binom{2n}{n}.\end{aligned}$$

Then proof follows. ■

[8] *Eugène Charles Catalan (1814 –1894, French and Belgian mathematician.*

From the theorem it immediately follows that

$$\lim_{n\to\infty}\left(\frac{\mathcal{C}_n}{\mathcal{C}_{n+1}}\right)=4.$$

We observe another interesting recursion satisfied by the first terms of the sequence.

$$\mathcal{C}_1=\mathcal{C}_0\mathcal{C}_0=1\cdot 1=1,$$

$$\mathcal{C}_2=\mathcal{C}_0\mathcal{C}_1+\mathcal{C}_1\mathcal{C}_0=1\cdot 1+1\cdot 1=2,$$

$$\mathcal{C}_3=\mathcal{C}_0\mathcal{C}_2+\mathcal{C}_1\mathcal{C}_1+\mathcal{C}_2\mathcal{C}_0=1\cdot 2+1\cdot 1+2\cdot 1=5,$$

$$\mathcal{C}_4=\mathcal{C}_0\mathcal{C}_3+\mathcal{C}_1\mathcal{C}_2+\mathcal{C}_2\mathcal{C}_1+\mathcal{C}_3\mathcal{C}_0=1\cdot 5+1\cdot 2+2\cdot 1+5\cdot 1=14,$$

$$\vdots$$

In fact, above property is satisfied by all the terms of the sequence.

THEOREM 3.3. *The sequence* $\{\mathcal{C}_n\}$ *of Catalan numbers satisfies the recursion*

$$\mathcal{C}_n=\mathcal{C}_0\mathcal{C}_{n-1}+\mathcal{C}_1\mathcal{C}_{n-1}+\mathcal{C}_2\mathcal{C}_{n-2}+\cdots+\mathcal{C}_{n-1}\mathcal{C}_0=\sum_{k=0}^{n-1}\mathcal{C}_k\mathcal{C}_{n-1-k}$$

for $n>0$.

Proof. Let $s_1, s_2, \cdots s_{2n}$ be a (n,n) Dyck sequence and let $S_1, S_2, \ldots, S_{2n}$ be the sequence of partial sums. Necessarily $S_1=s_1=0$ and $S_{2n}=n$. Let $2t$ be the smallest index for which $S_{2t}=t$. Note that it may happen that $t=n$. When $S_{2t}=t$, it is certain that $s_{2t}=1$. Then the sequence is of the form

$$0\,, s_2\,, \cdots, s_{2t-1}\,, 1\,, s_{2t+1}\,, \cdots, s_{2n.}$$

Subsequences $s_2\cdots s_{2t-1}$ and $s_{2t+1}\ \cdots\ s_{2n}$ are both Dyck sequences of lengths $2t-2$ and $2n-2t$ respectively. The number of such sequences is $\mathcal{C}_{t-1}\mathcal{C}_{n-t}$ for $t=1,2,\ldots,n$. Sum of all such terms is the right hand side of the given equality and this completes the proof. ■

COROLLARY 3.4. *If* $a_0, a_1, \ldots, a_n, \ldots$ *is a sequence such that* $a_n=\sum_{k=0}^{n-1}a_k a_{n-1-k}$ *for* $n>0$, *then the general term is given by* $a_n=\mathcal{C}_n a_0^{n+1}$.

Proof. (Induction on n) It is clear that the assertion holds for $n = 1$: $a_1 = a_0^2$. Now assume that $a_n = C_n a_0^{n+1}$ for all $n \leq n_0$ for some integer $n_0 \geq 1$. Then

$$\begin{aligned} a_{n_0+1} &= \sum_{k=0}^{n_0} a_k a_{n_0-k} \\ &= \sum_{k=0}^{n_0} C_k a_0^{k+1} \cdot C_{n_0-k} a_0^{n_0-k+1} \\ &= a^{n_0+2} \sum_{k=0}^{n_0-1} C_k C_{n_0-1-k}. \end{aligned}$$

Since $\sum_{k=0}^{n_0-1} C_k C_{n_0-1-k} = C_{n_0}$, claim follows. ■

2THEOREM 3.5. *Generating function of the sequence of Catalan numbers is*

$$C(x) = \frac{1-\sqrt{1-4x}}{2x}.$$

Proof. Let $\{w_n\}$ be the convolution of $\{C_n\}$ with itself: $\{w_n\} = \{C_n \circ C_n\}$. Since $w_k = \sum_{i=0}^{k} C_i C_{k-i}$, $k = 1,2,\ldots$ generating function of $\{w_n\}$ is

$$\begin{aligned} w(x) &= \sum_{k=0}^{\infty} w_k x^k \\ &= \sum_{k=0}^{\infty} \left(\sum_{i=0}^{k} C_i C_{k-i} \right) x^k. \end{aligned}$$

Using the previous theorem, we see that

$$\begin{aligned} w(x) &= \sum_{k=0}^{\infty} C_{k+1} x^k \\ &= \sum_{k=1}^{\infty} C_k x^{k-1} \\ &= (\,C(x) - 1) x^{-1} \end{aligned}$$

where $C(x) = 1 + C_1 x + C_2 x^2 + \cdots$ is the generating function of $\{C_n\}$. Since the generating function of a convolution of two sequences is the product of the generating functions of convoluted sequences we have

$$w(x) = C^2(x).$$

Equating the two expressions for $w(x)$, we obtain

$$xC^2(x) - C(x) + 1 = 0$$

which can be solved for $C(x)$ to give $2x\,C(x) = \left(1 \pm \sqrt{1-4x}\right)$. The condition $C(0) = 1$ forces us to choose the negative square root so that

$$C(x) = \frac{1 - \sqrt{1-4x}}{2x}.$$

Alternative proof. Recall that

$$\binom{1/2}{k} = \frac{\frac{1}{2} \cdot \left(-\frac{1}{2}\right) \cdot \left(-\frac{3}{2}\right) \cdots \left(-\frac{2k-3}{2}\right)}{k!}$$

$$= (-1)^{k-1} \frac{1 \cdot 3 \cdots (2k-3)}{2^k \cdot k!} \cdot \frac{2 \cdot 4 \cdot 6 \cdots (2k-2)}{2 \cdot 4 \cdot 6 \cdots (2k-2)}$$

$$= (-1)^{k-1} \frac{(2k-2)!}{k!\, 2^{2k-1}\, (k-1)!}$$

$$= \frac{(-1)^{k-1}}{4^k} \cdot \frac{2}{k} \binom{2k-2}{k-1} = \frac{(-1)^{k-1}}{4^k} \cdot 2\, C_{k-1}$$

so that

$$(1-4x)^{\frac{1}{2}} = \sum_{k=0}^{\infty} \binom{1/2}{k} (-4)^k x^k$$

$$= 1 + \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{4^k} \cdot 2\, C_{k-1} (-4)^k x^k$$

$$= 1 - 2 \sum_{k=1}^{\infty} C_{k-1} x^k.$$

Then,

$$\frac{1 - \sqrt{1-4x}}{2x} = \sum_{k=1}^{\infty} C_{k-1} x^{k-1}.$$

Thus we obtain the generating function $C(x)$. ■

COROLLARY 3.6. *Generating function of the sequence* $\{\binom{2n}{n}\}$ *is* $(1-4x)^{-\frac{1}{2}}$.

Proof. In the proof of above theorem, we have obtained that $(1-4x)^{\frac{1}{2}} = 1 - 2\sum_{k=1}^{\infty} C_{k-1}x^k$. Differentiating both sides of the equality we have

$$\begin{aligned} -2(1-4x)^{-\frac{1}{2}} &= -2\sum_{k=1}^{\infty} kC_{k-1}x^{k-1} \\ &= -2\sum_{k=0}^{\infty} (k+1)\, C_k x^k \\ &= -2\sum_{k=0}^{\infty} \binom{2k}{k} x^k. \quad \blacksquare \end{aligned}$$

An Approximation for C_n

Using the Stirling approximation $n! = \left(\frac{n}{e}\right)^n \sqrt{2\pi n}$ for $n!$, one obtains $\binom{2n}{n} \approx \frac{2^{2n}}{\sqrt{\pi n}}$

and

$$C_n \approx \frac{2^{2n}}{(n+1)\sqrt{\pi n}} \approx \frac{4^n}{n^{3/2}\sqrt{\pi}}.$$

Table of Catalan Numbers

Below table is a list of C_n and its approximations for $n = 0,1,2,\ldots,16$.

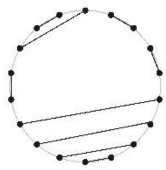
n	C_n	$\frac{2^{2n}}{(n+1)\sqrt{\pi n}}$	$\frac{4^n}{n^{3/2}\sqrt{\pi}}$.
0	1		
1	1	1,13	2,26
2	2	2,13	3,19
3	5	5,21	6,95
4	14	14,44	18,05
5	42	43,06	51,67
6	132	134,78	157,24
7	429	436,72	499,11
8	1.430	1.452,51	1.634,07
9	4.862	4.929,96	5.477,74
10	16.796	17.007,18	18.707,90
11	58.786	59.457,60	64.862,84
12	208.012	210.189,50	227.705,30
13	742.900	750.075,90	807.774,10
14	2.674.440	2.698.421	2.891.165
15	9.694.845	9.775.958	10.427.688
16	35.357.670	35.634.938	37.862.122

Problems Involving Catalan Numbers

There are many interesting problems whose solutions involve Catalan numbers. To give some examples, they are related with the number of binary bracketings of n letters, the ballot problem, the number of trivalent planted planar trees, the number of states possible in an n-flexagon, the number of different diagonals possible in a frieze pattern, the number of Dyck paths with n strokes, the number of ways of forming an n-fold exponential, the number of rooted planar binary trees with n internal nodes, the number of rooted plane bushes with n graph edges, the number of extended binary trees with n internal nodes, and the number of mountains which can be drawn with n upstrokes and n downstrokes, the number of noncrossing handshakes possible across a round table between n pairs of people. The book *Enumerative Combinatorics: Volume 2,* by Richard P. Stanley describes many different interpretations of the Catalan numbers.

Here we give several examples of such questions. Exercises at the end of this chapter also consider such problems.

Problem 1. There are $2n$ distinct points given on a circle. If no pair of chords intersect in or on the circle, find the number of drawing n chords whose end points are the given points.



in the clockwise sense, label each point either 0 or 1. Assign a 0 to the first point. For the following points, if the point is end point of a chord whose other end is already has been labeled assign 1, otherwise assign 0. In this manner we obtain a balanced Dyck sequence. It is not difficult to show that the drawings and labelings are in one-to-one correspondence. It follows that the number of such drawings is C_n.

Problem 2. Find the number of ways of distributing 20 balls to 20 labeled boxes such that the total number of balls in the first k boxes is at least k for $k = 1, \ldots, 20$.

Assume that the balls are distributed in the desired manner. Represent the k-th box with the array $u_k = 0 \cdots 01$ where the number of zeroes is equal to the number of balls in that box. Then the concatenation $u_1 \parallel \cdots \parallel u_{20}$ of these arrays is a balanced Dyck array. The number of distributions is C_n.

Problem 3. Find the number of ways of arranging the integers $1,2, \ldots, 2n$ as a $2 \times n$ array such that each row is in ascending order and k th term of the first row is larger than the k th term of second row for $k = 1,2, \ldots, n$.

Assume that we are given an arrangement satisfying the required conditions. Define the sequence $s_1, \dots, s_{2n}$ such that $s_i = a$ if i is in the first array or $s_i = b$ otherwise. Then necessarily, $\{s_n\}$ is a Dyck sequence. Then the arrangement given below

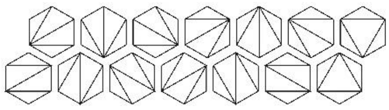
2	5	6	8	11	12
1	3	4	7	9	10

corresponds to the sequence $\{s_n\}$ as follows:

n	1	2	3	4	5	6	7	8	9	10	11	12
s_n	b	a	b	b	a	a	b	a	b	b	a	a

It follows that the number of such arrangements is C_n.

Problem 4. A triangulation of a plane region is a partition of the region into pairwise non-intersecting triangles. Find the number of all triangulations a convex n-gon by non-intersecting diagonals. The following hexagons illustrate the case $n = 6$.



Let T_n be the number of triangulations of a convex n-gon $n = 4,5, \dots$. It is obvious that $T_4 = 2$, $T_5 = 5, \dots$. By convention we set $T_2 = T_3 = 1$. Assume that $A_1 A_2 \cdots A_n$ is a given convex n-gon. In each triangulation of the n-gon, the side A_1A_2 will be a side of the triangle $A_1A_2A_k$ for some $k = 3, \dots, n$.

If $k = 3$, then n-gon is subdivided into two regions: a triangle $(A_1A_2A_3)$ and a $(n-1)$-gon $(A_1A_3A_4 \cdots A_n)$. Since the $(n-1)$-gon can be triangulated in T_{n-1} ways, $A_1A_2A_3$ appears in T_{n-1} triangulations.

If $k = 4, \dots, n-1$, then n-gon is subdivided into three regions: a $k-1$-gon $(A_2A_3 \cdots A_k)$, a triangle $(A_1A_2A_k)$ and a $(n-k+2)$-gon $(A_1A_k \dots A_n)$. Therefore the $A_1A_2A_k$ appears in $T_{k-1} \cdot T_{n-k+2}$ triangulations.

If $k = n$, then n-gon is subdivided into two regions: a $(n-1)$-gon $(A_2A_3 \cdots A_n)$ and a triangle $(A_1A_2A_n)$. Then, $A_1A_2A_n$ appears in T_{n-1} triangulations.

Since the cases for $k = 3, \dots, n$ are all pairwise disjoint, the number of all triangulations of the n-gon is the sum of numbers triangulations in all these cases.

Then we obtain the recursion

$$T_n = T_{n-1} + T_3T_{n-2} + T_4T_{n-3} + \cdots + T_{n-3}T_4 + T_{n-2}T_3 + T_{n-1}.$$

Using the convention $T_2 = 1$, the recursion can be written as

$$T_n = T_2T_{n-1} + T_3T_{n-2} + T_4T_{n-3} + \cdots + T_{n-3}T_4 + T_{n-2}T_3 + T_{n-1}T_2.$$

or

$$T_{n+2} = T_2T_{n+1} + T_3T_n + T_4T_{n-1} + \cdots + T_{n-1}T_4 + T_nT_3 + T_{n+1}T_2.$$

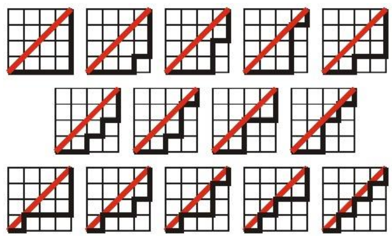
Now, substitution $C_n = T_{n+2}$ leads to write

$$C_n = C_0C_{n-1} + C_1C_{n-2} + C_2C_{n-3} + \cdots + C_{n-3}C_2 + C_{n-2}C_1 + C_{n-1}C_0.$$

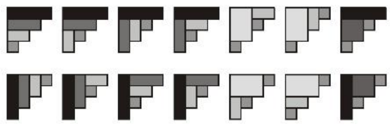
The last recursion we have obtained is the recursion satisfied by Catalan numbers. Since $C_0 = T_2 = 1 = \mathcal{C}_0$, we conclude that $C_n = \mathcal{C}_0$. It follows that the number of possible triangulations is $T_n = \mathcal{C}_{n-2}$.

Exercises

1. p students, each one having a single 10 TL banknote and q students, each one having a single 20 TL banknote, form a queue in front of a counter to buy 10 TL worth tickets. Each student is to buy only one ticket. At the beginning there are no banknotes at the counter. The counter quits selling tickets once he faces with a 20 TL banknote when he has no change to re-pay 10 TL. If we call a queue, 'good queue' if all the students buy tickets, find the number of good queues.
2. Suppose we have an election between two candidates and the ballots are counted one-by-one. Further suppose that the first candidate is never behind (she's always ahead or tied), but that the final count ends in a tie with each candidate getting n votes. How many ways can this happen?
3. Let $*$ be a non-associative binary operation defined on a set X. For $x_1, x_2, \dots, x_n \in X$, find the number of distinct ways of performing the operation $x_1 * x_2 * \cdots * x_n$.
4. Find the number of expressions containing n pairs of parentheses which are correctly matched: ((())) ()(()) ()()() (())() (()()).
5. A monotonic path along the edges of a grid with $n \times n$ square cells is one which starts in the lower left corner, finishes in the upper right corner, and consists entirely of edges pointing rightwards or upwards. Find the number of monotonic paths which do not pass above the diagonal. The following diagrams show the case $n = 4$



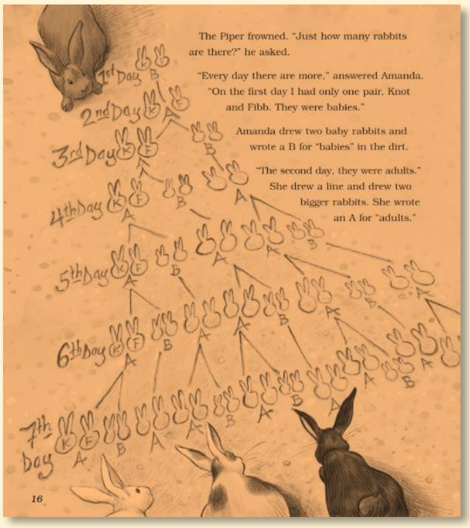
6. Find the number of ways to tile a stair step shape of height n with n rectangles. The following figure illustrates the case $n = 4$.



4. FIBONACCI NUMBERS

Fibonacci or Leonard of Pisa (1170-1250), played an important role in reviving ancient mathematics while making significant contributions of his own. He traveled to North Africa, Egypt, Syria, recognizing the advantages of the mathematical systems used in these countries.

Liber Abaci, published in 1202 after his return to Italy, is based on bits of arithmetic and algebra that Leonardo had accumulated during his travels. The Liber Abbaci introduced the Hindu-Arabic place-valued decimal system and the use of Arabic numerals into Europe. In Liber Abaci the following problem is posed: *How many pairs of rabbits will be produced in a year, beginning with a single pair, if in every month each pair bears a new pair which becomes productive from the second month on?*



Today the solution to this problem is known as the Fibonacci sequence, or Fibonacci numbers. A search of the Internet for "Fibonacci" will find dozens of Web sites and hundreds of pages of material. There is even a Fibonacci Association that publishes a scholarly journal, the Fibonacci Quarterly.

FIBONACCI'S RABBITS

The problem posed in Liber Abaci of Fibonacci can be formulated as follows. Beginning with a pair of new born rabbits, how many pairs of rabbits could be reached in a year assuming that

– each rabbit reaches maturity after one month,
– the gestation period of a rabbit is one month,
– each mature female rabbit gives birth to one male and one female rabbit every month,
– no rabbits die during the year.

After one month, the first pair is not mature and can't mate. At two months, the rabbits have mated but not yet given birth, resulting in only one pair of rabbits. After three months, the first pair will give birth to another pair, resulting in two pairs. At the fourth month mark, the original pair gives birth again, and the second pair mates but does not yet give birth, leaving the total at three pairs. This continues until a year has passed. Say that the number of newborn and mature pairs at month n are $\mathfrak{b}_n$ and $\mathfrak{m}_n$, respectively. Next month all these rabbits will be mature, i.e., $\mathfrak{m}_{n+1} = \mathfrak{m}_n + \mathfrak{b}_n$. For each mature pair of nth month, there will be a pair of newborn pair in next month: $\mathfrak{b}_{n+1} = \mathfrak{m}_n$. Now we have $\mathfrak{m}_{n+1} + \mathfrak{b}_{n+1} = (\mathfrak{m}_n + \mathfrak{b}_n) + \mathfrak{m}_n = (\mathfrak{m}_n + \mathfrak{b}_n) + (\mathfrak{m}_{n-1} + \mathfrak{b}_{n-1})$. Denoting the number of all rabbits in the nth month by $\mathfrak{r}_n$, that is, $\mathfrak{r}_n = \mathfrak{m}_n + \mathfrak{b}_n$, we get the relation $\mathfrak{r}_{n+1} = \mathfrak{r}_n + \mathfrak{r}_{n-1}$ for $n = 1,2,3,\ldots$. Since we have started with a pair of new born pair $\mathfrak{r}_1 = 1$. One month later we have a pair of mature rabbits, thus $\mathfrak{r}_2 = 1$. Then the remaining terms of the sequence $\{\mathfrak{r}_n\}$ can be computed as $1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233$. One year later there will be 233 pairs of rabbits.

FIBONACCI SEQUENCE

Let $\{a_n\}$ be a solution of the recursion

$$a_{n+2} = a_{n+1} + a_n. \tag{1}$$

We can write $\sum_{k=0}^{\infty}(a_{n+2} - a_{n+1} - a_n)\, x^{n+2} = 0$. By letting $a(x) = a_0 + a_1x + a_2x^2 + \cdots$ to be the generating function of $\{a_n\}$ we see that $a(x) - a_1x - a_0 - xa(x) + xa_0 - x^2a(x) = 0$ which gives

$$a(x) = \frac{a_0 + (a_1 - a_0)x}{1 - x - x^2}. \tag{2}$$

Now, factorization $(1 - \varphi x)(1 - \psi x)$ of $1 - x - x^2$, where $\varphi = \frac{1+\sqrt{5}}{2}$ (*) *and* $\psi = \frac{1-\sqrt{5}}{2}$ yields

$$\begin{aligned} a(x) &= \frac{1}{\sqrt{5}}\left(\frac{a_1 - \psi a_0}{1 - \varphi x} + \frac{\varphi a_0 - a_1}{1 - \varphi x}\right) \\ &= \frac{1}{\sqrt{5}}(a_1 - \psi a_0)\sum_{k=0}^{\infty}(\varphi x)^k + \frac{1}{\sqrt{5}}(\varphi a_0 - a_1)\sum_{k=0}^{\infty}(\psi x)^k \\ &= \frac{1}{\sqrt{5}}\sum_{k=0}^{\infty}\left((a_1 - \psi a_0)\varphi^k + (\varphi a_0 - a_1)\psi^k\right) x^k. \end{aligned}$$

Thus, the general term of $\{a_n\}$ is

$$a_n = \frac{1}{\sqrt{5}}\left((a_1 - \psi a_0)\varphi^n + (\varphi a_0 - a_1)\psi^n\right). \tag{3}$$

(*) $\varphi = \frac{1+\sqrt{5}}{2}$=1.618033⋯ *is known as the **golden ratio**.*

Fibonacci sequence $\{\mathfrak{f}_n\}$ is the solution of (1) with the initial values $\mathfrak{f}_0 = 0,\ \mathfrak{f}_1 = 1$. Each term of this sequence is called a ***Fibonacci number***. First few Fibonacci numbers are: $0, 1, 1,\ 2,\ 3,\ 5,\ 8,\ 13,\ 21,\ 34,\ 55,\ 89,\ 144,\ 233,\ 377,\ 610,\ 987, \cdots$. From (3), the general term of this sequence is

$$\mathfrak{f}_n = \frac{1}{\sqrt{5}}(\varphi^n - \psi^n).$$

This expression for the general term is known as ***Binet's formula***[9]. From (2) we obtain the generating function of $\{\mathfrak{f}_n\}$ as

$$\mathfrak{f}(x) = \frac{x}{1 - x - x^2}.$$

For $n \leq 1$, by writing the basic recursion as $\mathfrak{f}_{n-2} = \mathfrak{f}_n - \mathfrak{f}_{n-1}$, Fibonacci numbers $\mathfrak{f}_{-1}, \mathfrak{f}_{-2}, \dots$ with negative indices can be defined:

$$\text{For } n = 1,\quad \mathfrak{f}_{-1} = \mathfrak{f}_1 - \mathfrak{f}_0 = 1$$

$$\text{For } n = 0,\quad \mathfrak{f}_{-2} = \mathfrak{f}_0 - \mathfrak{f}_{-1} = -1$$

$$\text{For } n = -1, \mathfrak{f}_{-3} = \mathfrak{f}_{-1} - \mathfrak{f}_{-2} = 2$$

$$\text{For } n = -2, \mathfrak{f}_{-4} = \mathfrak{f}_{-2} - \mathfrak{f}_{-3} = -3$$

$$\text{For } n = -3, \mathfrak{f}_{-5} = \mathfrak{f}_{-3} - \mathfrak{f}_{-4} = 5$$

The resulting 'bidirectional' sequence $\{\mathfrak{f}_n\}_{n=-\infty}^{\infty}$ is

$$\cdots, -55, 34, -21, 13, -8, 5, -3, 2, -1, 1, 0, 1, 1, 2, 3, 5, \cdots$$

It can be shown that

$$\mathfrak{f}_{-n} = (-1)^{n+1}\mathfrak{f}_n\,.$$

Binet's formula holds for negative indices as well.

PROPOSITION 4.1. *General term of the solution* $\{a_n\}$ *of* (1) *with initial terms* a_0, a_1 *is*

$$a_n = a_1\mathfrak{f}_n + a_0\mathfrak{f}_{n-1}.$$

[9] *Jacques Philippe Marie Binet (1786 –1856), French mathematician, physicist and astronomer.*

Proof. From (3) we obtain

$$\begin{aligned} a_n &= \frac{1}{\sqrt{5}}\big((a_1 - \psi a_0)\varphi^n + (\varphi a_0 - a_1)\psi^n\big) \\ &= \frac{1}{\sqrt{5}}(a_1(\varphi^n - \psi^n) + a_0(-\psi\varphi^n + \varphi\psi^n)) \\ &= \frac{1}{\sqrt{5}}(a_1(\varphi^n - \psi^n) + a_0(\varphi^{n-1} - \psi^{n-1}))\,. \end{aligned}$$

Now the claim follows from Binet's formula. ■

Example 1 (*Lucas*[10] *Numbers*). The solution $\{\mathcal{L}_n\}$ of (1) with initial conditions $\mathcal{L}_0 = 2,\ \mathcal{L}_1 = 1$ is called the *Lucas sequence.* Then, the general term of this sequence is

$$\begin{aligned} \mathcal{L}_n &= \mathfrak{f}_n + 2\mathfrak{f}_{n-1} \\ &= \frac{1}{\sqrt{5}}(\varphi^n + 2\varphi^{n-1} - \psi^n - 2\psi^{n-1}) \\ &= \frac{1}{\sqrt{5}}\big(\varphi^{n-1}(\varphi + 2) - \psi^{n-1}(\psi + 2)\big) \end{aligned}$$

Note that $\varphi + 2 = \sqrt{5}\,\varphi$ and $\psi + 2 = -\sqrt{5}\,\psi$. Then

$$\mathcal{L}_n = \varphi^n + \psi^n.$$

Generating function of the Lucas sequence is $\mathcal{L}(x) = \frac{2-x}{1-x-x^2}$. Each term of this sequence is called a ***Lucas number***. First few Lucas numbers are:

$$2, 1, 3, 4, 7, 11, 18, 29, 47, 76, 123, 199, 322, 521, 843, 1364, 2207, \cdots$$

Matrix Representation of Fibonacci Numbers

For any integer n, basic recursion of Fibonacci sequence can be written as

$$\begin{pmatrix} \mathfrak{f}_{n+1} \\ \mathfrak{f}_n \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \mathfrak{f}_n \\ \mathfrak{f}_{n-1} \end{pmatrix}.$$

[10] *François Édouard Anatole Lucas (1842 –1891), French mathematician.*

It follows that

$$\begin{pmatrix} \mathfrak{f}_{n+1} \\ \mathfrak{f}_n \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}^n \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

It is easy to observe that

$$\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}^n = \begin{pmatrix} \mathfrak{f}_{n+1} & \mathfrak{f}_n \\ \mathfrak{f}_n & \mathfrak{f}_{n-1} \end{pmatrix}.$$

Basic Identities

In this section we prove that the following equalities hold for all integers n, m and r.

$$\mathfrak{f}_{n+m} = \mathfrak{f}_{n+1}\mathfrak{f}_m + \mathfrak{f}_n\mathfrak{f}_{m-1}$$

$$\mathfrak{f}_{2n} = \mathfrak{f}_{n+1}\mathfrak{f}_n + \mathfrak{f}_n\mathfrak{f}_{n-1}$$

$$\mathfrak{f}_{2n+1} = \mathfrak{f}_{n+1}^2 + \mathfrak{f}_n^2$$

$$\mathfrak{f}_n^2 - \mathfrak{f}_{n+1}\mathfrak{f}_{n-1} = (-1)^{n-1} \qquad \textit{(Cassini's[11] Identity)}$$

$$\mathfrak{f}_n^2 - \mathfrak{f}_{n+r}\mathfrak{f}_{n-r} = (-1)^{n-r}\mathfrak{f}_r^2 \qquad \textit{(Catalan's Identity)}$$

$$\mathfrak{f}_{m+1}\mathfrak{f}_n - \mathfrak{f}_m\mathfrak{f}_{n+1} = (-1)^n\mathfrak{f}_{m-n}. \qquad \textit{(D'Ocagne's[12] Identity)}$$

THEOREM 4.2. *For any integers m and n, Fibonacci numbers satisfy the following:*

$$\mathfrak{f}_{n+m} = \mathfrak{f}_{n+1}\mathfrak{f}_m + \mathfrak{f}_n\mathfrak{f}_{m-1}.$$

Proof. Since for any integers n and m, $A^nA^m = A^{n+m}$ for any square matrix A, we can write $\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}^{n+m} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}^n \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}^m$ which is equivalent to write

$$\begin{pmatrix} \mathfrak{f}_{n+m+1} & \mathfrak{f}_{n+m} \\ \mathfrak{f}_{n+m} & \mathfrak{f}_{n+m-1} \end{pmatrix} = \begin{pmatrix} \mathfrak{f}_{n+1} & \mathfrak{f}_n \\ \mathfrak{f}_n & \mathfrak{f}_{n-1} \end{pmatrix} . \begin{pmatrix} \mathfrak{f}_{m+1} & \mathfrak{f}_m \\ \mathfrak{f}_m & \mathfrak{f}_{m-1} \end{pmatrix}$$

$$= \begin{pmatrix} \mathfrak{f}_{n+1}\mathfrak{f}_{m+1} + \mathfrak{f}_n\mathfrak{f}_m & \mathfrak{f}_{n+1}\mathfrak{f}_m + \mathfrak{f}_n\mathfrak{f}_{m-1} \\ \mathfrak{f}_n\mathfrak{f}_{m+1} + \mathfrak{f}_{n-1}\mathfrak{f}_m & \mathfrak{f}_n\mathfrak{f}_m + \mathfrak{f}_{n-1}\mathfrak{f}_{m-1} \end{pmatrix}.$$

Comparing the corresponding entries of these matrices, desired identity follows. ■

[11] Giovanni Domenico Cassini (1625 –1712), Italian mathematician,astronomer, astrologer and engineer.

[12] Philbert Maurice d'Ocagne (1862 –1938), French engineer and mathematician.

COROLLARY 4.3. *For any integer n, the following equalities hold true:*

$$\mathfrak{f}_{2n} = \mathfrak{f}_{n+1}^2 - \mathfrak{f}_{n-1}^2,$$
$$\mathfrak{f}_{2n+1} = \mathfrak{f}_{n+1}^2 + \mathfrak{f}_n^2 .$$

Proof. In the theorem substitute $m = n$ to obtain the first identity:

$$\mathfrak{f}_{2n} = \mathfrak{f}_{n+1}\mathfrak{f}_n + \mathfrak{f}_n\mathfrak{f}_{n-1} = \mathfrak{f}_n(\mathfrak{f}_{n+1} + \mathfrak{f}_{n-1}) = (\mathfrak{f}_{n+1} - \mathfrak{f}_{n-1})(\mathfrak{f}_{n+1} + \mathfrak{f}_{n-1}) = \mathfrak{f}_{n+1}^2 - \mathfrak{f}_{n-1}^2.$$

The second identity is obtained directly from the theorem by substituting $m = n + 1$. ■

THEOREM 4.4 (*Cassini's Identity*). *For any integer n, Fibonacci numbers satisfy the following identity:*

$$\mathfrak{f}_n^2 - \mathfrak{f}_{n+1}\mathfrak{f}_{n-1} = (-1)^{n-1}.$$

Proof. Result directly follows from

$$\det\begin{pmatrix} \mathfrak{f}_{n+1} & \mathfrak{f}_n \\ \mathfrak{f}_n & \mathfrak{f}_{n-1} \end{pmatrix} = \mathfrak{f}_{n+1}\mathfrak{f}_{n-1} - \mathfrak{f}_n^2 = \det\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}^n = (-1)^n. \quad ■$$

LEMMA 4.5. *For any integer m, the following equality holds:*

$$\mathfrak{f}_{m+1}^2 - \mathfrak{f}_m\mathfrak{f}_{m+1} - \mathfrak{f}_m^2 = (-1)^m.$$

Proof.

$$\begin{aligned} f_{m+1}^2 - \mathfrak{f}_m\mathfrak{f}_{m+1} - \mathfrak{f}_m^2 &= (\mathfrak{f}_{m+1}+\mathfrak{f}_m)(\mathfrak{f}_{m+1} - \mathfrak{f}_m) - \mathfrak{f}_m\mathfrak{f}_{m+1} \\ &= \mathfrak{f}_{m+1}\mathfrak{f}_{m-1} + \mathfrak{f}_m\mathfrak{f}_{m-1} - \mathfrak{f}_m\mathfrak{f}_{m+1} \\ &= \mathfrak{f}_m^2 + (-1)^n - \mathfrak{f}_m(\mathfrak{f}_{m+1} - \mathfrak{f}_{m-1}) \\ &= \mathfrak{f}_m^2 + (-1)^m - \mathfrak{f}_m^2 \\ &= (-1)^m. \end{aligned}$$

The claim follows. ■

The following is a generalization of Cassini's identity.

THEOREM 4.6 (*Catalan's Identity*). *For all integers n and r the following identity is true.*

$$\mathfrak{f}_n^2 - \mathfrak{f}_{n+r}\mathfrak{f}_{n-r} = (-1)^{n-r}\mathfrak{f}_r^2.$$

Proof. For arbitrary integers m and a we have

$$\begin{aligned} \mathfrak{f}_{m+a}^2 - \mathfrak{f}_{m+2a}\mathfrak{f}_m &= (\mathfrak{f}_{m+1}\mathfrak{f}_a + \mathfrak{f}_m\mathfrak{f}_{a-1})^2 - (\mathfrak{f}_{m+1}\mathfrak{f}_{2a} + \mathfrak{f}_m\mathfrak{f}_{2a-1})\mathfrak{f}_m \\ &= \mathfrak{f}_{m+1}^2\mathfrak{f}_a^2 + f_m^2\mathfrak{f}_{a-1}^2 + 2\mathfrak{f}_{m+1}\mathfrak{f}_a\mathfrak{f}_m\mathfrak{f}_{a-1} \\ &\quad -\mathfrak{f}_m[\mathfrak{f}_{m+1}(\mathfrak{f}_{a+1}\mathfrak{f}_a + \mathfrak{f}_a\mathfrak{f}_{a-1}) + \mathfrak{f}_m(\mathfrak{f}_a^2 + \mathfrak{f}_{a-1}^2)] \\ &= \mathfrak{f}_{m+1}^2\mathfrak{f}_a^2 + \mathfrak{f}_{m+1}\mathfrak{f}_a\mathfrak{f}_m\mathfrak{f}_{a-1} - \mathfrak{f}_m\mathfrak{f}_{m+1}\mathfrak{f}_a\mathfrak{f}_{a+1} - \mathfrak{f}_m^2\mathfrak{f}_a^2 \end{aligned}$$

$$= \mathfrak{f}_{m+1}^2\mathfrak{f}_a^2 - \mathfrak{f}_m\mathfrak{f}_{m+1}\mathfrak{f}_a^2 + \mathfrak{f}_m^2\mathfrak{f}_a^2$$
$$= (\mathfrak{f}_{m+1}^2 - \mathfrak{f}_m\mathfrak{f}_{m+1} - \mathfrak{f}_m^2)\mathfrak{f}_a^2$$
$$= (-1)^m\,\mathfrak{f}_a^2.$$

Now substitute $m = n - r$ and $r = a$. ■

THEOREM 4.7 (*d'Ocagne's Identity*). *For all integers m and n the following identity is true.*

$$\mathfrak{f}_m\mathfrak{f}_{n+1} - \mathfrak{f}_{m+1}\mathfrak{f}_n = (-1)^n\mathfrak{f}_{m-n}.$$

Proof.

$$\mathfrak{f}_m\mathfrak{f}_{n+1} - \mathfrak{f}_{m+1}\mathfrak{f}_n = \frac{1}{\sqrt{5}}[(\varphi^m - \psi^m)(\varphi^{n+1} - \psi^{n+1}) - (\varphi^{m+1} - \psi^{m+1})(\varphi^n - \psi^n)]$$
$$= \frac{1}{\sqrt{5}}(-\varphi^m\psi^{n+1} - \psi^m\varphi^{n+1} + \varphi^{m+1}\psi^n + \psi^{m+1}\varphi^n)$$
$$= \frac{1}{\sqrt{5}}[\varphi^m\psi^n(\varphi - \psi) - \varphi^n\psi^m(\varphi - \psi)]$$
$$= \frac{1}{\sqrt{5}}\varphi^n\psi^n[\varphi^{m-n} - \psi^{m-n}]$$

Now the result follows from Binet's formula and the fact that $\varphi\psi = -1$. ■

DECIMATED SUBSEQUENCES

In this section we investigate the properties of decimated subsequences of $\{\mathfrak{f}_n\}$. Since a regular decimation of a sequence does not increase the linear complexity, any decimated subsequence of $\{\mathfrak{f}_n\}$ has linear complexity at most 2.

Example 2. Let $\{\mathfrak{g}_n\}$ be the 2 −decimated subsequence of $\{\mathfrak{f}_n\}$, that is $\mathfrak{g}_n = \mathfrak{f}_{2n}, n = 1,2, \ldots$. If $\mathfrak{g}(x)$ is the generating function of $\{\mathfrak{g}_n\}$, then

$$\mathfrak{g}(x) = \frac{1}{2}[\mathfrak{f}(\sqrt{x}) + \mathfrak{f}(-\sqrt{x})]$$
$$= \frac{1}{2}\left(\frac{\sqrt{x}}{1 - \sqrt{x} - x} + \frac{-\sqrt{x}}{1 + \sqrt{x} - x}\right)$$
$$= \frac{x}{1 - 3x - x^2}.$$

It follows that $\mathfrak{g}_{n+2} = 3\mathfrak{g}_{n+1} - \mathfrak{g}_n$.

LEMMA 4.8. *For any positive integer k, Fibonacci numbers satisfy the relation*

$$\mathfrak{f}_{n+k} + (-1)^k \mathfrak{f}_{n-k} = \mathcal{L}_k \mathfrak{f}_n$$

$n = k, k+1, \ldots$ *where* $\mathcal{L}_k$ *is the k th Lucas number.*

Proof. By Binet's formula

$$\mathcal{L}_k \mathfrak{f}_n = \frac{1}{\sqrt{5}}(\varphi^n - \psi^n)(\varphi^k + \psi^k)$$
$$= \frac{1}{\sqrt{5}}\left(\varphi^{n+k} - \psi^{n+k} + \varphi^k \psi^k (\varphi^{n-k} - \psi^{n-k})\right).$$

Since $\varphi\psi = -1$, claim follows. ■

THEOREM 4.9. *If* $\{w_n\}$ *is a* λ *−decimated subsequence of* $\{\mathfrak{f}_n\}$, *that is* $w_n = \mathfrak{f}_{\lambda n+\mu}$ *for some integers* $0 \leq \mu < \lambda$, *then terms of* $\{w_n\}$ *satisfy the linear recursion*

$$w_{n+1} = \mathcal{L}_\lambda w_n - (-1)^\lambda w_{n-1}.$$

Proof. Using the lemma, we write

$$w_{n+1} + (-1)^\lambda w_{n-1} = \mathfrak{f}_{\lambda n+\mu+\lambda} + (-1)^\lambda \mathfrak{f}_{\lambda n+\mu-\lambda} = \mathcal{L}_\lambda \mathfrak{f}_{\lambda n+\mu}.$$

Since $\mathfrak{f}_{\lambda n+\mu} = w_n$, the desired equality is obtained. ■

Example 3. Below table demonstrates subsequences $\{\mathfrak{f}_{\lambda n+\mu}\}$ for several values of λ and μ together with the recursion of the subsequence.

(λ, μ)	$\{\mathfrak{f}_{\lambda n+\mu}\}$	recursion
(2,0)	0 1 3 8 21 55 144 377 987 2584 ⋯	$u_n = 3u_{n-1} - u_{n-2}$
(3,0)	0 2 8 34 144 610 2584 10946 46368 196418 ⋯	$u_n = 4u_{n-1} + u_{n-2}$
(3,1)	1 3 13 55 233 987 4181 17711 75025 317811 ⋯	$u_n = 4u_{n-1} + u_{n-2}$
(3,2)	1 5 21 89 377 1597 6765 28657 121393 514229 ⋯	$u_n = 4u_{n-1} + u_{n-2}$
(4,3)	1 8 55 377 2584 17711 121393 832040 5702887 39088169 ⋯	$u_n = 7u_{n-1} - u_{n-2}$
(5,0)	0 5 55 610 6765 75025 832040 9227465 102334155 ⋯	$u_n = 11u_{n-1} + u_{n-2}$
(5,3)	2 21 233 2584 28657 317811 3524578 39088169 433494437 ⋯	$u_n = 11u_{n-1} + u_{n-2}$
(6,2)	1 21 377 6765 121393 2178309 39088169 701408733 ⋯	$u_n = 18u_{n-1} - u_{n-2}$
(7,0)	0 13 377 10946 317811 9227465 267914296 7778742049 ⋯	$u_n = 29u_{n-1} + u_{n-2}$
(8,1)	1 34 1597 75025 3524578 165580141 7778742049 ⋯	$u_n = 47u_{n-1} - u_{n-2}$

FINITE SUMS INVOLVING $\mathfrak{f}_n$

In this section we compute the following finite sums:

$$\sum_{k=0}^{n} \mathfrak{f}_k = \mathfrak{f}_{n+2} - 1,$$

$$\sum_{k=0}^{n} k\mathfrak{f}_k = (n+1)\mathfrak{f}_{n+2} - \mathfrak{f}_{n+4} + 2,$$

$$\sum_{k=0}^{n} \mathfrak{f}_{2k} = \mathfrak{f}_{2n+1} - 1,$$

$$\sum_{k=0}^{n} \mathfrak{f}_{2k+1} = \mathfrak{f}_{2n+2},$$

$$\sum_{k=0}^{n} \mathfrak{f}_k^2 = \mathfrak{f}_n\mathfrak{f}_{n+1}.$$

First recall that if u and U are functions defined on integers such that $\Delta U(k) = U(k+1) - U(k) = u(k)$, then

$$\sum_{k=a}^{b} u_k = U(b+1) - U(a).$$

The following proposition computes two useful differences concerning Fibonacci numbers.

PROPOSITION 4.10. *For any integer k we have*

$$\Delta\mathfrak{f}_{k+1} = \mathfrak{f}_k,$$

$$\Delta[k\mathfrak{f}_{k+1} - \mathfrak{f}_{k+3}] = k\mathfrak{f}_k.$$

Proof. Since for any $k \geq 1$, $\Delta\mathfrak{f}_k = \mathfrak{f}_{k+1} - \mathfrak{f}_k = \mathfrak{f}_{k-1}$ we obtain the first equality. For the second equality, first observe that $\Delta k\mathfrak{f}_{k+1} = (k+1)\mathfrak{f}_{k+2} - k\mathfrak{f}_{k+1} = k(\mathfrak{f}_{k+2} - \mathfrak{f}_{k+1}) + \mathfrak{f}_{k+2} = k\mathfrak{f}_k + \mathfrak{f}_{k+2}$. On the other hand, $\Delta\mathfrak{f}_{k+3} = \mathfrak{f}_{k+2}$, so $\Delta[k\mathfrak{f}_{k+1} - \mathfrak{f}_{k+3}] = k\mathfrak{f}_k$. ■

Using the above proposition we immediately obtain $\sum_{k=1}^{n} \mathfrak{f}_k = \mathfrak{f}_{n+2} - 1$ and similarly

$$\sum_{k=0}^{n} k\mathfrak{f}_k = (n+1)\mathfrak{f}_{n+2} - \mathfrak{f}_{n+4} + \mathfrak{f}_3 = (n+1)\mathfrak{f}_{n+2} - \mathfrak{f}_{n+4} + 2.$$

Each of the remaining three sums can be written as a telescoping series:

$$\sum_{k=0}^{n} \mathfrak{f}_{2k} = \mathfrak{f}_0 + \mathfrak{f}_2 + \mathfrak{f}_4 + \mathfrak{f}_6 + \cdots + \mathfrak{f}_{2n-2} + \mathfrak{f}_{2n}$$

$$= 0 + (\mathfrak{f}_3 - \mathfrak{f}_1) + (\mathfrak{f}_5 - \mathfrak{f}_3) + (\mathfrak{f}_7 - \mathfrak{f}_5) + \cdots + (\mathfrak{f}_{2n-1} - \mathfrak{f}_{2n-3}) + (\mathfrak{f}_{2n+1} - \mathfrak{f}_{2n-1})$$

$$= \mathfrak{f}_{2n+1} - \mathfrak{f}_1.$$

$$\sum_{k=0}^{n} \mathfrak{f}_{2k+1} = \mathfrak{f}_1 + \mathfrak{f}_3 + \mathfrak{f}_5 + \mathfrak{f}_7 + \cdots + \mathfrak{f}_{2n-1} + \mathfrak{f}_{2n+1}$$

$$= (\mathfrak{f}_2 - \mathfrak{f}_0) + (\mathfrak{f}_4 - \mathfrak{f}_2) + (\mathfrak{f}_6 - \mathfrak{f}_4) + (\mathfrak{f}_8 - \mathfrak{f}_6) + \cdots + (\mathfrak{f}_{2n+1} - \mathfrak{f}_{2n-2})$$
$$+ (\mathfrak{f}_{2n+2} - \mathfrak{f}_{2n})$$

$$= \mathfrak{f}_{2n+2}.$$

$$\sum_{k=0}^{n} \mathfrak{f}_k^2 = \mathfrak{f}_1^2 + \mathfrak{f}_2^2 + \mathfrak{f}_3^2 + \cdots + \mathfrak{f}_{n-1}^2 + \mathfrak{f}_n^2$$

$$= (\mathfrak{f}_1\mathfrak{f}_2 - \mathfrak{f}_1\mathfrak{f}_0) + (\mathfrak{f}_2\mathfrak{f}_3 - \mathfrak{f}_2\mathfrak{f}_1) + (\mathfrak{f}_3\mathfrak{f}_4 - \mathfrak{f}_3\mathfrak{f}_2) + \cdots + (\mathfrak{f}_{n-1}\mathfrak{f}_n - \mathfrak{f}_{n-1}\mathfrak{f}_{n-2})$$
$$+ (\mathfrak{f}_n\mathfrak{f}_{n+1} - \mathfrak{f}_n\mathfrak{f}_{n-1})$$

$$= \mathfrak{f}_n\mathfrak{f}_{n+1}.$$

Convergence of Ratios and the Generating Function

For any positive integer n, consider the ratio $\frac{\mathfrak{f}_{n+1}}{\mathfrak{f}_n}$ which can be written as $\frac{\mathfrak{f}_{n+1}}{\mathfrak{f}_n} = \frac{\varphi^{n+1}-\psi^{n+1}}{\varphi^n-\psi^n} = \frac{\varphi-\psi\,(\psi/\varphi)^n}{1-(\psi/\varphi)^n}$. Since $|\varphi| > |\psi|$, we have $\lim_{n\to\infty}\left(\frac{\psi}{\varphi}\right) = 0$, then

$$\lim_{n\to\infty}\left(\frac{\mathfrak{f}_{n+1}}{\mathfrak{f}_n}\right) = \varphi.$$

Similarly, for any positive integer k, the ratio $\mathfrak{f}_{n+k}/\mathfrak{f}_n$ can be written as $\frac{\varphi^{n+k}-\psi^{n+k}}{\varphi^n-\psi^n} = \frac{\varphi^k-\psi^k\,(\psi/\varphi)^n}{1-(\psi/\varphi)^n}$.

Then

$$\lim_{n\to\infty}\left(\frac{\mathfrak{f}_{n+k}}{\mathfrak{f}_n}\right) = \varphi^k.$$

The series $\sum_{k=0}^{\infty} \mathfrak{f}_k x^k$ is convergent for $-\frac{1}{\varphi} < x < \frac{1}{\varphi}$ and for such x, the sum is given by

$$\sum_{k=0}^{\infty} \mathfrak{f}_k x^k = \frac{x}{1-x-x^2}.$$

Example 4. By substituting $x = \frac{1}{2}$ in the generating function of $\{\mathfrak{f}_n\}$ we have

$$\sum_{k=0}^{\infty} \frac{\mathfrak{f}_k}{2^k} = 2.$$

Note that $\frac{\mathfrak{f}_n}{\mathfrak{f}_{n+1}} < \frac{1}{\varphi}$ if and only if n is even. Then the sequence $\sum_{k=0}^{\infty} \mathfrak{f}_k x^k$ is convergent for $x = \frac{\mathfrak{f}_{2n}}{\mathfrak{f}_{2n+1}}$ where n is some fixed integer. We obtain

$$\sum_{k=0}^{\infty} \left(\frac{\mathfrak{f}_{2n}}{\mathfrak{f}_{2n+1}}\right)^k \mathfrak{f}_k = \frac{\mathfrak{f}_{2n}\mathfrak{f}_{2n+1}}{\mathfrak{f}_{2n+1}^2 - \mathfrak{f}_{2n}\mathfrak{f}_{2n+1} - \mathfrak{f}_{2n}^2}$$

But since $\mathfrak{f}_{m+1}^2 - \mathfrak{f}_m\mathfrak{f}_{m+1} - \mathfrak{f}_m^2 = (-1)^m$ for any integer m, denominator of the righthandside is 1. S we get

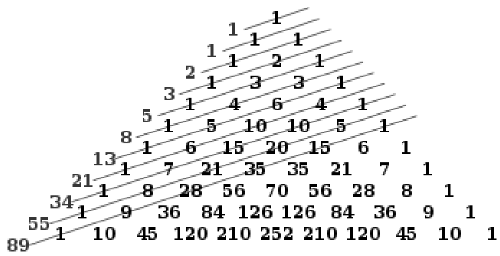
$$\sum_{k=0}^{\infty} \left(\frac{\mathfrak{f}_{2n}}{\mathfrak{f}_{2n+1}}\right)^k \mathfrak{f}_k = \mathfrak{f}_{2n}\mathfrak{f}_{2n+1}.$$

Some Other Properties

- For each positive integer n, the sequence $\binom{n}{0}, \binom{n-1}{1}, \binom{n-2}{2}, \binom{n-3}{3}, \ldots \binom{n-k}{k}$ where $k = \lfloor n/2 \rfloor$ is called a shallow diagonal of the Pascal's triangle. The sum of all terms on a shallow diagonal is a Fibonacci number:

$$\sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n-k}{k} = \mathfrak{f}_n.$$

(See Theorem 1.5.)



- We have two more examples which involve Fibonacci numbers and binomial coefficients:

$$\sum_{k=0}^{n} \binom{n}{k} \mathfrak{f}_k = \mathfrak{f}_{2n}, \qquad \sum_{k=0}^{n} \binom{n}{k} 2^k \mathfrak{f}_k = \mathfrak{f}_{3n}.$$

- Applying binomial inversion to above expressions we have:

$$\sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} \mathfrak{f}_{2k} = \mathfrak{f}_n, \qquad \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} \mathfrak{f}_{3k} = 2^n \mathfrak{f}_n.$$

- We can compute the nth Fibonacci number by rounding up $\frac{\varphi^n}{\sqrt{5}}$ to the closest integer. In Binet's formula $\mathfrak{f}_n = \frac{1}{\sqrt{5}}(\varphi^n - \psi^n)$, since $\frac{|\psi|}{\sqrt{5}} < \frac{1}{2}$, for any integer n, $\mathfrak{f}_n$ is the closest integer to $\frac{\varphi^n}{\sqrt{5}}$. That is,

$$\mathfrak{f}_n = \left[\frac{\varphi^n}{\sqrt{5}}\right].$$

- Any power of the golden ratio can be written as a linear combination of φ and 1 with integer coefficients as $\varphi^n = \varphi\mathfrak{f}_n + \mathfrak{f}_{n-1}$.

- Golden ratio is the limit of a continuous fraction:

$$\varphi = 1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \ddots}}}}.$$

- Reciprocal Fibonacci constant: $\sum_{k=1}^{\infty} \frac{1}{\mathfrak{f}_k} = 3.359885 \cdots$.

- Golden ratio is the limit of a nested square roots:

$$\varphi = \sqrt{1 + \sqrt{1 + \sqrt{1 + \cdots}}}.$$

- An integer m is a Fibonacci number if and only if at least one of $5m^2 + 4$ and $5m^2 - 4$ is a perfect square.

- If it is known that a given integer is a Fibonacci number, then the index of F in the sequence $\{\mathfrak{f}_n\}$ is

$$n(F) = \left\lfloor \log_{\varphi}\left(\frac{1}{2} + F \cdot \sqrt{5}\right) \right\rfloor.$$

- For any positive integers m and n, if $m|n$, then $\mathfrak{f}_m|\mathfrak{f}_n$.

- For any positive integers m and n, $\gcd(\mathfrak{f}_m, \mathfrak{f}_n) = \mathfrak{f}_{\gcd(m,n)}$.

- For any integer $n > 1$, $(\mathfrak{f}_{2n-1}, 2\mathfrak{f}_n\mathfrak{f}_{n-1}, \mathfrak{f}_n^2 - \mathfrak{f}_{n-1}^2)$ is a Phytogoran triple.

EXERCISES

1. Show that

a) $\mathfrak{f}_{n-1} + \mathfrak{f}_{n+1} = \mathcal{L}_n$,

b) $\mathcal{L}_{n-1} + \mathcal{L}_{n+1} = 5\mathfrak{f}_n$,

c) $\mathfrak{f}_{2n} = \mathcal{L}_n \mathfrak{f}_n$,

d) $\mathfrak{f}_{nm} = \mathcal{L}_n \mathfrak{f}_{n(m-1)} - (-1)^n \mathfrak{f}_{n(m-2)}$,

e) $\mathfrak{f}_n \mathfrak{f}_m = \frac{1}{5}[\mathcal{L}_{n+m} - (-1)^m \mathcal{L}_{n-m}]$.

2. Prove that greatest common divisor of two Fibonacci numbers is again a Fibonacci numbers. Specifically

$$\gcd(\mathfrak{f}_n, \mathfrak{f}_m) = \mathfrak{f}_{\gcd(m,n)}.$$

3. Prove that

$$\sum_{k=0}^{n} \binom{n+k}{2k} = \mathfrak{f}_{2n+1}.$$

4. Let $\{\mathfrak{f}_n\}$ be the standard Fibonacci sequence ($\mathfrak{f}_0 = \mathfrak{f}_1 = 1$). For each of the following, write first few terms of $\{u_n\}$, find the generating function of $\{u_n\}$, find a constant coefficient, linear, homogeneous linear recursion of smallest possible value satisfied by $\{u_n\}$, evaluate $\lim\limits_{n\to\infty} u_{n+1}/u_n$ and determine whether $\{u_n\}$ is periodic.

a) $u_n = (\mathfrak{f}_n)^2$,

b) $u_n = \mathfrak{f}_n + n^2$,

c) $u_n = \mathfrak{f}_n + 3$,

d) $u_n = \mathfrak{f}_{2n}$,

e) $u_n = \mathfrak{f}_0 + \mathfrak{f}_1 + \cdots + \mathfrak{f}_n$,

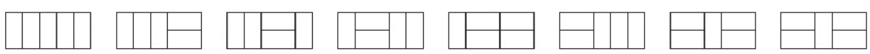
f) $u_n = \begin{cases} 1 + \mathfrak{f}_n & 2|n \\ n + \mathfrak{f}_n & \text{otherwise} \end{cases}$,

g) $u_n = \begin{cases} 1 + \mathfrak{f}_n & 2|n \\ n + \mathfrak{f}_n & \text{otherwise} \end{cases}$,

h) $u_n = \begin{cases} \mathfrak{f}_n + 1 & 2|n \\ \mathfrak{f}_n - 1 & \text{otherwise} \end{cases}$,

i) $u_n = \begin{cases} 3 + \mathfrak{f}_n & 2|n \quad \text{or/and} \quad 3|n \\ n + \mathfrak{f}_n & \text{otherwise} \end{cases}$.

5. Show that the number of ways to cover a $2 \times n$ checkerboard by 2×1 dominoes is $\mathfrak{f}_{n+1}$. The figure shows all possible $\mathfrak{f}_6 = 8$ ways of covering a 2×5 checkerboard.



6. Find the number of subsets of $\{1,2,\dots,n\}$ which do not contain any pair of two consecutive numbers.

7. Compute the probability of not getting two heads in a row of n tosses of a coin.

8. Find the number of ways in which n coin tosses can be made such that there are not three consecutive heads or tails.

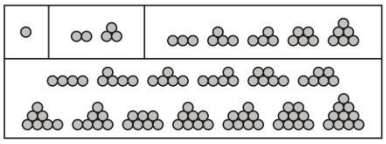
9. Find the number of binary strings of length n without consecutive1's. List of 13 such strings of length 5 is: 00000, 00001, 00010, 00100, 00101, 01000, 01001, 01010, 10000, 10001, 10010, 10100, 10101.

10. Show that the number of binary strings of length n without an odd number of consecutive 1's is the Fibonacci number $\mathfrak{f}_{n+1}$.

11. Find the number of binary strings of length n without an even number of consecutive 0's or 1's.

12. Find the number of permutations $\sigma_1\sigma_2\cdots\sigma_n$ of $\{1,2,\dots,n\}$ such that $|\sigma_i - i| \leq 1$ for $i = 1,2,\dots n$.

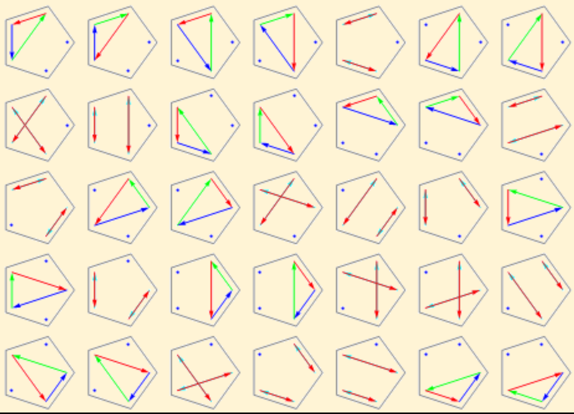
13. Find the number of ways of arranging coins in rows such that there are no gaps between blocks of coins in each row, each coin except ones on the bottom row touches two coins on the row below, and there are n coins in the bottom row. The figure shows all possible ways for $n = 1,2,3,4$.



5. STIRLING NUMBERS

James Stirling[13] (1692-1770), remembered for the Stirling numbers, the Stirling's interpolation formula, and the formula for the Gamma function among many other things, was one of the great minds of classic numerical analysis. Among Stirling's goals was to find methods to speed up series convergence. The studies yield interesting number sequences that are now known as Stirling numbers. Stirling numbers have applications in various fields of study, particularly in combinatorial problems. Generalized definitions and implementations for the two types of Stirling numbers are desired.

Following types of problems are related with Stirling numbers: finding the number of ways to distribute n distinct objects into k non-empty, indistinct bins; number of partitions of a set of n objects into k non-empty subsets; number of equivalence relations with k equivalence classes, defined on a set with n elements; number of factorizations, each with exactly k factors greater than 1, of a square-free positive integer that has exactly n different prime factors.



There are two kinds of Stirling numbers and in most cases it is more convenient to start with the second kind.

[13] *James Stirling (1692-1770), Scottish mathematician.*

Stirling Numbers of the Second Kind

The number of ways of partitioning an n −set into m disjoint subsets is called a ***Stirling number of the second kind*** and is denoted by $\left\{ n \atop m \right\}$, by convention we set $\left\{ 0 \atop 0 \right\} = 1$ and $\left\{ n \atop 0 \right\} = 0$ for any positive integer n.

Example 1. Partitions of {1,2,3,4}

1 part	{1,2,3,4}
2 parts	{1,2,3}{4} {1,2,4}{3} {1,3,4}{2} {2,3,4}{1}
	{1,2}{3,4} {1,3}{2,4} {1,4}{2,3}
3 parts	{1,2}{3}{4} {1,3}{2}{4} {1,4}{2}{3}
	{2,3}{1}{4} {2,4}{1}{3} {3,4}{1}{2}
4 parts	{1}{2}{3}{4}

Then we conclude

$$\left\{ 4 \atop 1 \right\} = 1 \quad \left\{ 4 \atop 2 \right\} = 7 \quad \left\{ 4 \atop 3 \right\} = 6 \quad \left\{ 4 \atop 4 \right\} = 1.$$

It is easy to observe that $\left\{ n \atop 1 \right\} = 1$, because the only option is to place all the objects into the single subset. Similarly $\left\{ n \atop n \right\} = 1$, for we must place each object in a different subset. Thus

$$\left\{ n \atop 1 \right\} = 1, \qquad \left\{ n \atop n \right\} = 1.$$

Choose any nonempty proper subset of a given set X with n elements. This choice, together with its complement, constitutes a partition of X into two subsets. In this way each partition is counted twice, so

$$\left\{ n \atop 2 \right\} = 2^{n-1} - 1.$$

Any partitioning of X into $n-1$ subsets consists of $n-2$ subsets with one element and one subset with two elements. Such a partition is completely determined when the subset with two elements is determined. Then

$$\left\{ n \atop n-1 \right\} = \binom{n}{2}.$$

THEOREM 5.1 (*Basic recursion of the Stirling numbers of the second kind*). *For any integers $n \geq 2$ and $1 \leq m \leq n$, Stirling numbers of the second kind satisfy the recurrence*

$$\left\{ {n \atop m} \right\} = m \left\{ {n-1 \atop m} \right\} + \left\{ {n-1 \atop m-1} \right\}.$$

Proof. In a partition of $\{1, 2, \ldots, n\}$ into m nonempty subsets, the element n either appears as a singleton or it is in of the k nonempty subsets with more than one element. In the first case, the partition is $\{n\}$ together with a partition of $\{1, 2, \ldots, n-1\}$ into $m-1$ nonempty parts. There are $\left\{ {n-1 \atop m-1} \right\}$ of them. In the second case, take a collection of m nonempty subsets partitioning $\{1, 2, \ldots, n-1\}$. There are $\left\{ {n-1 \atop m} \right\}$ of them. The number n can be put back into any of the m parts. ■

Example 2. Compute $\left\{ {5 \atop 3} \right\}$.

We will obtain $\left\{ {5 \atop 3} \right\}$ by counting all partitions of the set $\{A, B, C, D, E\}$ into three parts. Below we see that each partition of $\{A, B, C, D, E\}$ into three parts can be obtained from partitions of the set $\{A, B, C, D\}$ into two parts and three parts:

First recall that $\left\{ {4 \atop 2} \right\} = 7$ and $\left\{ {4 \atop 3} \right\} = 6$ (Example 1).

We have $\left\{ {4 \atop 3} \right\} = 6$ and all partitions of $\{A, B, C, D\}$ into 3 parts are:

$$\{A,B\}\{C\}\{D\} \quad \{A,C\}\{B\}\{D\} \quad \{A,D\}\{B\}\{C\}$$
$$\{B,C\}\{A\}\{D\} \quad \{B,D\}\{A\}\{C\} \quad \{C,D\}\{A\}\{B\}$$

We can insert the fifth element 'E', into any one of the existing parts so that

$$\begin{array}{llll}
\{A,B\}\{C\}\{D\} & \mapsto & \{A,B,\boldsymbol{E}\}\{C\}\{D\} \quad \{A,B\}\{C,\boldsymbol{E}\}\{D\} \quad \{A,B\}\{C\}\{D,\boldsymbol{E}\} \\
\{A,C\}\{B\}\{D\} & \mapsto & \{A,C,\boldsymbol{E}\}\{B\}\{D\} \quad \{A,C\}\{B,\boldsymbol{E}\}\{D\} \quad \{A,C\}\{B\}\{D,\boldsymbol{E}\} \\
\{A,D\}\{B\}\{C\} & \mapsto & \{A,D,\boldsymbol{E}\}\{B\}\{C\} \quad \{A,D\}\{B,\boldsymbol{E}\}\{C\} \quad \{A,D\}\{B\}\{C,\boldsymbol{E}\} \\
\{B,C\}\{A\}\{D\} & \mapsto & \{B,C,\boldsymbol{E}\}\{A\}\{D\} \quad \{B,C\}\{A,\boldsymbol{E}\}\{D\} \quad \{B,C\}\{A\}\{D,\boldsymbol{E}\} \\
\{B,D\}\{A\}\{C\} & \mapsto & \{B,D,\boldsymbol{E}\}\{A\}\{C\} \quad \{B,D\}\{A,\boldsymbol{E}\}\{C\} \quad \{B,D\}\{A\}\{C,\boldsymbol{E}\} \\
\{C,D\}\{A\}\{B\} & \mapsto & \{C,D,\boldsymbol{E}\}\{A\}\{B\} \quad \{C,D\}\{A,\boldsymbol{E}\}\{B\} \quad \{C,D\}\{A\}\{B,\boldsymbol{E}\}
\end{array}$$

and in this manner we obtain $3 \cdot \left\{ {4 \atop 3} \right\} = 18$ partitions of $\{A, B, C, D, E\}$ into 3 parts.

Now we consider the partitions of $\{A, B, C, D\}$ into two parts. We have $\left\{ {4 \atop 2} \right\} = 7$ and all such partitions are:

{A,B,C}{D} {A,B,D}{C} {A,C,D}{B} {B,C,D}{A} {A,B}{C,D} {A,C}{B,D} {A,D}{B,C}

By appending $\{E\}$ as an additional new part in each of these partitions we get

{A,B,C}{D}$\{E\}$ {A,B,D}{C}$\{E\}$ {A,C,D}{B}$\{E\}$ {B,C,D}{A}$\{E\}$

{A,B}{C,D}$\{E\}$ {A,C}{B,D}$\{E\}$ {A,D}{B,C}$\{E\}$

$\left\{ {4 \atop 2} \right\} = 7$ more partitions of $\{A, B, C, D, E\}$ into 3 parts. Then

$$\left\{ {5 \atop 3} \right\} = 3\left\{ {4 \atop 3} \right\} + \left\{ {4 \atop 2} \right\} = 3 \cdot 6 + 7 = 25.$$

THEOREM 5.2. *Let X and Y be two sets with $|X| = n$ and $|Y| = m$. The number of functions $f: X \to Y$ such that $|f(X)| = k$ is*

$$\frac{m!}{(m-k)!}\left\{ n \atop m \right\}.$$

Proof. Assume that the size of image set $f(X)$ of f is k. If $f(X) = \{y_1, y_2, \cdots, y_k\}$, then $f^{-1}(y_1), \cdots, f^{-1}(y_k)$ is a partition of X into k pairwise disjoint subsets. Consequently any function which has k points in the image set describes a partition of X into k subsets. On the other hand, assume that we have given a partition of X into k subsets, say $X_1, \dots, X_k$. Each arrangement $y_1 y_2 \cdots y_k$ of k elements of Y describes a function $X \to Y$ by setting $f(x) = y_i$ for all $x \in X_i$, $i = 1, \dots, k$. Thus, each partition of X into k subsets defines $\binom{m}{k} k! = \frac{m!}{(m-k)!}$ functions each of which has k points in the image set. ■

COROLLARY 5.3. *Let X and Y be two sets with $|X| = n$ and $|Y| = m$. The number of onto functions $f: X \to Y$ is $m! \left\{ n \atop m \right\}$.*

Proof. Just write $k = m$ in the theorem. ■

Since the number of onto functions is given by $\sum_{k=0}^{m} (-1)^{m-k} \binom{m}{k} k^n$, a closed form expression for $\left\{ n \atop m \right\}$ is

$$\left\{ n \atop m \right\} = \frac{1}{m!} \sum_{k=0}^{m} (-1)^{m-k} \binom{m}{k} k^n.$$

COROLLARY 5.4. *For any integers $0 \le m \le n$, the following equality holds*

$$\sum_{k=1}^{m} k! \binom{m}{k} \left\{ n \atop k \right\} = m^n.$$

Proof. Each side of the equality counts the number of all functions from a set with n elements into a set of m elements in another way. ■

Now we obtain ordinary and exponential generating functions of Stirling numbers of the second kind.

LEMMA 5.5. *If* $k \geq 0$, *then*

$$\frac{x^k}{(1-x)(1-2x)\cdots(1-kx)} = \frac{1}{k!}\sum_{j=0}^{k}(-1)^{k-j}\binom{k}{j}\frac{1}{1-jx}.$$

Proof. (By induction on k.) For $k = 1$ equality holds: left hand side is $\frac{x}{1-x}$ and right hand side is $-1 + \frac{1}{1-x} = \frac{x}{1-x}$. Assume that the equality holds for some integer $k \geq 0$ then

$$\frac{x^{k+1}}{(1-x)(1-2x)\cdots(1-(k+1)x)} = \frac{1}{k!}\sum_{j=0}^{k}(-1)^{k-j}\binom{k}{j}\frac{x}{1-(k+1)x}\cdot\frac{1}{1-jx}.$$

Now we can write the quotient in the summation as

$$\frac{x}{(1-(k+1)x)(1-jx)} = \frac{1}{k+1-j}\left(\frac{1}{1-(k+1)x} - \frac{1}{1-jx}\right)$$

to obtain

$$\frac{x^{k+1}}{(1-x)(1-2x)\cdots(1-(k+1)x)} = \frac{1}{k!}\sum_{j=0}^{k}(-1)^{k-j}\binom{k}{j}\frac{1}{k+1-j}\left(\frac{1}{1-(k+1)x} - \frac{1}{1-jx}\right)$$

$$= \frac{1}{(k+1)!}\sum_{j=0}^{k}\left[(-1)^{k-j}\binom{k+1}{j}\left(\frac{1}{1-(k+1)x} - \frac{1}{1-jx}\right)\right]$$

$$= \frac{1}{(k+1)!}\frac{1}{1-(k+1)x}\sum_{j=0}^{k}(-1)^{k-j}\binom{k+1}{j}$$

$$+\frac{1}{(k+1)!}\sum_{j=0}^{k}(-1)^{k+1-j}\binom{k+1}{j}\frac{1}{1-jx}$$

$$= \frac{1}{(k+1)!}\left[\frac{1}{1-(k+1)x} + \sum_{j=0}^{k}(-1)^{k+1-j}\binom{k+1}{j}\frac{1}{1-jx}\right]$$

$$= \frac{1}{(k+1)!}\sum_{j=0}^{k+1}(-1)^{k+1-j}\binom{k+1}{j}\frac{1}{1-jx}$$

which completes the proof. ■

THEOREM 5.6. *For any fixed integer* $k \geq 0$ *, generating function of the sequence* $\left\{ {n \brace k} \right\}_{n=k}^{\infty}$ *is*

$$G(x) = \sum_{n \geq k}^{\infty} {n \brace k} x^n = \frac{x^k}{(1-x)(1-2x)\cdots(1-kx)}.$$

Proof. From the lemma we have

$$\begin{aligned}\frac{x^k}{(1-x)(1-2x)\cdots(1-kx)} &= \frac{1}{k!}\sum_{j=0}^{k}(-1)^{k-j}\binom{k}{j}\frac{1}{1-jx} \\ &= \frac{1}{k!}\sum_{j=0}^{k}(-1)^{k-j}\binom{k}{j}\sum_{n=0}^{\infty} j^n x^n \\ &= \sum_{n=0}^{\infty}\left(\frac{1}{k!}\sum_{j=0}^{k}(-1)^{k-j}\binom{k}{j} j^n\right)x^n\end{aligned}$$

Since $\frac{1}{k!}\sum_{j=0}^{k}(-1)^{k-j}\binom{k}{j} j^n$ is the closed form of ${n \brace k}$ we obtain the desired equality. ■

Note that the generating function given in the theorem can also be written as

$$G(x) = \frac{1}{x(k+1)!\binom{1/x}{k+1}}.$$

THEOREM 5.7. *For any fixed integer* $k \geq 0$ *, exponential generating function of the sequence* $\left\{ {n \brace k} \right\}_{n=k}^{\infty}$ *is*

$$H(x) = \sum_{n \geq k}^{\infty} {n \brace k} \frac{x^n}{n!} = \frac{(e^x-1)^k}{k!}.$$

Proof. We consider the power series expansion of $\frac{(e^x-1)^k}{k!}$:

$$\begin{aligned}\frac{(e^x-1)^k}{k!} &= \frac{1}{k!}\sum_{i=0}^{k}(-1)^{k-i}\binom{k}{i}e^{ix} \\ &= \frac{1}{k!}\sum_{i=0}^{k}(-1)^{k-i}\binom{k}{i}\sum_{n=0}^{\infty}\frac{i^n x^n}{n!} \\ &= \sum_{n=0}^{\infty}\left(\frac{1}{k!}\sum_{i=0}^{k}(-1)^{k-i}\binom{k}{i} i^n\right)\frac{x^n}{n!}\end{aligned}$$

Since $\frac{1}{k!}\sum_{i=0}^{k}(-1)^{k-i}\binom{k}{i} i^n = {n \brace k}$ we obtain the desired equality. ■

THEOREM 5.8. *Bivariate exponential generating function for the Stirling numbers of the second kind is*

$$F(x,y) = \sum_{n,k=0}^{\infty} \left\{ {n \atop k} \right\} \frac{x^n y^k}{n!} = e^{y(e^x-1)}.$$

Proof. We have to show that coefficient of $\frac{x^n y^k}{n!}$ in power series representation of $e^{y(e^x-1)}$ is $\left\{ {n \atop k} \right\}$.

$$\begin{aligned} e^{y(e^x-1)} &= \sum_{k=0}^{\infty} \frac{1}{k!} y^k (e^x - 1)^k \\ &= \sum_{k=0}^{\infty} \frac{1}{k!} y^k \sum_{j=0}^{k} (-1)^{k-j} \binom{k}{j} e^{jx} \\ &= \sum_{k=0}^{\infty} \frac{1}{k!} y^k \sum_{j=0}^{k} (-1)^{k-j} \binom{k}{j} \sum_{n=0}^{\infty} \frac{j^n x^n}{n!} \\ &= \sum_{k=0}^{\infty} \sum_{n=0}^{\infty} \left(\frac{1}{k!} \sum_{j=0}^{k} (-1)^{k-j} \binom{k}{j} j^n \right) \frac{x^n}{n!} y^k \\ &= \sum_{n,k=0}^{\infty} \left\{ {n \atop k} \right\} \frac{x^n}{n!} y^k. \end{aligned}$$

Thus, proof is completed. ∎

STIRLING NUMBERS OF THE FIRST KIND

We denote the set of all permutations of $\{1, 2, \cdots, n\}$ by $S(n)$. A permutation $\sigma \in S(n)$ is a bijective mapping whose 'word representation' is $\sigma = \sigma_1 \cdots \sigma_n$ where $\sigma_i = \sigma(i)$ for $i = 1, \dots, n$. Under the operation of composition $S(n)$ forms the symmetric group of order n. An alternative way to describe a permutation is its ***cycle decomposition***. For every i, the sequence $i, \sigma(i), \sigma^2(i), \dots$ eventually terminates with i again. If k is the smallest positive integer such that $\sigma^k(i) = i$, we denote the cycle containing i by $[i\ \sigma(i)\ \sigma^2(i) \cdots \sigma^{k-1}(i)]$. If $\sigma^t(i) = j$, then the cycles containing i and j are defined to be the same. Cycle decomposition of σ is the list of all distinct cycles of σ.

Example 3. Let $\sigma = 936478251 \in S(9)$. Then σ is the bijection from $\{1,2,3,4,5,6,7,8,9\}$ to itself which is defined as $\sigma(1) = 9$, $\sigma(2) = 3$, $\sigma(3) = 6, \sigma(4) = 4, \sigma(5) = 7, \sigma(6) = 8, \sigma(7) = 2, \sigma(8) = 5, \sigma(9) = 1$. Observe that $\sigma(3) = 6, \sigma(6) = 8, \sigma(8) = 5, \sigma(5) = 7$, $\sigma(7) = 2$ and $\sigma(2) = 3$. Thus, starting with 3 we have the chain $3 \xrightarrow{\sigma} 6 \xrightarrow{\sigma} 8 \xrightarrow{\sigma} 5 \xrightarrow{\sigma} 7 \xrightarrow{\sigma} 2 \xrightarrow{\sigma} 3$. The cycle containing 3 is [368572]. It is clear that the cycle containing 5 is [572368] and by definition these cycles are the same. The cycle containing 1 is [19] and the cycle containing 4 is [4]. Then cycle decomposition of σ is $\sigma =$ [19][236857][4].

To represent a cycle, we may start with any element in the cycle. Order of cycles is irrelevant. Cycles of length 1 are ***fixed points*** in σ. Cycles of length 2 are ***transpositions*** $i \leftrightarrow j$. In the above example 4 is a fixed point of σ and and $1 \leftrightarrow 9$ $(\sigma(1) = 9, \sigma(9) = 1)$ is a transposition.

Example 4. For the permutation σ of the previous example, 4 is a fixed point and [19] is a transposition.

In the identity permutation each element is a fixed point thus, it has the most crowded cycle decomposition: $id = [1][2][3] \cdots [n]$. On the other etreme we may have permutations whoso cycle decomposition consists of a single cycle. Such a permutation is called ***cyclic***. There are $(n-1)!$ cyclic permutations of $S(n)$.

Example 5. Let $\tau = 3421 \in S(4)$, then $1 \xrightarrow{\tau} 3 \xrightarrow{\tau} 2 \xrightarrow{\tau} 4 \xrightarrow{\tau} 1$ and the cycle containing 1 is [1324]. Since this cycle contains all of the elements, cycle decomposition of τ consists of a single cycle: $\tau = [1324]$. All cyclic permutations of $S(4)$ are

$$[1234], \quad [1243], \quad [1324], \quad [1342], \quad [1423], \quad [1432].$$

The number of permutations of $\{1,2,\dots,n\}$ with m −cycles is called a ***Stirling Number of the first kind*** and is denoted by $\left[{n \atop m}\right]$. By convention we set $\left[{0 \atop 0}\right] = 1$ and $\left[{n \atop 0}\right] = 0$ for $n > 0$.

$$\begin{aligned} &\text{As subsets} \quad \{1,2,3,4\} = \{2,3,4,1\} = \{1,2,4,3\}, \\ &\text{As cycles} \quad [1,2,3,4] = [2,3,4,1] \neq [1,2,4,3]. \end{aligned}$$

Example 6. Permutations and cycle decompositions of $\{1,2,3,4\}$:

1234	[1][2][3][4]	4 cycles	3124	[132] [4]	2 cycles
1243	[1][2][34]	3 cycles	3142	[1342]	1 cycle
1324	[1][23][4]	3 cycles	3214	[13] [2] [4]	3 cycles
1342	[1][234]	2 cycles	3241	[134] [2]	2 cycles
1423	[1][243]	2 cycles	3412	[1342]	1 cycle
1432	[1][42][3]	3 cycles	3421	[1324]	1 cycle
2134	[12][3][4]	3 cycles	4123	[1432]	1 cycle
2143	[12][34]	2 cycles	4132	[142] [3]	2 cycles
2314	[123][4]	2 cycles	4213	[14] [2]	2 cycles
2341	[1234]	1 cycle	4231	[14] [2] [3]	3 cycles
2413	[1243]	1 cycle	4312	[1423]	1 cycle
2431	[124] [3]	2 cycles	4321	[14] [23]	4 cycles

1 cycle	[1,2,3,4] [1,2,4,3] [1,3,2,4] [1,3,4,2] [1,4,2,3] [1,4,3,2]
2 cycles	[1,2,3][4] [1,3,2][4] [1,2,4][3] [1,4,2][3] [1,3,4][2] [1,4,3][2] [2,3,4][1] [2,4,3][1] [1,2][3,4] [1,3][2,4] [1,4][2,3]
3 cycles	[1,2][3][4] [1,3][2][4] [1,4][2][3] [2,3][1][4] [2,4][1][3] [3,4][1][2]
4 cycles	[1][2][3][4]

Then we conclude

$$\begin{bmatrix}4\\1\end{bmatrix} = 6, \quad \begin{bmatrix}4\\2\end{bmatrix} = 11, \quad \begin{bmatrix}4\\3\end{bmatrix} = 6 \quad \begin{bmatrix}4\\4\end{bmatrix} = 1$$

It is obvious that $\begin{bmatrix}n\\1\end{bmatrix} = (n-1)!$, because a single cycle consisting of n elements is a cyclic permutation of these elements and there are $(n-1)!$ such permutations. Similarly $\begin{bmatrix}n\\n\end{bmatrix} = 1,$ since only way of defining n cycles with n elements is to take each element as a cycle:

$$\begin{bmatrix}n\\1\end{bmatrix} = (n-1)!\,, \qquad \begin{bmatrix}n\\n\end{bmatrix} = 1.$$

If we wish to define $n-1$ cycles, we have to pick two arbitrary elements to form one of the cycles and all the remaining cycles will consist of a single element. Then

$$\begin{bmatrix}n\\n-1\end{bmatrix} = \binom{n}{2}.$$

THEOREM 5.9 (*Basic recursion of the Stirling numbers of the first kind*). *For any integers $n \geq 2$ and $1 \leq m \leq n$, Stirling numbers of the second kind satisfy the recurrence*

$$\begin{bmatrix}n\\m\end{bmatrix} = (n-1)\begin{bmatrix}n-1\\m\end{bmatrix} + \begin{bmatrix}n-1\\m-1\end{bmatrix}$$

with initial conditions $\begin{bmatrix}0\\0\end{bmatrix} = 1$ and $\begin{bmatrix}n\\0\end{bmatrix} = \begin{bmatrix}0\\n\end{bmatrix} = 0$ for $n > 0$.

Proof. Consider forming a new permutation with n objects from a permutation of $n-1$ objects by inserting an 'n'. There are exactly two ways in which this can be accomplished. First, we could form a singleton cycle, leaving the extra object fixed. This increases the number of cycles by 1 and so accounts for the $\begin{bmatrix}n-1\\m-1\end{bmatrix}$ term in the recurrence. Second, we could insert the object into one of the existing cycles. Consider an arbitrary permutation of $n-1$ objects with m cycles. To form the new permutation, we insert the new object before any of the $n-1$ objects already present. This explains the $(n-1)\begin{bmatrix}n-1\\m\end{bmatrix}$term of the recurrence. These two cases include all of the possibilities, so the recurrence relation follows with.

Example 7. Compute $\left[{5 \atop 3}\right]$.

First recall that $\left[{4 \atop 3}\right] = 6$ and all permuations of $\{A, B, C, D, E\}$ with three cycles are:

$[A, B][C][D] \quad [A, C][B][D] \quad [A, D][B][C] \quad [B, C][A][D] \quad [B, D][A][C] \quad [C, D][A][B]$.

We can insert 'E' to one of the cycles of each these permutations in 4 different ways :

$$
\begin{array}{llllll}
[A,B][C][D] & \to & [A,B,E][C][D] & [A,E,B][C][D] & [A,B][C,E][D] & [A,B][C][D,E] \\
[A,C][C][D] & \to & [A,C,E][C][D] & [A,E,C][C][D] & [A,C][C,E][D] & [A,C][C][D,E] \\
[A,D][B][C] & \to & [A,D,E][B][C] & [A,E,D][B][C] & [A,D][B,E][C] & [A,D][B][C,E] \\
[B,C][A][D] & \to & [B,C,E][A][D] & [B,E,C][A][D] & [B,C][A,E][D] & [B,C][A][D,E] \\
[B,D][A][C] & \to & [B,D,E][A][C] & [B,E,D][A][C] & [B,D][A,E][C] & [B,D][A][C,E] \\
[C,D][A][B] & \to & [C,D,E][A][B] & [C,E,D][A][B] & [C,D][A,E][B] & [C,D][A][B,E]
\end{array}
$$

To obtain the remaining permutations, we add the cycle $[E]$ to each permutation of $\{A, B, C, D\}$ with two cycles. There are $\left[{4 \atop 2}\right] = 11$ such permutations:

$[A, B, C][D] \quad [A, C, B][D] \quad [A, B, D][C] \quad [A, D, B][C] \quad [A, C, D][B] \quad [A, D, C][B]$

$[B, C, D][A] \quad [B, D, C][A] \quad [A, B][C, D] \quad [A, C][B, D] \quad [A, D][B, C]$

Thus we acquire 11 cycles of $\{A, B, C, D, E\}$ with 3 cycles where $[E]$ appears as a single cycle:

$$
\begin{array}{cccc}
[A,B,C][D][E] & [A,C,B][D][E] & [A,B,D][C][E] & [A,D,B][C][E] \\
[A,C,D][B][E] & [A,D,C][B][E] & [B,C,D][A][E] & [B,D,C][A][E]
\end{array}
$$
$$
[A,B][C,D][E] \quad [A,C][B,D][E] \quad [A,D][B,C][E]
$$

Then we conclude $\left[{5 \atop 3}\right] = 4\left[{4 \atop 3}\right] + \left[{4 \atop 2}\right] = 4 \cdot 6 + 11 = 35$.

THEOREM 5.10 (*Basic Identities*). *For nonnegative integers n and r let H_n be the harmonic number and $H_n^{(r)}$ be generalized harmonic numbers. The following identities hold.*

i. $\sum_{k=1}^{n} \left[{n \atop k}\right] = n!$,

ii. $\left[{n \atop 2}\right] = (n-1)!\, H_{n-1}$,

iii. $\left[{n \atop 3}\right] = \frac{1}{2}(n-1)! \left[H_{n-1}^2 - H_{n-1}^{(2)}\right]$,

iv. $\left[{n \atop 4}\right] = \frac{1}{3!}(n-1)! \left[H_{n-1}^3 - 3H_{n-1}H_{n-1}^{(2)} + 2H_{n-1}^{(3)}\right]$,

v. $\left[{n \atop n-2}\right] = \frac{1}{4}(3n-1)\binom{n}{3}$,

vi. $\left[{n \atop n-3}\right] = \binom{n}{2}\binom{n}{4}$.

Proof. Part ***i.*** follows immediately from the definition. Parts ***ii.***, ***iii.*** and ***iv.*** can be proved by similar methods. We prove parts ***ii.*** and ***iii.***; leave the proof of part ***iv.*** as exercise. Parts ***v.*** and ***vi.*** are proved by direct computation.

i. $\left[{n \atop k}\right]$ is the number of permutations with k cycles,so the sum $\left[{n \atop 1}\right] + \left[{n \atop 2}\right] + \cdots + \left[{n \atop n}\right]$ counts each permutation once and only once.

ii. Define the sequence $\{u_n\}$ by setting $u_n = \frac{1}{(n-1)!}\left[{n \atop 2}\right]$. Since $\left[{n-1 \atop 1}\right] = (n-2)!$ for $n > 2$, from the basic recursion we write

$$u_n = \frac{1}{(n-1)!}\left((n-1)\left[{n-1 \atop 2}\right] + (n-2)!\right)$$
$$= \frac{1}{(n-2)!}\left[{n-1 \atop 2}\right] + \frac{1}{n-1}.$$

As $u_{n-1} = \frac{1}{(n-2)!}\left[{n-1 \atop 2}\right]$, we see that $u_n = u_{n-1} + \frac{1}{n-1}$. Now

$$\sum_{k=2}^{n} u_k = \sum_{k=1}^{n-1} u_k + \sum_{k=1}^{n-1} \frac{1}{k}$$

so, $u_n = H_{n-1}$ and consequently $\left[{n \atop 2}\right] = (n-1)!\, H_{n-1}$.

iii. Let $u_n = \frac{1}{(n-1)!}\left[{n \atop 3}\right]$, $n = 3,4, \ldots$ and use the basic recursion

$$\left[{n \atop 3}\right] = (n-1)\left[{n-1 \atop 3}\right] + \left[{n-1 \atop 2}\right]$$
$$= (n-1)\left[{n-1 \atop 3}\right] + (n-2)!\, H_{n-2}$$

for Stirling numbers of the first kind to write

$$u_n = \frac{1}{(n-1)!}\left((n-1)\left[{n-1 \atop 3}\right] + (n-2)!\, H_{n-2}\right)$$
$$= \frac{1}{(n-2)!}\left[{n-1 \atop 3}\right] + \frac{1}{n-1} H_{n-2}.$$

It follows that $u_n = u_{n-1} + \frac{1}{n-1} H_{n-2}$. Now we consider the sum over all $n \geq 3$

$$\sum_{k=3}^{n} u_k = \sum_{k=3}^{n} u_{k-1} + \sum_{k=3}^{n} \frac{1}{k-1} H_{k-2}$$
$$= \sum_{k=2}^{n-1} u_k + \sum_{k=2}^{n-1} \frac{1}{k} H_{k-1.}$$

Since $u_2 = 0$, we obtain $u_n = \sum_{k=2}^{n-1} \frac{1}{k} H_{k-1}$ and

$$u_n = \sum_{k=2}^{n-1} \frac{1}{k}\left(H_k - \frac{1}{k}\right)$$
$$= \left(\sum_{k=1}^{n-1} \frac{1}{k}(H_k - 1)\right) - \left(\sum_{k=1}^{n-1} \frac{1}{k^2} - 1\right)$$
$$= \frac{1}{2}\left((H_{n-1})^2 + H_{n-1}^{(2)}\right) - H_{n-1}^{(2)}$$
$$= \frac{1}{2}\left((H_{n-1})^2 - H_{n-1}^{(2)}\right).$$

v. A permutation $\sigma \in S(n)$ decomposes into $n-2$ cycles in one of two ways:

- A cycle of size 3 and $n-3$ cycles of size 1. There are $2!\binom{n}{3} = \frac{n!}{3(n-3)}$ such permutations.

- Two cycles of size 2 elements and $n-4$ cycles of size 1. There are $\frac{1}{2}\binom{n}{2}\binom{n-2}{2} = \frac{n!}{8(n-4)!}$ such permutations.

It follows that $\left[{n \atop n-2}\right] = \frac{n!}{3(n-3)} + \frac{n!}{8(n-4)!}$ which simplifies into

$$\left[{n \atop n-2}\right] = \binom{n}{3}\left(2 + \frac{3}{4}(n-3)\right)$$
$$= \binom{n}{3}\frac{3n-1}{4}.$$

vi. A permutation $\sigma \in S(n)$ decomposes into $n-3$ cycles in one of three ways:

- Lengths of cycles: 4, $\underbrace{1, 1, \dots, 1}_{n-4 \ \ cycles}$. Number of such cycles is $3!\binom{n}{4}$,

- Lengths of cycles: 3, 2, $\underbrace{1, 1, \dots, 1}_{n-5 \ \ cycles}$. Number of such cycles is $2!\binom{n}{3}\binom{n-3}{2} = 4(n-5)\binom{n}{4}$,

- Lengths of cycles: 2, 2,2 $\underbrace{1, 1, \dots, 1}_{n-6 \ \ cycles}$. Number of such cycles is $\frac{1}{3!}\binom{n}{2}\binom{n-2}{2} \ \binom{n-4}{2} =$ $\frac{(n-4)(n-5)}{2}\binom{n}{4}$.

Then we have

$$\left[{n \atop n-3}\right] = \binom{n}{4}(6 + 4(n-4) + (n-4)(n-5)/2)$$
$$= \binom{n}{4}\frac{n(n-1)}{2}.$$

As $\frac{n(n-1)}{2} = \binom{n}{2}$, claim follows. ■

THEOREM 5.11. *Bivariate exponential generating function for the Stirling numbers of the first kind is*

$$\left(\frac{1}{1-x}\right)^y = \sum_{n=0}^{\infty}\sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} \frac{x^n}{n!} y^k .$$

For any fixed integer $k \geq 0$, *generating function of the sequence* $\left\{\begin{bmatrix} n \\ k \end{bmatrix}\right\}_{n=k}^{\infty}$ *is*

$$-\frac{(\ln(1+z))^k}{k!} = \sum_{n=0}^{\infty} \begin{bmatrix} n \\ k \end{bmatrix} \frac{x^n}{n!}.$$

Relations Between Two Kinds of Stirling Numbers

The following inequality is a direct consequence of the definition

$$\begin{bmatrix} n \\ m \end{bmatrix} \geq \left\{ \begin{matrix} n \\ m \end{matrix} \right\} \qquad n, m \geq 0 \quad \text{(integer).}$$

THEOREM 5.12 (Falling and rising factorial powers). For any integer $n \geq 0$, the following equalities hold

$$x^{\underline{n}} = (-1)^n (-x)^{\overline{n}},$$

$$x^{\overline{n}} = (-1)^n (-x)^{\underline{n}},$$

$$x \cdot x^{\underline{n}} = x^{\underline{n+1}} + n x^{\underline{n}}.$$

Proof.

$$\begin{aligned}
(-x)^{\overline{n}} &= (-x)(-x+1)\cdots(-x+n-1) \\
&= (-1)^n x(x-1)\cdots(x-n+1) \\
&= (-1)^n x^{\underline{n}} \\
(-x)^{\underline{n}} &= (-x)(-x-1)\cdots(-x-n+1) \\
&= (-1)^n x(x+1)\cdots(x+n-1) \\
&= (-1)^n x^{\overline{n}} \\
x^{\underline{n+1}} &= x(x-1)\cdots(x-n+1)(x-n) \\
&= x^{\underline{n}}(x-n) \\
&= x x^{\underline{n}} - n x^{\underline{n}}.
\end{aligned}$$

This completes the proof. ■

THEOREM 5.13. *For any integer $n \geq 0$, factorial powers can be written in terms of ordinary powers as follows:*

$$x^{\underline{n}} = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} (-1)^{n-k} x^k,$$
$$x^{\overline{n}} = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} x^k.$$

Proof. (Mathematical induction on n) For $n = 1$ the claim holds: $x^{\underline{1}} = x(x-1) = x^2 - x = \begin{bmatrix} 1 \\ 0 \end{bmatrix} x^2 - \begin{bmatrix} 1 \\ 1 \end{bmatrix} x$. Now assume that $x^{\underline{n}} = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} (-1)^{n-k} x^k$ for some $n \in \mathbb{N}$, then

$$\begin{aligned} x^{\underline{n+1}} = x^{\underline{n}}(x-n) &= \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} (-1)^{n-k} x^{k+1} - \sum_{k=0}^{n} n \begin{bmatrix} n \\ k \end{bmatrix} (-1)^{n-k} x^k \\ &= \sum_{k=1}^{n+1} \begin{bmatrix} n \\ k-1 \end{bmatrix} (-1)^{n-k+1} x^k - \sum_{k=0}^{n} n \begin{bmatrix} n \\ k \end{bmatrix} (-1)^{n-k} x^k \\ &= x^{n+1} + \sum_{k=1}^{n} (-1)^{n-k+1} \left(\begin{bmatrix} n \\ k-1 \end{bmatrix} + n \begin{bmatrix} n \\ k \end{bmatrix} \right) x^k \\ &= \sum_{k=1}^{n+1} (-1)^{n-k+1} \begin{bmatrix} n+1 \\ k \end{bmatrix} x^k. \end{aligned}$$

This proves the first equality. Second one can be obtained as follows

$$\begin{aligned} x^{\overline{n}} &= (-1)^n (-x)^{\underline{n}} \\ &= (-1)^n \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} (-1)^{n-k} (-x)^k \\ &= \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} x^k. \end{aligned}$$

The desired result is obtained. ∎

Example 8.

$$x^{\underline{4}} = x(x-1)(x-2)(x-3) = x^4 - 6x^3 + 11x^2 - 6x,$$
$$x^{\overline{4}} = x(x+1)(x+2)(x+3) = x^4 + 6x^3 + 11x^2 + 6x.$$

THEOREM 5.14. *For any integer $n \geq 0$, ordinary powers can be written in terms of factorial powers as follows*

$$x^n = \sum_{k=0}^{n} \begin{Bmatrix} n \\ k \end{Bmatrix} x^{\underline{k}},$$
$$x^n = \sum_{k=0}^{n} \begin{Bmatrix} n \\ k \end{Bmatrix} (-1)^{n-k} x^{\overline{k}}.$$

Proof. (Mathematical induction on n) For $n = 1$ claim holds:

$$x^1 = \sum\nolimits_{k=0}^{1} \left\{ {1 \atop k} \right\} x^{\underline{k}}$$
$$= x^{\underline{1}}$$
$$= x.$$

Now assume that $x_n = \sum_{k=0}^{n} \left\{ {n \atop k} \right\} x^{\underline{k}}$ for some $n \in \mathbb{N}$, then

$$x^{n+1} = x \sum_{k=0}^{n} \left\{ {n \atop k} \right\} x^{\underline{k}}$$
$$= \sum_{k=0}^{n} \left\{ {n \atop k} \right\} \left(x^{\underline{k+1}} + kx^{\underline{k}} \right)$$
$$= \sum_{k=0}^{n} \left\{ {n \atop k} \right\} x^{\underline{k+1}} + \sum_{k=0}^{n} \left\{ {n \atop k} \right\} kx^{\underline{k}}$$
$$= \sum_{k=1}^{n+1} \left\{ {n \atop k-1} \right\} x^{\underline{k}} + \sum_{k=0}^{n} \left\{ {n \atop k} \right\} kx^{\underline{k}}$$
$$= \sum_{k=0}^{n} \left\{ {n+1 \atop k} \right\} x^{\underline{k}}.$$

This proves the first equality, for the second one:

$$x^n = (-1)^n (-x)^n$$
$$= \sum_{k=0}^{n} \left\{ {n \atop k} \right\} (-1)^n (-x)^{\underline{k}}$$
$$= \sum_{k=0}^{n} \left\{ {n \atop k} \right\} (-1)^{n-k} (x)^{\overline{k}}.$$

The desired result is obtained. ■

Example 9. We compute $\sum_{x=1}^{n} x^5$.

$$x^5 = \sum_{k=0}^{5} \left\{ {5 \atop k} \right\} x^{\underline{k}}$$
$$= \left\{ {5 \atop 0} \right\} x^{\underline{0}} + \left\{ {5 \atop 1} \right\} x^{\underline{1}} + \left\{ {5 \atop 2} \right\} x^{\underline{2}} + \left\{ {5 \atop 3} \right\} x^{\underline{3}} + \left\{ {5 \atop 4} \right\} x^{\underline{4}} + \left\{ {5 \atop 5} \right\} x^{\underline{5}}$$
$$= x^{\underline{1}} + 15x^{\underline{2}} + 25x^{\underline{3}} + 10x^{\underline{4}} + x^{\underline{5}}.$$

Example 10. Recall that $\sum_{x=1}^{n} x^{\underline{m}} = \sum_{1}^{n+1} x^{\underline{m}}\,\delta x = \frac{x^{\underline{m+1}}}{m+1}\Big|_{x=1}^{n+1} = \frac{(n+1)^{\underline{m+1}}}{m+1}$. Then,

$$\sum_{k=1}^{n} k^m = \sum_{k=1}^{n}\sum_{j=1}^{m} \begin{Bmatrix} m \\ j \end{Bmatrix} k^{\underline{j}}$$

$$= \sum_{j=1}^{m} \begin{Bmatrix} m \\ j \end{Bmatrix} \sum_{k=1}^{n} k^{\underline{j}}$$

$$= \sum_{j=1}^{m} \begin{Bmatrix} m \\ j \end{Bmatrix} \frac{(n+1)^{\underline{j+1}}}{j+1}$$

$$= \sum_{j=1}^{m} \begin{Bmatrix} m \\ j \end{Bmatrix} \frac{(n+1)^{\underline{j+1}}}{j+1}$$

$$= \sum_{l=1}^{m+1} \sum_{j=l}^{m+1} \frac{1}{j} \begin{Bmatrix} m \\ j-1 \end{Bmatrix} \begin{bmatrix} j \\ l \end{bmatrix} (-1)^{j-l}(n+1)^{l}\,.$$

Then

$$\sum_{x=1}^{n} x^5 = \sum_{x=1}^{n} (x^{\underline{1}} + 15x^{\underline{2}} + 25x^{\underline{3}} + 10x^{\underline{4}} + x^{\underline{5}})$$

$$= \frac{1}{2}(x+1)^{\underline{2}} + 5(x+1)^{\underline{3}} + \frac{25}{4}(x+1)^{\underline{4}} + 2(x+1)^{\underline{5}} + \frac{1}{6}(x+1)^{\underline{6}}$$

$$= \frac{x^6}{6} + \frac{x^5}{5} + \frac{7x^4}{12} + \frac{x^3}{3} + \frac{x^2}{12}$$

$$= \frac{1}{12}x^2(x+1)^2(2x^2+2x+1).$$

THEOREM 5.15. *For any integers $n, \geq m \geq 0$, the following equality holds*

$$\sum_{k=0}^{n} \begin{Bmatrix} n \\ k \end{Bmatrix} \begin{bmatrix} k \\ m \end{bmatrix} (-1)^{n-k} = \delta_m^n\,.$$

Proof.

$$x^n = \sum_{k=0}^{n} \begin{Bmatrix} n \\ k \end{Bmatrix} (-1)^{n-k} x^{\overline{k}}$$

$$= \sum_{k=0}^{n} \begin{Bmatrix} n \\ k \end{Bmatrix} (-1)^{n-k} \left(\sum_{m=0}^{k} \begin{bmatrix} k \\ m \end{bmatrix} x^m \right)$$

$$= \sum_{m=0}^{n} \left(\sum_{k=0}^{n} \begin{Bmatrix} n \\ k \end{Bmatrix} \begin{bmatrix} k \\ k \end{bmatrix} (-1)^{n-k} \right) x^m$$

Comparing the coefficients of powers of x the desired result is obtained. ■

Using the basic recursions in reverse direction, for the Stirling numbers of both kinds can be defined for negative arguments. In this case we have

$$\begin{bmatrix} n \\ m \end{bmatrix} = \left\{ \begin{matrix} -n \\ -m \end{matrix} \right\}.$$

BELL NUMBERS

Bell numbers count the number of partitions of a set. Namely, the ***n th Bell number*** $\mathcal{B}_n$ counts the number of ways to partition a set with n elements into pairwise disjoint nonempty subsets. In other words, $\mathcal{B}_n$ is the number of equivalence relations on a set with n elements. From these definitions it follows that

$$\mathcal{B}_n = \sum_{k=1}^{n} \left\{ \begin{matrix} n \\ k \end{matrix} \right\}.$$

THEOREM 5.16 (*Dobinski's Formula*). *For any integer $n > 0$, Bell number $\mathcal{B}_n$ can be represented with the infinite sum*

$$\mathcal{B}_n = \frac{1}{e} \sum_{k=0}^{\infty} \frac{k^n}{k!}.$$

Proof. Define the function $\mathcal{P}(x)$ as follows:

$$\mathcal{P}(x) = e^{-x} \sum_{k=0}^{\infty} \frac{k^n}{k!} x^n.$$

We have to show that $\mathcal{B}_n = \mathcal{P}(1)$. Now we have

$$\mathcal{P}(x) = \left(\sum_{i=0}^{\infty} (-1)^i \frac{x^i}{i!} \right) \cdot \left(\sum_{j=0}^{\infty} \frac{j^n}{j!} x^n \right)$$

$$= \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} (-1)^i \frac{1}{i!\,j!} j^n x^{i+j}$$

$$= \sum_{k=0}^{\infty} \left(\sum_{j=0}^{k} (-1)^{k-j} \frac{1}{(k-j)!\,j!} j^n \right) x^k$$

$$= \sum_{k=0}^{\infty} \left(\frac{1}{k!} \sum_{j=0}^{k} (-1)^{k-j} \binom{k}{j} j^n \right) x^k$$

$$= \sum_{k=0}^{\infty} \left\{ {n \atop k} \right\} x^k.$$

Since $\left\{ {n \atop k} \right\} = 0$ for $k \geq n$ we have $\mathcal{P}(x) = \sum_{k=1}^{n} \left\{ {n \atop k} \right\} x^k$ and consequently $\mathcal{P}(1) = \mathcal{B}_n$. ■

THEOREM 5.17. *Exponential generating function of Bell numbers is*

$$\mathcal{B}(x) = e^{e^x - 1} = \sum_{n=0}^{\infty} \frac{\mathcal{B}_n}{n!} x^n.$$

Proof. Recall that $\sum_{n=k}^{\infty} \left\{ {n \atop k} \right\} \frac{x^n}{n!} = \frac{(e^x-1)^k}{k!}$. Then

$$\mathcal{B}(x) = \sum_{n=0}^{\infty} \mathcal{B}_n \frac{x^n}{n!}$$

$$= \sum_{n=0}^{\infty} \sum_{k=1}^{n} \left\{ {n \atop k} \right\} \frac{x^n}{n!}$$

$$= \sum_{k=0}^{\infty} \sum_{n=k}^{\infty} \left\{ {n \atop k} \right\} \frac{x^n}{n!}$$

$$= \sum_{k=0}^{\infty} \frac{(e^x - 1)^k}{k!}.$$

But the last term we have obtained is just e^{e^x-1}. ■

THEOREM 5.18. *Bell numbers satisfy the following recursive relations*

$$\mathcal{B}_{n+1} = \sum_{k=0}^{n} \binom{n}{k} \mathcal{B}_k$$

and

$$\mathcal{B}_{n+m} = \sum_{k=0}^{n} \sum_{j=0}^{m} \left\{ {m \atop j} \right\} \binom{n}{k} j^{n-k} \mathcal{B}_k.$$

Proof. We prove the first relation and leave the second as an exercise. We first compute $e^x\mathcal{B}(x)$:

$$\begin{aligned} e^x\mathcal{B}(x) &= \sum_{i=0}^{\infty}\frac{x^i}{i!}\cdot\sum_{j=0}^{\infty}\frac{\mathcal{B}_j}{j!}x^j \\ &= \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\frac{1}{i!\,j!}\mathcal{B}_j\,x^{i+j} \\ &= \sum_{n=0}^{\infty}\sum_{k=0}^{n}\frac{1}{(n-k)!\,k!}\mathcal{B}_k\,x^n \\ &= \sum_{n=0}^{\infty}\frac{1}{n!}\sum_{k=0}^{n}\binom{n}{k}\mathcal{B}_k\,x^n \end{aligned}$$

Derivative of $\mathcal{B}(x)$ is given by

$$\begin{aligned} \mathcal{B}'(x) &= \sum_{n=1}^{\infty}\mathcal{B}_n\frac{x^{n-1}}{(n-1)!} \\ &= \sum_{n=0}^{\infty}\mathcal{B}_{n+1}\frac{x^n}{n!} \end{aligned}$$

On the other hand $\mathcal{B}(x) = e^{e^x-1}$ so $\mathcal{B}'^{(x)} = e^x e^{e^x-1} = e^x\mathcal{B}(x)$. Thus

$$\sum_{n=0}^{\infty}\mathcal{B}_{n+1}\frac{x^n}{n!} = \sum_{n=0}^{\infty}\frac{1}{n!}\sum_{k=0}^{n}\binom{n}{k}\mathcal{B}_k\,x^n.$$

Now, comparing the coefficients of $\frac{x^n}{n!}$ leads the desired equality. ■

Bell Triangle

The Bell numbers can easily be calculated by the Bell triangle,

- The first row consists of a single 1,
- Each row starts with the last element of the previous row,
- Each element is equal to the sum of elements on its left and left-top,
- Each row has one more element than the previous row,
- The last element of n th row is $\mathcal{B}_n$.

1					
1	**2**				
2	3	**5**			
5	7	10	**15**		
15	20	27	37	**52**	
52	67	87	114	151	**203**

We close this section by stating a double sum expression and an integral representation of Bell numbers.

$$\mathcal{B}_n = \sum_{k=1}^{n}\sum_{j=1}^{k}(-1)^{k-i}\frac{i^n}{k!},$$

$$\mathcal{B}_n = \frac{n!}{2\pi i e}\int_{\gamma}\frac{e^{e^t}}{t^{n+1}}dt.$$

Table of Stirling Numbers

$n\backslash k$	1	2	3	4	5	6	7	8	9	10	11	12	13
1	1												
2	1	1											
3	1	3	1										
4	1	7	6	1									
5	1	15	25	10	1								
6	1	31	90	65	15	1							
7	1	63	301	350	140	21	1						
8	1	127	966	1701	1050	266	28	1					
9	1	255	3025	7770	6951	2646	462	36	1				
10	1	511	9330	34105	42525	22827	5880	750	45	1			
11	1	1023	28501	145750	246730	179487	63987	11880	1155	55	1		
12	1	2047	86526	611501	1379400	1323652	627396	159027	22275	1705	66	1	
13	1	4095	261625	2532530	7508501	9321312	5715424	1899612	359502	39325	2431	78	1

Table of Stirling Numbers $\left\{ n \atop k \right\}$ for $1 \le n \le 13$ and $1 \le k \le n$

$n\backslash k$	1	2	3	4	5	6	7	8	9	10	11
1	1										
2	1	1									
3	2	3	1								
4	6	11	6	1							
5	24	50	35	10	1						
6	120	274	225	85	15	1					
7	720	1764	1624	735	175	21	1				
8	5040	13068	13132	6769	1960	322	28	1			
9	40320	109584	118124	67284	22449	4536	546	36	1		
10	362880	1026576	1172700	723680	269325	63273	9450	870	45	1	
11	3628800	10628640	12753576	8409500	3416930	902055	157773	18150	1320	55	1

Table of Stirling Numbers $\left[n \atop k \right]$ for $1 \le n \le 11$ and $1 \le k \le n$

Exercises

1. In how many ways can 30 distinguishable balls be placed in four identical boxes so that

a) There is at least one ball in each box,

b) Some boxes may be empty?

2. Prove the following identities without using induction:

$$\frac{x^{\overline{n}}}{n!} = \sum_{i=1}^{n} \binom{n-1}{i-1} \frac{x_{\underline{i}}}{i!} \,,$$

$$\frac{x_{\underline{n}}}{n!} = \sum_{i=1}^{n} \binom{n-1}{i-1} \frac{x^{\overline{i}}}{i!} \,.$$

3. Show that $\left\{ {n \atop 3} \right\} = \frac{1}{2}(3^{n-1} + 1) - 2^{n-1}$ for $n \geq 1$.

4. Show that $\sum_{k=0}^{n} \left\{ {n \atop k} \right\} x^{\underline{k}} = x^n$.

5. Let u_n be the number of ways of partitioning a set with $n > 0$ elements into subsets of sizes not exceeding 2. Show that $u_{n+1} = u_n + n u_{n-1}$.

6. Show that $\sum_{j=0}^{n} \left\{ {n \atop j} \right\} \left[{j \atop k} \right] = \sum_{j=0}^{n} \left[{n \atop j} \right] \left\{ {j \atop k} \right\} = \delta_n^k$.

7. Show that

a) $\left[{n \atop 1} \right] = (n-1)!$

b) $\left[{n \atop n-1} \right] = \binom{n}{2}$,

c) $\left[{n \atop 2} \right] = (n-1)!\, H_{n-1}$,

d) $\left[{n \atop n-2} \right] = \frac{1}{4}(3n-1)\binom{n}{3}$.

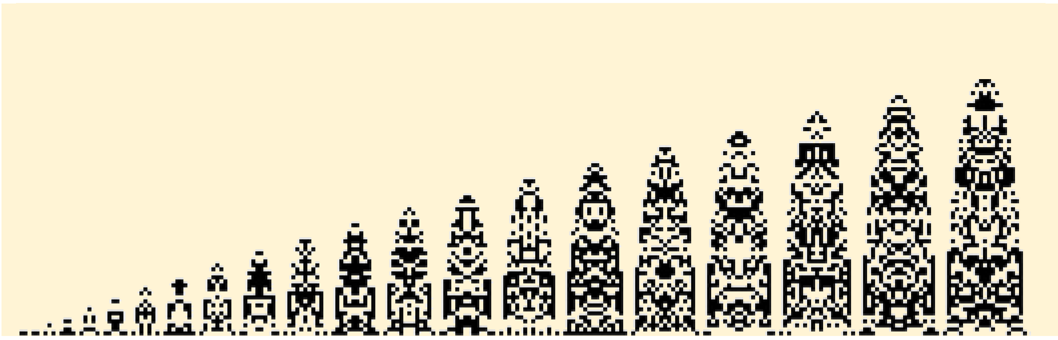
8. For any nonnegative integer n prove that the following are true:

a) $\mathcal{B}_n < n!$,

b) $\mathcal{B}_n = \frac{1}{e} \sum_{k=1}^{n} \frac{k^n}{k!}$.

9. Below figure is for $n = 4$. Draw a figure for $n = 5$.

6. EULERIAN NUMBERS



The number of permutations of $\{1,2,\ldots,n\}$ that have m −ascents is called an ***Eulerian number*** and is denoted by $\left\langle {n \atop m} \right\rangle$. By convention $\left\langle {0 \atop 0} \right\rangle = 0$ and $\left\langle {n \atop 0} \right\rangle = 1$ for $n > 0$.

Example 1. The following permutation has 5 ascents:

4 10 3 9 1 5 6 2 8 7

↗ ↘ ↗ ↘ ↗ ↗ ↘ ↗ ↘

Example 2. In the following table we see all permutations of {1,2,3,4} together with the number of ascents they contain.

1234	↗↗↗	3	2134	↘↗↗	2	3124	↘↗↗	2	4123	↘↗↗	2
1243	↗↗↘	2	2143	↘↗↘	1	3142	↘↗↘	1	4132	↘↗↘	1
1324	↗↘↗	2	2314	↗↘↗	2	3214	↘↘↗	1	4213	↘↘↗	1
1342	↗↗↘	2	2341	↗↗↘	2	3241	↘↗↘	1	4231	↘↗↘	1
1432	↗↘↘	1	2413	↗↘↗	2	3412	↗↘↗	2	4312	↘↘↗	1
1423	↗↘↗	2	2431	↗↘↘	1	3421	↗↘↘	1	4321	↘↘↘	0

It is seen that $\left\langle {4 \atop 0} \right\rangle = 1$, $\left\langle {4 \atop 1} \right\rangle = 11$, $\left\langle {4 \atop 2} \right\rangle = 11$, $\left\langle {4 \atop 3} \right\rangle = 1$.

There cannot be any permutation of $\{1, \dots, n\}$ which has n ascents, therefore

$$\left\langle {n \atop n} \right\rangle = 0, \quad n > 0.$$

For fixed n, $(1,2, \dots, n-2, n-1, n)$ has n-1 ascents and there is no other permutation with $n-1$ ascents, thus

$$\left\langle {n \atop n-1} \right\rangle = 1, \qquad n > 0.$$

For fixed n, there is only one permutation without any ascents: $(n, n-1, n-2, \cdots 2,1)$, then

$$\left\langle {n \atop 0} \right\rangle = 1, \qquad n > 0$$

A permutation has k ascents if and only if the reverse permutation has $n-1-k$ ascents, so

$$\left\langle {n \atop m} \right\rangle = \left\langle {n \atop n-m-1} \right\rangle, \qquad n > 0$$

THEOREM 6.1 (*Basic recursion of Eulerian numbers*). *For any integers $0 < n$ and $0 \le m < n$, Eulerian numbers satisfy the recursive relation*

$$\left\langle {n \atop m} \right\rangle = (m+1) \left\langle {n-1 \atop m} \right\rangle + (n-m) \left\langle {n-1 \atop m-1} \right\rangle.$$

Proof. Consider any permutation $\sigma_1\sigma_2 \cdots \sigma_n$ of $\{1, 2, \dots, n\}$ with k ascents. We have $\sigma_i = n$ for some $1 \le i \le n$, and removing this σ_i yields a permutation $\tilde{\sigma}$ of $\{1, 2, . \dots, n-1\}$ with either k or $k-1$ ascents. Every permutation of $\{1, 2, \dots, n\}$ with k ascents is therefore built from a unique permutation of $\{1, 2, \dots, n-1\}$ with k or $k-1$ ascents by inserting n. There are now two cases. Given a permutation of $\{1, 2, \dots, n-1\}$ with $k-1$ ascents, we gain an ascent by inserting n only when we do so at a descent or at the end of the permutation. There are $n-k-1$ descents, so this produces $n-k$ permutations of $\{1, 2, \dots, n\}$ with k ascents. Similarly, given a permutation of $\{1, 2, \dots, n-1\}$ with k ascents, we want to preserve the number of ascents when inserting n. To do this, the insertion must happen at one of the k ascents, or at the beginning of the permutation. This produces $k+1$ permutations of $\{1, 2, \dots, n\}$ with k ascents. Combining these two cases yields the desired recurrence. ■

Example 3. It is given that $\left\langle {5 \atop 2} \right\rangle = 66$ and $\left\langle {5 \atop 3} \right\rangle = 26$, Compute $\left\langle {6 \atop 3} \right\rangle$.

Any permutation of $\{1,2,3,4,5,6\}$ can be obtained from a unique permutation $\sigma_1\sigma_2\sigma_3\sigma_4\sigma_5$ of $\{1,2,3,4,5\}$ by inserting '6' in the slot between two successive terms or by appending to one of the ends:

$$\mathbf{6}\ \sigma_1\sigma_2\sigma_3\sigma_4\sigma_5 \qquad \sigma_1\mathbf{6}\ \sigma_2\sigma_3\sigma_4\sigma_5 \qquad \cdots \qquad \sigma_1\sigma_2\sigma_3\sigma_4\ \sigma_5\mathbf{6}$$

Let $\sigma_1\sigma_2\sigma_3\sigma_4\sigma_5$ be a permutation of $\{1,2,3,4,5\}$. Appending '6' to the left end of $\sigma_1\sigma_2\sigma_3\sigma_4\sigma_5$ or inserting it in one of the ascending slots does not increase the number of ascents. So, each permutation of $\{1,2,3,4,5\}$ with 3 ascents produces 4 permutations of $\{1,2,3,4,5,6\}$ with 3 ascents. Contribution is $(m+1)\left\langle {n-1 \atop m} \right\rangle = 4 \cdot \left\langle {5 \atop 3} \right\rangle$.

Let $\sigma_1\sigma_2\sigma_3\sigma_4\sigma_5$ be a permutation of $\{1,2,3,4,5\}$. Since appending '6' to the right end of $\sigma_1\sigma_2\sigma_3\sigma_4\sigma_5$ or inserting it in one of descending slots increases the number of ascents by one. So, each permutation of $\{1,2,3,4,5\}$ with 2 ascents produces 3 permutations of $\{1,2,3,4,5,6\}$ with 3 ascents. Contribution is $(n-m)\left\langle {n-1 \atop m-1} \right\rangle = 3 \cdot \left\langle {5 \atop 2} \right\rangle$.

Finally, $\left\langle {6 \atop 3} \right\rangle = 4\left\langle {5 \atop 3} \right\rangle + 3\left\langle {5 \atop 2} \right\rangle = 302$.

THEOREM 6.2. *For integers $0 < m < n$, a closed form expression for Eulerian numbers is*

$$\left\langle {n \atop m} \right\rangle = \sum_{j=0}^{n} (-1)^j \binom{n+1}{j} (m+1-j)^n.$$

■

THEOREM 6.3 (Basic Identities). *For any positive integer n and real number x, the following equalities hold:*

i. $\sum_{k=0}^{n} \left\langle {n \atop k} \right\rangle = n!$

ii. $x^n = \sum_{k=0}^{n} \left\langle {n \atop k} \right\rangle \binom{x+k}{n}$ *(Worpitzky's identity)*

iii. $\sum_{k=1}^{n} k^m = \sum_{i=1}^{m} \left\langle {m \atop i} \right\rangle \binom{n+i+1}{m+1}$.

Proof. i. $\left\langle {n \atop k} \right\rangle$ is the number of permutations with k ascents, so the sum $\left\langle {n \atop 0} \right\rangle + \left\langle {n \atop 1} \right\rangle + \cdots + \left\langle {n \atop n-1} \right\rangle$ counts each permutation once and only once. Thus, the sum of the Eulerian numbers for a fixed value of n is the total number of permutations.

ii. (Induction on n) Claim is true for $n = 0$: $x^0 = \left\langle {0 \atop 0} \right\rangle \binom{x}{0} = 1$. Assume that equality holds for some non-negative integer n. Then

$$\sum_{k=0}^{n+1} \left\langle {n+1 \atop k} \right\rangle \binom{x+k}{n+1} = \sum_{k=0}^{n+1} (k+1) \left\langle {n \atop k} \right\rangle \binom{x+k}{n+1} + \sum_{k=1}^{n+1} (n+1-k) \left\langle {n \atop k-1} \right\rangle \binom{x+k}{n+1}$$

$$= \sum_{k=0}^{n} (k+1) \left\langle {n \atop k} \right\rangle \binom{x+k}{n+1} + \sum_{k=0}^{n} (n-k) \left\langle {n \atop k} \right\rangle \binom{x+k+1}{n+1}$$

$$= \sum_{k=0}^{n} \left(\frac{(k+1)(x+k-n) + (n-k)(x+k+1)}{n+1} \right) \left\langle {n \atop k} \right\rangle \binom{x+k}{n}$$

$$= x \sum_{k=0}^{n} \left\langle {n \atop k} \right\rangle \binom{x+k}{n} = x^{n+1}.$$

iii. Recall that $\sum_{k=0}^{n} \binom{k+j}{m} = \binom{n+j+1}{m+1}$.

$$\sum_{k=0}^{n} k^m = \sum_{k=1}^{n} \sum_{j=0}^{m} \left\langle {m \atop j} \right\rangle \binom{k+j}{m} = \sum_{j=1}^{m} \left\langle {m \atop j} \right\rangle \sum_{k=0}^{n} \binom{k+j}{m}$$

$$= \sum_{j=1}^{m} \left\langle {m \atop j} \right\rangle \binom{n+j+1}{m+1}. \quad \blacksquare$$

THEOREM 6.4. *For integer n, the following identities are true.*

$$\left\langle {n \atop 1} \right\rangle = 2^n - n - 1,$$

$$\left\langle {n \atop 2} \right\rangle = 3^n - 2^{n(n+1)} + \frac{1}{2} n(n+1).$$

Proof. Define the sequence $\{u_n\}$ by setting $u_n = \left\langle {n \atop 1} \right\rangle$. By the basic recursion we have $u_n = 2u_{n-1} + n - 1$. Then

$$\sum_{n=1}^{\infty} u_n x^n = 2 \sum_{n=1}^{\infty} u_{n-1} x^n + \sum_{n=1}^{\infty} (n-1) x^n$$

$$= 2x\sum_{n=1}^{\infty} u_n x^n + \frac{x^2}{(1-x)^2}.$$

Rearranging the terms we have

$$\sum_{n=1}^{\infty} u_n x^n = \frac{1}{(1-2x)(1-x)^2}$$

$$= \frac{1}{1-2x} - \frac{1}{(1-x)^2}$$

$$= \left(\sum_{n=1}^{\infty} 2^n x^n\right) - \left(\sum_{n=1}^{\infty} (n+1) x^n\right)$$

$$= \sum_{n=1}^{\infty} (2^n - n - 1) x^n$$

thus we obtain the first identity. The second one can be obtained similarly. ■

The following are interesting infinite sums related with Eulerian numbers:

$$\sum_{k=1}^{\infty} k x^k = \frac{x}{(1-x)^2},$$

$$\sum_{k=1}^{\infty} k^2 x^k = \frac{x}{(1-x)^2}(1+x),$$

$$\sum_{k=1}^{\infty} k^3 x^k = \frac{x}{(1-x)^2}(1+4x+x^2),$$

and in general

$$\sum_{k=1}^{\infty} k^n x^k = \frac{x}{(1-x)^{n+1}} \sum_{j=0}^{n-1} \left\langle {n \atop j} \right\rangle x^j.$$

THEOREM 6.5. *A bivariate exponential generating function for Eulerian numbers is*

$$\frac{(t-1)e^x}{te^x - e^{xt}} = \sum_{k=0}^{\infty} \sum_{n=0}^{\infty} \left\langle {n \atop k} \right\rangle \frac{x^n}{n!} \frac{t^k}{k!}.$$

Table of Eulerian Numbers

$n\backslash k$	1	2	3	4	5	6	7	8	9	10	11
1	1										
2	1	1									
3	1	4	1								
4	1	11	11	1							
5	1	26	66	26	1						
6	1	57	302	302	57	1					
7	1	120	1191	2416	1191	120	1				
8	1	247	4293	15619	15619	4293	247	1			
9	1	502	14608	88234	156190	88234	14608	502	1		
10	1	1013	47840	455192	1310354	1310354	455192	47840	1013	1	
11	1	2036	152637	2203488	9738114	15724248	9738114	2203488	152637	2036	1

Table of Eulerian Numbers $\left\langle {n \atop k} \right\rangle$ for $1 \le n \le 11$ and $1 \le k \le n$

7. DERANGEMENTS

There are several well-known problems which are used to introduce derangements. One of them is 'Old Hats Problem':

A group of n men enter a restaurant and check their hats. The hat-checker is absent minded, and upon leaving, she redistributes the hats back to the men at random. What is the probability P_n that no man gets his correct hat, and how does P_n behave as n approaches infinity?

Let $n = 4$ and call the men A, B, C, D. Then the ordering $ABCD$ of hats means that each man gets his correct hat, whereas $ABDC$ means that A and B get their own hats but those of C and D are swapped. The ordering $BCDA$ means that no one gets his own hat. To find the number of all such orderings, we list all possible orderings of four hats:

ABCD	ABDC	ACBD	ACDB	ADBC	ADCB
BACD	**BADC**	BCAD	**BCDA**	**BDAC**	BDCA
CABD	**CADB**	CBAD	CBDA	**CDAB**	**CDBA**
DABC	DACB	DBAC	DBCA	**DCAB**	**DCBA**

The orderings for which no man gets his own hat are in bold typeset. There are 9 such orderings out of total 24 orderings. Thus, $P_n = 9/24 = 0{,}375$.

If we let $n = 10$ or $n = 100$ how would this probabiliy change? Would it be smaller, too smaller or larger than 0.375? To answer thes questions, since we know that the number of all orderings is $n!$, we have to find the number of orderings for which no letter is in its original position.

Let $X = \{x_1, x_2, \dots, x_n\}$ be an ordered set. In a permutation of elements of X an element which appears in its original position is called a ***fixed point*** of the permutation. A ***derangement*** is a permutation which has no fixed points. We denote the number of derangements of n objects with $\mathcal{D}_n$. By convention $\mathcal{D}_0 = 1$.

Example 1. For $n = 1$, there is only one permutation of $\{1\}$ and this permutation has a fix point. Thus $\mathcal{D}_1 = 0$.

For $n = 2$, there are two permutations of $\{1,2\}$, namely 12 and 21. Since only one of these has no fixed points, $\mathcal{D}_2 = 1$.

For $n = 3$, the list of all permutations are **123**, **1**32, 21**3**, 231, 312, 3**2**1 where the bold characters show fixed points. It follows that there are only 2 derangements. $\mathcal{D}_3 = 2$.

Denote the number of permutations of n objects with k fixed points with $\mathcal{D}_{n,k}$. To construct such a permutation, the fixed points can be chosen in $\binom{n}{k}$ ways and the remaining elements can be arranged in $\mathcal{D}_{n-k}$ ways to form a derangement among themselves. Then

$$\mathcal{D}_{n,k} = \binom{n}{k}\mathcal{D}_{n-k}.$$

THEOREM 7.1. *For any positive integer n*

$$\sum_{k=0}^{n} \mathcal{D}_{n,k} = n!.$$

Proof. Since $\mathcal{D}_{n,k}$ is the number of permutations with k fixed points, the sum $\sum_{i=0}^{n} \mathcal{D}_{n,k}$ counts each permutation exactly once. ■

From the theorem it follows that

$$\sum_{k=0}^{n} \binom{n}{k}\mathcal{D}_k = n!$$

In all permutations of $\{1, \dots, n\}$, the element 1 is at the first position in exactly $(n-1)!$ permutations. This means that n appears as a fixed point in $(n-1)!$ permutations. Since this is the same for all elements of the set, the total number of fixed points in all permutations is $n \cdot (n-1)! = n!$. It follows that the average number of fixed points of all permutations of $\{1, \dots, n\}$ is 1. Another way of writing the total number of fixed points in all permutations is $\sum_{k=0}^{n} k\mathcal{D}_{n,k}$, then

$$\sum_{k=0}^{n} k \binom{n}{k} \mathcal{D}_{n-k} = n!.$$

THEOREM 7.2 (*Second Order Recursion for Derangements*). *For $n > 2$, the number of derangements satisfy the second order recursion*

$$\mathcal{D}_n = (n-1)(\mathcal{D}_{n-1} + \mathcal{D}_{n-2}).$$

Proof. Suppose that there are n balls numbered $1,2, \dots, n$ and let there be n boxes also numbered $1,2, \dots, n$. We wish to put one ball to each boxes such that the number of each box is different from the number of the ball in it. Clearly, the number of such distributions is $\mathcal{D}_n$. There are $n-1$ ways to put a ball in the first box. Let us assume that the first box contains the ball i. Now there are two possibilities, depending on whether or not box i contains ball 1:

- Box i does not contain the ball 1. This case is equivalent to solving the problem with $n-1$ boxes $n-1$ balls.

- Box i contains the ball 1. Now the problem reduces to $n-2$ balls and $n-2$ boxes.

Then we obtain the desired recursion. ■

THEOREM 7.3 (*First Order Recursion for Derangements*). *For $n > 1$, the number of derangements satisfy the first order recursion*

$$\mathcal{D}_n = n\mathcal{D}_{n-1} + (-1)^n .$$

Proof. Rearrange the second order recursion as $\mathcal{D}_n - n\mathcal{D}_{n-1} = -(\mathcal{D}_{n-1} - (n-1)\mathcal{D}_{n-2})$. Now define the sequence $\{u_n\}$ by $u_n = \mathcal{D}_n - n\mathcal{D}_{n-1}$ for $n = 1,2, \ldots$. We observe that $u_n = -u_{n-1}$. Since $u_1 = \mathcal{D}_1 - \mathcal{D}_0 = -1$, we conclude that $u_n = (-1)^n$. Now we have $u_n = (-1)^n = \mathcal{D}_n - n\mathcal{D}_{n-1}$. ∎

THEOREM 7.4. *Exponential generating function of the number of derangements is*

$$\sum_{n=1}^{\infty} \mathcal{D}_n \frac{x^n}{n!} = \frac{e^{-x}}{1-x}.$$

Proof. Multiply both sides of the first order recursion by x^n and form the sum for $n = 1,2, \ldots$

$$\sum_{n=1}^{\infty} \mathcal{D}_n \frac{x^n}{n!} = \sum_{n=1}^{\infty} \mathcal{D}_{n-1} \frac{x^n}{(n-1)!} + \sum_{n=1}^{\infty} (-1)^n \frac{x^n}{n!}.$$

Then

$$\sum_{n=0}^{\infty} \mathcal{D}_n \frac{x^n}{n!} - 1 = x \sum_{n=1}^{\infty} \mathcal{D}_{n-1} \frac{x^{n-1}}{(n-1)!} + \sum_{n=0}^{\infty} \frac{(-x)^n}{n!} - 1$$

which means

$$\mathcal{D}(x) = x\mathcal{D}(x) + e^{-x}.$$

We obtain the result by rearranging the last equality. ∎

THEOREM 7.5. *For any nonnegative integer n, the number of derangements of n objects is*

$$\mathcal{D}_n = n! \sum_{k=0}^{n} \frac{(-1)^k}{k!}.$$

Proof. We give three different proofs.

(Principle of Inclusion-Exclusion.) Define the condition C_i to be 'i is at position i' for $i = 1, \ldots, n$. Then $\mathcal{D}_n$ is the number of permutations of $\{1, \ldots, n\}$ for which none of the conditions C_i is satisfied. In terms of inclusion-exclusion principle $\mathcal{D}_n = \overline{N}$.

For any $i = 1, \ldots n$, the number of permutations which satisfy C_i is $(n-1)!$, then $N_1 = n \cdot (n-1)!$. Analogously, the number of permutations which satisfy k of the conditions is $(n-k)!$, then $N_k = \binom{n}{k} \cdot (n-k)! = \frac{n!}{k!}$.

It follows that $N_0 = n!$ and for any $k = 1,2,\dots,n$, $N_k = \frac{n!}{k!}$. We obtain

$$\begin{aligned}\mathcal{D}_n &= \overline{N} = N_0 - N_1 + N_2 - \cdots \\ &= n!\sum_{k=0}^{n}\frac{(-1)^k}{k!}.\end{aligned}$$

(Exponential Generating Function) We consider the exponential generating function of the number of derangements

$$\begin{aligned}\sum_{n=1}^{\infty}\mathcal{D}_n\frac{x^n}{n!} &= \frac{e^{-x}}{1-x} \\ &= \left(\sum_{i=0}^{\infty}x^i\right)\left(\sum_{j=0}^{\infty}\frac{(-x)^j}{j!}\right) \\ &= \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\frac{(-1)^j}{j!}x^{i+j} \\ &= \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\frac{(-1)^k}{k!}\right)x^n.\end{aligned}$$

Comparing the coefficients of $\frac{x^n}{n!}$ we obtain the desired expression.

(Binomial Inversion) Since $\sum_{k=0}^{n}\binom{n}{k}\mathcal{D}_k = n!$, from binomial inversion theorem we have

$$\begin{aligned}\mathcal{D}_n &= \sum_{k=0}^{n}(-1)^{n-k}\binom{n}{k}k! \\ &= \sum_{k=0}^{n}(-1)^{n-k}\frac{n!}{(n-k)!} \\ &= n!\sum_{k=0}^{n}(-1)^k\frac{1}{k!}.\end{aligned}$$

This completes proof. ■■■

We return back to the 'Old Hats Problem'. The number of orderings for which no man gets his own hat is $\mathcal{D}_n$ and the probability P_n that no man gets his correct hat is

$$P_n = \frac{\mathcal{D}_n}{n!} = \sum_{k=0}^{n} (-1)^k \frac{1}{k!}.$$

Now we observe that

$$\lim_{n\to\infty} \left(\sum_{k=0}^{n} (-1)^k \frac{1}{k!} \right) = \sum_{k=0}^{\infty} (-1)^k \frac{1}{k!} = \frac{1}{e}.$$

Then, as $n \to \infty$, the probability converges to $1/e$ and consequentely

$$P_n \approx \frac{1}{e}.$$

Since the series $\sum_{k=0}^{n}(-1)^k \frac{1}{k!}$ converges to $1/e$ very rapidly, above approximation can be used even for small values of n. For comparing $1/e = 0.36787944\cdots$ to actual probability values, we give the list

n	$\mathcal{D}_n/n!$
2	0.5000
5	0.3667
10	0.3679

At the beginning of this section we have computed that $P_4 = 0.375$. Now we see that as the number of men gets larger the probability that no one gets his own hat decreases very slightly. For practical purposes, independent of n, we say that this probability is $1/e$.

Then, The number of derangements can be approximated by $\frac{n!}{e}$. In fact, for any nonnegative integer we have

$$\mathcal{D}_n = \left[\frac{n!}{e}\right]$$

where [ ] is the closest integer function.

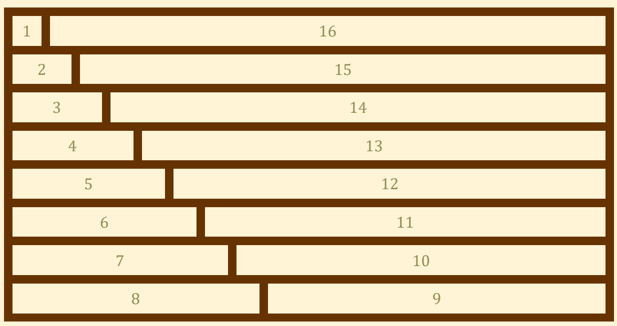
Appendix A

FINITE SUMS

In the late 18th century, Büttner, a German schoolmaster, gave —with the intention of keeping his pupils busy for another hour— the task to sum hundred terms of an arithmetic progression to a class of little boys who, of course, had never heard of arithmetic progressions. The youngest pupil, however, wrote down the answer instantaneously and waited gloriously, with his arms folded, for the next hour while his classmates toiled: at the end it turned out that little Johann Friederich Carl Gauss had been the only one to hand in the correct answer. Young Gauss had seen instantaneously how to sum such a series analytically: the sum equals the number of terms multiplied by the average of the first and the last term. (To quote E.T.Bell: "The problem was of the following sort, 81297 + 81495 + 81693 + ... + 100899, where the step from one number to the next is the same all along (here 198), and a given number of terms (here 100) are to be added.")

To compute the sum easily Gauss made a pair of first and last terms, that's 182196, Then multiplied by 50 pairs, to find 9109800 and there, the job is done.

In two respects this is a classical example: firstly young Gauss produced his answer about a thousand times as fast as his classmates, secondly he was the only one to produce the correct answer. So much for the effective ordering of one's thoughts!



$$1+2+3+\cdots+16=(1+16)+(2+15)+\cdots+(8+9)=17\times\frac{16}{2}$$

Concern of this section is to introduce a collection of methods for the evaluation of finite sums whose summands are given as a sequence, either in functional form $f(k)$, or in subscript form a_k. The last part of the section is devoted to 'Finite Calculus' which mimics the methods of calculus for computing the finite sums.

Let $\{a_n\} = a_0, a_1, \ldots$ be a sequence of real (or complex) numbers. We denote the sum $a_0 + a_1 + \cdots + a_n$ by S_n, that is

$$S_n = a_0 + a_1 + \cdots + a_n = \sum_{k=0}^{n} a_k.$$

More generally, the sum $a_m + a_{m+1} + \cdots + a_n$ for any m with $0 \leq m \leq n$ is written as

$$\sum_{k=m}^{n} a_k \quad \text{or} \quad \sum\nolimits_{k=m}^{n} a_k.$$

In the above notation k is called the ***index variable***. Note that index variable is dummy, in the sense that it does not make any harm replacing k with any other variable: $\sum_{k=m}^{n} a_k = \sum_{j=m}^{n} a_j$. m is called the ***lower bound*** (or lower limit); b is called the ***upper bound*** (or upper limit) and a_k, for each $k = m, \ldots, n$, is called a ***summand***.

An alternative notation is $\sum_{k \in A} a_k$ which means that the terms a_k are summed up for all values of $k \in A$. For the given context, if there is no doubt about the bounds, one may write $\sum_k a_k$ or even $\sum a_k$.

After listing some basic properties of the finite summation, we examine several methods for computing S_n.

Properties of Summation

For any sequences $\{a_n\}$, $\{b_n\}$, and $\lambda \in \mathbb{C}$ the following properties hold for finite sums:

$$\sum_{k=m}^{n} \lambda = \lambda(n-m+1),$$

$$\sum_{k=m}^{n} \lambda(a_k + b_k) = \lambda \sum_{k=m}^{n} a_k + \sum_{k=m}^{n} b_k\,,$$

$$\sum_{k=m}^{n} a_k = \sum_{k=m}^{\ell} a_k + \sum_{k=\ell+1}^{n} a_k \quad \text{for any integer } m \le \ell \le n,$$

$$\sum_{k=m}^{n} a_k = \sum_{k=n}^{m} a_{n+m-k},$$

$$\sum_{k=m}^{n} a_k = \sum_{k=m}^{n} a_{\varphi(k)}, \text{ for any permutation } \varphi \text{ of } \{m, \dots, n\}$$

$$\sum_{k=m}^{n} a_k = \sum_{k=m-\ell}^{n-\ell} a_{k+\ell} \quad \text{for any integer } \ell,$$

$$\sum_{k=m}^{n} (a_k + b_k)^2 = \sum_{k=m}^{n} a_k^2 + 2\sum_{k=m}^{n} a_k b_k + \sum_{k=m}^{n} b_k^2,$$

Double Sums

Let $\{a_{ij}\}$ be a two dimensional array, say

$$\begin{matrix} a_{00} & a_{01} & a_{02} & \cdots & a_{0j} & \cdots & a_{0n} \\ a_{10} & a_{11} & a_{12} & \cdots & a_{1j} & \cdots & a_{1n} \\ a_{20} & a_{21} & a_{22} & \cdots & a_{2j} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \cdots & \vdots & \cdots & \vdots \\ a_{i0} & a_{i1} & a_{i2} & \cdots & a_{ij} & \cdots & a_{in} \\ \vdots & \vdots & \vdots & \cdots & \vdots & \cdots & \vdots \\ a_{m0} & a_{m1} & a_{m2} & \cdots & a_{mj} & \cdots & a_{mn} \end{matrix}.$$

If we wish to compute the sum S of all terms of this array, we can first find sum of each row, then compute the sum of row sums. Say that the sum of terms on row i is R_i for $i = 0, \dots, m$, that is

$$R_i = \sum_{j=0}^{n} a_{ij}$$

and consequently

$$S = \sum_{i=0}^{m} R_i$$

$$= \sum_{i=0}^{m} \left(\sum_{j=0}^{n} a_{ij} \right).$$

Dropping the braces, we write this sum as

$$\sum_{i=0}^{m} \sum_{j=0}^{n} a_{ij}.$$

This iterated sum is called a ***double sum*.**

To compute the sum S of all terms of this array, we could first compute the sums of columns rather than that of rows. Assume that the sum of terms on column j is C_j for $i = 0, \dots, n$, that is

$$C_j = \sum_{i=0}^{m} a_{ij}$$

and sum of all elements of the array $\{a_{ij}\}$ is then,

$$S = \sum_{j=0}^{n} C_j$$

$$= \sum_{j=0}^{n} \left(\sum_{i=0}^{m} a_{ij} \right)$$

$$= \sum_{j=0}^{n} \sum_{i=0}^{m} a_{ij}.$$

We observe that

$$\sum_{i=0}^{m} \sum_{j=0}^{n} a_{ij} = \sum_{j=0}^{n} \sum_{i=0}^{m} a_{ij} .$$

PERTURBATION OF SUMMATION

The sum S_n can be written as $S_n = a_0 + \sum_{k=1}^{n} a_k$. Then by replacing the index k with $k+1$, we get $S_n = a_0 + \sum_{k=0}^{n-1} a_{k+1}$. Finally, adding $a_{n+1} - a_{n+1}$ to the sum we obtain the following ***perturbation*** of the original sum:

$$\begin{aligned} S_n &= a_0 + \sum_{k=0}^{n-1} a_{k+1} + a_{n+1} - a_{n+1} \\ &= a_0 + \underbrace{\overbrace{a_1 + a_2 + \cdots + a_n}^{\sum_{k=0}^{n-1} a_{k+1}} + a_{n+1}}_{\sum_{k=0}^{n} a_{k+1}} - a_{n+1} \end{aligned}$$

or

$$S_n = a_0 - a_{n+1} + \sum_{k=0}^{n} a_{k+1}$$

If the general term a_k has a suitable relation with a_{k+1}, the last summation can help us in computing S_n.

Example 1. Compute $S_n = \sum_{k=0}^{n} r^k$.

If $r = 0$, then $S_n = \sum_{i=0}^{n} r^k = 0$ and if $r = 1$, then $S_n = \sum_{i=0}^{n} 1 = n+1$. Now assume that $r \in \mathbb{C} - \{0,1\}$. As $a_0 = r^0 = 1$, $a_{n+1} = r^{n+1}$ and $a_{k+1} = r^{k+1} = r \cdot r^k$, perturbed sum is

$$\begin{aligned} S_n &= 1 - r^{n+1} + \sum_{k=0}^{n} a_{k+1} \\ &= 1 - r^{n+1} + r \underbrace{\sum_{k=0}^{n} r^k}_{S_n} \end{aligned}$$

which gives $(1-r)S_n = 1 - r^{n+1}$, that is,

$$\sum_{k=0}^{n} r^k = \frac{1 - r^{n+1}}{1 - r}.$$

Example 2. Compute $S_n = \sum_{i=1}^{n} kr^k$ where $r \in \mathbb{C} - \{1\}$.

There is no harm in writing $S_n = \sum_{i=0}^{n} kr^k$. If $r = 0$, then $S_n = \sum_{i=0}^{n} r^k = 0$. Assume that $r \neq 0$, then we have $a_0 = 0$, $a_{n+1} = (n+1)r^{n+1}$ and $a_{k+1} = (k+1)r^{k+1} = rkr^k + rr^k$. Then

$$S_n = \sum_{k=0}^{n} kr^k$$

$$= -(n+1)r^{n+1} + r\underbrace{\sum_{k=0}^{n} kr^k}_{S_n} + r\underbrace{\sum_{k=0}^{n} r^k}_{\frac{1-r^{n+1}}{1-r}}$$

$$= -(n+1)r^{n+1} + rS_n + r\frac{(1-r^{n+1})}{1-r}$$

which simplifies into

$$(1-r)S_n = \frac{r-(n+1)r^{n+1}+nr^{n+2}}{1-r}.$$

So we obtain

$$\sum_{k=1}^{n} kr^k = \frac{(nr-n-1)r^{n+1}+r}{(1-r)^2}.$$

Example 3. Compute $S_n = \sum_{k=1}^{n} k^2$.

We have $a_1 = 1$, $a_{n+1} = (n+1)^2$ and $a_{k+1} = (k+1)^2 = k^2 + 2k + 1$. Thus

$$S_n = 1-(n+1)^2 + \underbrace{\sum_{k=1}^{n} k^2}_{S_n} + 2\sum_{k=1}^{n} k + \underbrace{\sum_{k=1}^{n} 1}_{n}$$

$$= 1-(n+1)^2 + S_n + 2\sum_{k=1}^{n} k + n.$$

which results in $\sum_{k=1}^{n} k = \frac{1}{2}((n+1)^2 - 1 - n) = \frac{n(n+1)}{2}$. Direct application of the method did not work properly and resulted in the sum $\sum k$ rather than $\sum k^2$. Then we try our chance by attempting to compute the sum $T_n = \sum_{k=1}^{n} k^3$.

Perturbed sum is

$$T_n = 1 - (n+1)^3 + \sum_{k=1}^{n} (k+1)^3$$

$$= 1 - (n+1)^3 + \underbrace{\sum_{k=1}^{n} k^3}_{T_n} + 3\underbrace{\sum_{k=1}^{n} k^2}_{S_n} + 3\underbrace{\sum_{k=1}^{n} k}_{\frac{1}{2}n(n+1)} + \underbrace{\sum_{k=1}^{n} 1}_{n}$$

$$= 1 - (n+1)^3 + T_n + 3S_n + \frac{3}{2}n(n+1) + n.$$

As expected, T_n is vanished and we are left with S_n:

$$3S_n = (n+1)^3 - \frac{3}{2}n(n+1) - n - 1$$

$$= \frac{1}{2}n(n+1)(2n+1).$$

Consequently

$$\sum_{k=1}^{n} k^2 = \frac{1}{6}n(n+1)(2n+1).$$

Example 4. Compute $S_n = \sum_{k=0}^{n} k \cdot k!$.

Since $a_0 = 0, a_{n+1} = (n+1)(n+1)!$ and $a_{k+1} = (k+1)(k+1)! = (k+1)^2 k! = k^2 k! + 2kk! + k!$, perturbed sum is $S_n = -(n+1)(n+1)! + \sum_{k=1}^{n} k^2 k! + 2S_n + \sum_{k=1}^{n} k!$. Now, S_n is not vanished, but we are faced with two sums $\sum_{k=1}^{n} k^2 k!$ and $\sum_{k=1}^{n} k!$, neither of which is easier than the sum we have started with. In fact, there is not a known simple closed form to express the sum $P_n = \sum_{k=0}^{n} k!$. We can try to compute P_n hoping to obtain S_n. Perturbation of P_n gives

$$P_n = 1 - (k+1)! + \sum_{k=1}^{n} (k+1)!$$

$$= 1 - (k+1)! + \sum_{k=1}^{n} (k+1)k!$$

$$= 1 - (k+1)! + S_n + P_n.$$

So we get

$$\sum_{k=0}^{n} k \cdot k! = (n+1)! - 1.$$

Example 5. Compute $S_n = \sum_{k=0}^{n} k\binom{n}{k}$.

We have $a_0 = 0, a_{n+1} = 0$ and $a_{k+1} = (k+1)\binom{n}{k+1} = (n-k)\binom{n}{k}$. Then

$$S_n = \sum_{k=0}^{n} (n-k)\binom{n}{k}$$

$$= n\sum_{k=0}^{n}\binom{n}{k} - S_n$$

$$= n2^n - S_n.$$

Hence $S_n = n2^{n-1}$. Note that $\sum_{k=0}^{n} k\binom{n}{k} = n2^{n-1}$ is the total number of elements in all subsets of a set with n elements.

Example 6. Compute $S_n = \sum_{k=0}^{n} k^2\binom{n}{k}$.

We have $a_0 = 0, a_{n+1} = 0$ and $a_{k+1} = (k+1)^2\binom{n}{k+1} = (n-k)(k+1)\binom{n}{k}$. Then

$$S_n = \sum_{k=0}^{n} (n-k)(k+1)\binom{n}{k}$$

$$= (n-1)\underbrace{\sum_{k=0}^{n} k\binom{n}{k}}_{n2^{n-1}} - \underbrace{\sum_{k=0}^{n} k^2\binom{n}{k}}_{S_n} + n\underbrace{\sum_{k=0}^{n}\binom{n}{k}}_{2^n}$$

$$= n(n-1)2^{n-1} + n2^n - S_n$$

$$= n(n+1)2^{n-1} - S_n.$$

Hence $S_n = \sum_{k=0}^{n} k^2\binom{n}{k} = n(n+1)2^{n-2}$.

Example 7. Compute $S_n = \sum_{k=0}^{n} \sin(kx)$ where $x \in \mathbb{R}$.

First we define an auxiliary summation as $C_n = \sum_{k=0}^{n} \cos(kx)$. Since $a_0 = 0,\ a_{n+1} = \sin\big((n+1)x\big)$ and $a_{k+1} = \sin\big((k+1)x\big) = \cos x \sin(kx) + \sin x \cos(kx)$, perturbed sum is

$$S_n = -\sin\big((n+1)x\big) + \cos x\, S_n + \sin x\, C_n.$$

In a similar manner, perturbation of C_n gives

$$C_n = 1 - \cos\big((n+1)x\big) + \cos x\, C_n - \sin x\, S_n.$$

We obtain the system of equations

$$[\cos x - 1]S_n + \sin x\, C_n = \sin\big((n+1)x\big)$$

$$-\sin x\, S_n + [\cos x - 1]C_n = \cos\big((n+1)x\big) - 1$$

whose solution is

$$S_n = \frac{[\cos x - 1]\sin\big((n+1)x\big) - \big(\cos\big((n+1)x\big) - 1\big)\sin(x)}{2(1-\cos x)}$$

$$C_n = \frac{[\cos x - 1]\big(\cos\big((n+1)x\big) - 1\big) + \sin\big((n+1)x\big)\sin(x)}{2(1-\cos x)}.$$

Making use of trigonometric identities, solution can be expressed as

$$S_n = \frac{\sin\left(\frac{n+1}{2}x\right)\sin\frac{nx}{2}}{\sin\frac{x}{2}},$$

$$C_n = 1 + \frac{\cos\left(\frac{n+1}{2}x\right)\sin\frac{nx}{2}}{\sin\frac{x}{2}}.$$

Note that, using these sums, interesting trigonometric identities can be obtained such as

$$\sin 2° + \sin 4° + \sin 6° + \cdots + \sin 180° = \frac{\sin 91°}{\sin 1°}$$

$$\sin 2° + \sin 4° + \sin 6° + \cdots + \sin 60° = \frac{\sin 31°}{2\sin 1°}$$

$$\sin 10° + \sin 20° + \sin 30° + \cdots + \sin 180° = \frac{\sin 95°}{\sin 5°}$$

or

$$\sin 20° + \sin 40° + \sin 60° + \cdots + \sin 180° = \frac{\sin 100°}{\sin 10°}.$$

Converting a Single Sum to a Double Sum

If it is possible to write the general term a_k as a finite sum $a_k = \sum_{j=0}^{k} b_{kj}$, then the sum $S_n = \sum_{k=0}^{n} a_k$ can be written as a double sum $S_n = \sum_{k=0}^{n} \sum_{j=0}^{k} b_{kj}$. If we change the order of the double sum we get $S_n = \sum_{j=0}^{n} \sum_{k=j}^{n} b_{kj}$:

$$\sum_{k=0}^{n} \sum_{j=0}^{k} b_{kj} = b_{00} + (b_{10} + b_{11}) + (b_{20} + b_{21} + b_{22}) + \cdots + (b_{n0} + b_{n1} + \cdots b_{nn})$$

$$= (b_{00} + b_{10} + \cdots + b_{n0}) + (b_{11} + b_{21} + \cdots + b_{n1}) + (b_{22} + \cdots + b_{n2}) \cdots + b_{nn}$$

$$= \sum_{j=0}^{n} \sum_{k=j}^{n} b_{kj}.$$

Note that, above property can be generalized as

$$\sum_{k=a}^{n} \sum_{j=a}^{k} b_{kj} = \sum_{j=a}^{n} \sum_{k=j}^{n} b_{kj}.$$

It may be the case that computing $\sum_{j=0}^{n} \sum_{k=j}^{n} b_{kj}$ is easier than computing $\sum_{k=0}^{n} \sum_{j=0}^{k} b_{kj}$.

Example 8. Compute $S_n = \sum_{k=1}^{n} k^2$.

First note that $\sum_{j=1}^{k} 1 = k$ and $\sum_{j=1}^{k} k = k^2$. Then the given sum can be written as

$$S_n = \sum_{k=1}^{n} k^2 = \sum_{k=1}^{n} \underbrace{\sum_{j=1}^{k} k}_{k^2}$$

$$= \sum_{j=1}^{n} \sum_{k=j}^{n} k$$

$$= \frac{1}{2} \sum_{j=1}^{n} \big(n(n+1) - j(j-1)\big)$$

$$= \frac{1}{2} \left[n^2(n+1) - \sum_{j=1}^{n} j^2 - \sum_{j=1}^{n} j \right] = \frac{1}{2} n^2(n+1) - \frac{1}{2} S_n - \frac{1}{4} n(n+1)$$

and finally

$$S_n = \frac{1}{6} n(n+1)(2n+1).$$

Example 9. Compute $S_n = \sum_{k=1}^{n} k2^k$.

$$\begin{aligned} S_n &= \sum_{k=1}^{n} k2^k \\ &= \sum_{k=1}^{n}\sum_{j=1}^{k} 2^k \\ &= \sum_{j=1}^{n}\sum_{k=j}^{n} 2^k \\ &= \sum_{j=1}^{n} (2^{n+1} - 2^j) \\ &= n2^{n+1} - (2^{n+1} - 2) \\ &= (n-1)2^{n+1} + 2 \end{aligned}$$

Example 10. Compute $S_n = \sum_{k=1}^{n} H_k$ where H_k is the harmonic number: $H_k = 1 + \frac{1}{2} + \cdots + \frac{1}{k}$.

$$\begin{aligned} S_n &= \sum_{k=1}^{n} H_k \\ &= \sum_{k=1}^{n}\sum_{j=1}^{k} \frac{1}{j} \\ &= \sum_{j=1}^{n}\sum_{k=j}^{n} \frac{1}{j} \\ &= \sum_{j=1}^{n} \frac{(n+1-j)}{j} \\ &= (n+1)\sum_{j=1}^{n}\frac{1}{j} - \sum_{j=1}^{n} 1 \\ &= (n+1)H_n - n. \end{aligned}$$

Example 11. Compute $S_n = \sum_{k=1}^{n} k\, H_k$.

We replace k with $\sum_{j=1}^{k} 1$:

$$\begin{aligned} S_n &= \sum_{j=1}^{n}\sum_{k=j}^{n} H_k \\ &= \sum_{j=1}^{n} \big((n+1)H_n - n - jH_{j-1} + j - 1\big) \\ &= n(n+1)H_n - n^2 - \sum_{j=1}^{n} jH_{j-1} + \frac{1}{2}n(n+1) - n \end{aligned}$$

$$= n(n+1)\left(H_{n+1} - \frac{1}{n+1}\right) - \frac{n^2}{2} - \frac{n}{2} - \sum_{j=1}^{n} j\left(H_j - \frac{1}{j}\right)$$

$$= n(n+1)H_{n+1} - n - \frac{1}{2}n(1+n) - S_n + n$$

$$= n(n+1)H_{n+1} - \frac{n(n+1)}{2} - S_n.$$

Then

$$\sum_{k=1}^{n} k\, H_k = \frac{1}{2}n(n+1)\left(H_{n+1} - \frac{1}{2}\right).$$

Alternatively, we could try to replace H_k with $\sum_{j=1}^{k} \frac{1}{j}$:

$$\sum_{k=1}^{n} kH_k = \sum_{k=1}^{n}\sum_{j=1}^{k} \frac{k}{j}$$

$$= \sum_{j=1}^{n} \frac{1}{j} \sum_{k=j}^{n} k$$

$$= \frac{1}{2}\sum_{j=1}^{n} \frac{1}{j}\left(n(n+1) - j(j-1)\right)$$

$$= \frac{1}{2}\left(n(n+1)H_n - \frac{n(n+1)}{2} + n\right)$$

$$= \frac{1}{2}\left(n(n+1)H_{n+1} - \frac{n(n+1)}{2}\right)$$

$$= \frac{1}{2}n(n+1)\left(H_{n+1} - \frac{1}{2}\right).$$

Converting the Sum to a Power Series

If the function $S: \mathbb{R} \to \mathbb{R}$ is defined by $S(x) = \sum_{k=0}^{n} a_k x^k$ where $a_i \in \mathbb{R}$, $i = 1, \dots, n$, then

$$S'(x) = \sum_{k=0}^{n} k a_k x^{k-1}$$

$$= \frac{1}{x}\sum_{k=0}^{n} k a_k x^k$$

and

$$S''(x) = \sum_{k=0}^{n} k(k-1) a_k x^{k-2}$$

$$= \frac{1}{x^2}\sum_{k=0}^{n} k(k-1) a_k x^k.$$

Then

$$\sum_{k=0}^{n} a_k = S(1),$$
$$\sum_{k=0}^{n} k a_k = S'(1),$$
$$\sum_{k=0}^{n} k^2 a_k = S''(1) + S'(1).$$

Example 12. Compute $S_n = \sum_{k=0}^{n} \binom{n}{k}$.

Define $S(x) = \sum_{k=0}^{n} \binom{n}{k} x^k = (1+x)^n$, then $S_n = S(1) = 2^n$.

Example 13. Compute $S_n = \sum_{k=0}^{n} k\binom{n}{k}$.

Define $S(x) = \sum_{k=0}^{n} \binom{n}{k} x^k = (1+x)^n$, then $S'(x) = \sum_{k=0}^{n} k\binom{n}{k} x^{k-1}$ so that we have $S'(1) = S(n)$. On the other hand $S'(x) = n(1+x)^{n-1}$, hence $S_n = S'(1) = n2^{n-1}$.

In certain cases, expressing the general term a_k as a function of x and employing methods of calculus can be helpful fo computing finite sums.

Example 14. Compute $S_n = \sum_{k=0}^{n} k r^k$.

Define $S(r) = \sum_{k=0}^{n} r^k$ and differentiate $S(r)$ with respect to r:

$$\frac{dS(r)}{dr} = \sum_{k=0}^{n} k r^{k-1}$$
$$= \frac{1}{r}\sum_{k=0}^{n} k r^k.$$

On the other hand $S(r) = \frac{1-r^{n+1}}{1-r}$ and

$$\frac{dS(r)}{dr} = \frac{nr^{n+1} - (n+1)r^n + 1}{(1-r)^2}.$$

Comparing the two expressions obtained for $dS(r)/dr$ we get

$$\sum_{k=0}^{n} k r^k = r\,\frac{nr^{n+1} - (n+1)r^n + 1}{(1-r)^2}.$$

FINITE CALCULUS

The ***difference*** (or ***finite derivative***) operator Δ maps a function $f:\mathbb{R}\to\mathbb{R}$ to the function $\Delta f:\mathbb{R}\to\mathbb{R}$ which is defined as

$$\Delta f(x) = f(x+1) - f(x).$$

It is seen that $\Delta p = 0$ if and only if $p(x)$ is a function which is periodic with 1, for example $\Delta(\sin(2\pi x)) = 0$. In particular, difference of a constant function is zero. It follows that $\Delta f = \Delta g$ if and only if the functions f and g differ by a function which is periodic with 1.

Difference of the function $f(x) = x$ is $(x+1) - x = 1$, that is

$$\Delta x = 1.$$

Since $2^{x+1} - 2^x = 2^x$, difference of the function $f(x) = 2^x$ is itself:

$$\Delta 2^x = 2^x.$$

Differences of some frequently used functions are as follows

$$\Delta x^2 = (x+1)^2 - x^2 = 2x + 1\,,$$

$$\Delta \frac{1}{x} = \frac{1}{x+1} - \frac{1}{x} = \frac{1}{x(x+1)}\,.$$

If $x \in \mathbb{N}$, then

$$\Delta x! = x \cdot x!\,,$$

$$\Delta \binom{x}{k} = \binom{x}{k-1}\,,$$

$$\Delta H_x = \frac{1}{x+1}\,,$$

$$\Delta\left(\sum_{k=1}^{x} a_k\right) = a_{x+1}\,.$$

Linearity of difference operator is obvious, that is

$$\Delta(af + g) = a\Delta f + \Delta g$$

for any $f, g:\mathbb{R}\to\mathbb{R}$ and $a \in \mathbb{R}$.

If the difference of F is f, that is, $\Delta F = f$, the function F is called an ***anti-difference*** of f. The operation which sends f to an anti-difference is denoted by Σ. Thus, Σ is inverse of the operator Δ. For a given function f, the class of all anti-differences is denoted by $\sum f(x)\delta x$ and it is called the ***indefinite sum*** of f. The ***definite sum*** of f is defined by setting $\sum_a^b f(x)\delta x = F(b) - F(a) = F|_a^b$ where F is any antiderivative of f.

For any integers a, b and c such that $a \le b \le c$, the definite sum satisfies the following equalities:

$$\sum\nolimits_a^a f(x)\delta x = 0\,,$$
$$\sum\nolimits_a^b f(x)\delta x = -\sum\nolimits_b^a f(x)\delta x\,,$$
$$\sum\nolimits_a^c f(x)\delta x = \sum\nolimits_a^b f(x)\delta x + \sum\nolimits_b^c f(x)\delta x\,.$$

Let $\Delta F = f$, then the relation between the finite sum $\sum_{k=a}^b f(k)$ and the definite sum $\sum_a^b f(x)\Delta x$ is obtained as follows:

$$\begin{aligned}\sum_{k=a}^{b} f(k) &= f(a) + f(a+1) + f(a+2) + \cdots + f(b-2) + f(b-1) + f(b)\\ &= \underbrace{F(a+1) - F(a)}_{f(a)} + \underbrace{F(a+2) - F(a+1)}_{f(a+1)} + \underbrace{F(a+3) - F(a+2)}_{f(a+2)} + \cdots +\\ &\quad \underbrace{F(b-1) - F(b-2)}_{f(b-2)} + \underbrace{F(b) - F(b-1)}_{f(b-1)} + \underbrace{F(b+1) - F(b)}_{f(b)}\\ &= F(b+1) - F(a)\\ &= \sum\nolimits_a^{b+1} f(x)\delta x\end{aligned}$$

Example 15. Let r be a nonzero real number, then $\sum_{k=0}^n r^k = \sum_0^{n+1} r^x \delta x$.

We compute the difference of r^x: $\Delta r^x = r^{x+1} - r^x = r^x(r-1)$ which implies that $\Delta\left(\frac{r^x}{r-1}\right) = r^x$. Then

$$\begin{aligned}\sum_{k=0}^{n} r^k &= \sum\nolimits_0^{n+1} r^x \delta x\\ &= \frac{r^x}{r-1}\Big|_0^{n+1} = \frac{r^{n+1}-1}{r-1}.\end{aligned}$$

Example 16. Since $\Delta\left(\frac{1}{x}\right) = \frac{1}{x+1} - \frac{1}{x} = -\frac{1}{x(x+1)}$ we obtain

$$\sum_{k=1}^{n} \frac{1}{k(k+1)} = \sum_{0}^{n+1} \frac{1}{x(x+1)}\,\delta x$$

$$= -\frac{1}{x}\Big|_{0}^{n+1}$$

$$= \frac{n}{n+1}.$$

Example 17. Using $\Delta x! = x \cdot x!$,

$$\sum_{k=0}^{n} k \cdot k! = \sum\nolimits_{0}^{n+1} x \cdot x!\,\delta x = x!|_0^{n+1} = (n+1)! - 1.$$

Example 18. As $\Delta\binom{x}{m+1} = \binom{x}{m}$, we have

$$\sum_{k=m}^{n} \binom{k}{m} = \sum\nolimits_{m}^{n+1} \binom{x}{m}\,\delta x = \binom{x}{m+1}\Big|_0^{n+1} = \binom{n+1}{m+1}.$$

Falling factorial powers

For $x \in \mathbb{R}$ and $n \in \mathbb{N}$, the product $\underbrace{x(x-1)\cdots(x-n+1)}_{n \text{ factors}}$ is called a ***falling factorial power*** of x and it is denoted by $x^{\underline{n}}$, that is, for the positive integer n

$$x^{\underline{n}} = x(x-1)\cdots(x-n+1).$$

Some examples are $x^{\underline{1}} = x$, $x^{\underline{2}} = x^2 - x$, $1^{\underline{2}} = 0$. If n is a positive integer then $n^{\underline{n}} = n!$ and $n^{\underline{m}} = n!/(n-m)!$ for integer $m \le n$.

By convention

$$x^{\underline{0}} = 1$$

and for negative falling factorial powers we define

$$x^{\underline{-n}} = \frac{1}{(x+1)(x+2)\cdots(x+n)}.$$

Difference of $x^{\underline{n}}$ is

$$\begin{aligned}\Delta(x^{\underline{n}}) &= (x+1)^{\underline{n}} - x^{\underline{n}} \\ &= [(x+1)x(x-1)\cdots(x-n+2)] - [x(x-1)\cdots(x-n+1)] \\ &= x(x-1)\cdots(x-n+2)[(x+1)-(x-n+1)] \\ &= nx^{\underline{n-1}}.\end{aligned}$$

It follows that for any integer $a \neq -1$

$$\sum\nolimits_0^n x^{\underline{a}}\delta x = \frac{1}{a+1}x^{\underline{a+1}}\Big|_0^n = \frac{1}{a+1}n^{\underline{a+1}}.$$

Since $\Delta H_x = \frac{1}{(x+1)} = x^{\underline{-1}}$, we have $\sum_0^n x^{\underline{-1}}\delta x = H_x|_0^n = H_n$. We combine these two sums as

$$\sum\nolimits_0^n x^{\underline{a}}\delta x = \begin{cases} \dfrac{1}{a+1}n^{\underline{a+1}} & \text{if} \quad a \neq -1 \\ H_n & \text{if} \quad a = -1 \end{cases}.$$

In terms of finite sums, we can write

$$\sum_{k=0}^{n} k^{\underline{a}} = \sum\nolimits_0^{n+1} x^{\underline{a}}\delta x = \begin{cases} \dfrac{1}{a+1}(n+1)^{\underline{a+1}} & \text{if} \quad a \neq -1 \\ H_{n+1} & \text{if} \quad a = -1 \end{cases}.$$

Some particular cases are

$$\sum_{k=1}^{n} k = \sum\nolimits_1^{n+1} x^{\underline{1}}\delta x = \frac{1}{2}x^{\underline{2}}\Big|_1^{n+1} = \frac{1}{2}n(n+1),$$

$$\sum_{k=1}^{n}(k-1)k = \frac{1}{3}(n-1)n(n+1),$$

$$\sum_{k=1}^{n}(k-2)(k-1)k = \frac{1}{4}(n-2)(n-1)n(n+1).$$

Example 19. Compute $\sum_{k=1}^{n} k^3$. First observe that $k^{\underline{3}}+3k^{\underline{2}}+k^{\underline{1}} = k^3$. Then

$$\begin{aligned}\sum_{k=1}^{n} k^3 &= \sum_{k=1}^{n} k^{\underline{3}} + 3k^{\underline{2}} + k^{\underline{1}} \\ &= \frac{1}{4}(n+1)^{\underline{4}} + (n+1)^{\underline{3}} + \frac{1}{2}(n+1)^{\underline{2}} \\ &= \frac{1}{4}(n+1)n\big((n-1)(n-2) + 4(n-1) + 2\big) \\ &= \frac{1}{4}(n+1)^2n^2.\end{aligned}$$

For the product of two functions we have

$$\Delta(fg)(n) = (fg)(n+1) - (fg)(n)$$
$$= f(n+1)g(n+1) - f(n)g(n).$$

To have a product rule which is analogous of the product rule for derivatives, we define a new operator, namely the ***shift operator*** E which maps a function $f: \mathbb{R} \to \mathbb{R}$ to the function $\mathrm{E}f: \mathbb{R} \to \mathbb{R}$ which is defined as $\mathrm{E}f(x) = f(x+1)$. Using the shift operator, we can write

$$\Delta(fg)(x) = f(x+1)g(x+1) - f(x)g(x) + g(x+1)f(x) - g(x+1)f(x)$$
$$= g(n+1)\big(f(x+1) - f(x)\big) + f(x)\big(g(x+1) - g(x)\big)$$
$$= (Eg \cdot \Delta f)(x) - (f \cdot \Delta g)(x)$$

so that

$$\Delta(fg) = Ef \cdot \Delta g + f \cdot \Delta g$$

which leads to

$$\sum u\Delta v = uv - \sum Ev\, \Delta u.$$

Above rule is known as the ***summation by parts.***

Example 20. Compute $\sum_{k=1}^{n} k^2$.

First write $\sum_{k=1}^{n} k^2 = \sum_{1}^{n+1} x^2\, \delta x$ and let $u(x) = x$ and $\Delta v(x) = x\delta x = x^{\underline{1}}\delta x$ so that $\Delta u(x) = \delta x$ and $v(x) = \frac{1}{2}x^{\underline{2}}$ and $Ev(x) = \frac{1}{2}(x+1)^{\underline{2}}$. Then

$$\sum x^2 \delta x = \frac{1}{2}x^{\underline{2}} - \frac{1}{2}\sum (x+1)^{\underline{2}}\delta x$$
$$= \frac{1}{2}x \cdot x^{\underline{2}} - \frac{1}{6}(x+1)^{\underline{3}}$$
$$= \frac{1}{6}x(x+1)\big(3x - (x-1)\big)$$
$$= \frac{1}{6}x(x+1)(2x+1).$$

Then

$$\sum_{k=1}^{n} k^2 = \sum\nolimits_{1}^{n+1} x^2 \delta x$$

$$= \frac{1}{6} x(x-1)(2x-1) \Big|_{0}^{n+1}$$

$$= \frac{1}{6} n(n+1)(2n+1).$$

Example 21. Compute $\sum_{k=1}^{n} kH_k$.

First write $\sum_{k=1}^{n} kH_k = \sum_{1}^{n+1} xH_x \, \delta x$ and let $u(x) = H_x$ and $\Delta v(x) = x^{\underline{1}} \delta x$ so that $\Delta u(x) = x^{\underline{-1}} \delta x$ and $v(x) = \frac{1}{2} x^{\underline{2}}$ and $Ev(x) = \frac{1}{2}(x+1)^{\underline{2}}$. Then

$$\sum_{1}^{n+1} xH_x \, \delta x = \frac{1}{2} x^{\underline{2}} H_x \Big|_{1}^{n+1} - \frac{1}{2} \sum_{1}^{n+1} x^{\underline{-1}} (x+1)^{\underline{2}} \delta x$$

$$= \frac{1}{2}(n+1)^{\underline{2}} H_{n+1} - \frac{1}{2} \sum_{1}^{n+1} x^{\underline{1}} \delta x$$

$$= \frac{1}{2}(n+1)^{\underline{-2}} H_{n+1} - \frac{1}{4} x^{\underline{2}} \Big|_{1}^{n+1}$$

$$= \frac{1}{2}(n+1)^{\underline{-2}} H_{n+1} - \frac{1}{4}(n+1)^{\underline{2}} \Big|_{1}^{n+1}$$

$$= \frac{1}{2}(n+1)^{\underline{-2}} \left(H_{n+1} - \frac{1}{2} \right)$$

$$= \frac{1}{2} n(n+1) \left(H_{n+1} - \frac{1}{2} \right).$$

All the finite sums we computed throughout the examples in this section are listed in the following table. The boldface numbers in the table are numbers of examples in which the corresponding sum is computed. Italicized numbers refer to exercises.

	Perturbation	Double Sums	Power Series	Finite Calculus
$\sum_{k=0}^{n} \boldsymbol{k} = \frac{1}{2}n(n+1)$	**3**	*2. a)*		
$\sum_{k=0}^{n} \boldsymbol{k}^2 = \frac{1}{6}n(n+1)(2n+1)$	**3**	**8**		**20**
$\sum_{k=0}^{n} \boldsymbol{k}^3 = \frac{1}{4}n^2(n+1)^2$		*2. b)*		**19**
$\sum_{k=0}^{n} \boldsymbol{r}^{\boldsymbol{k}} = \frac{1-r^{n+1}}{1-r}$	**1**			**15**
$\sum_{k=0}^{n} \boldsymbol{k}\boldsymbol{r}^{\boldsymbol{k}} = \frac{(nr-n-1)r^{n+1}+r}{(1-r)^2}$	**2**	*2. c)*	**14**	
$\sum_{k=1}^{n} \boldsymbol{k}\mathbf{2}^{\boldsymbol{k}} = (n-1)2^{n+1}+2$		**9**		
$\sum_{k=0}^{n} \boldsymbol{k}\cdot\boldsymbol{k}! = (n+1)!-1$	**4**			**17**
$\sum_{k=0}^{n} \binom{\boldsymbol{n}}{\boldsymbol{k}} = 2^n$			**12**	
$\sum_{k=0}^{n} \boldsymbol{k}\binom{\boldsymbol{n}}{\boldsymbol{k}} = n2^{n-1}$	**5**		**13**	
$\sum_{k=0}^{n} \boldsymbol{k}^2\binom{\boldsymbol{n}}{\boldsymbol{k}} = n(n+1)2^{n-2}$	**6**			
$\sum_{k=m}^{n} \binom{\boldsymbol{k}}{\boldsymbol{m}} = \binom{n+1}{m+1}$	*1. a)*			**18**
$\sum_{k=0}^{n} \sin(\mathbf{2}\boldsymbol{\pi}\boldsymbol{k}) = \frac{\sin\left(\frac{n+1}{2}x\right)\sin\frac{nx}{2}}{\sin\frac{x}{2}}$ $\sum_{k=0}^{n} \cos(\mathbf{2}\boldsymbol{\pi}\boldsymbol{k}) = 1+\frac{\cos\left(\frac{n+1}{2}x\right)\sin\frac{nx}{2}}{\sin\frac{x}{2}}$	**7**			
$\sum_{k=1}^{n} \boldsymbol{H}_{\boldsymbol{k}} = (n+1)H_n - n.$	*1. b)*	**10**		
$\sum_{k=1}^{n} \boldsymbol{k}\,\boldsymbol{H}_{\boldsymbol{k}} == \frac{1}{2}n(n+1)\left(H_{n+1}-\frac{1}{2}\right)$	*1. c)*	**11**		**21**
$\sum_{k=1}^{n} \frac{\mathbf{1}}{\boldsymbol{k}(\boldsymbol{k}+\mathbf{1})} = \frac{n}{n+1}$				**16**

EXERCISES

1. Compute the following sums by perturbing the sum:

a) $S_n = \sum_{k=m}^{n} \binom{k}{m}$,

b) $S_n = \sum_{k=1}^{n} H_k$,

c) $\sum_{k=1}^{n} k\, H_k$,

d) $\sum_{k=1}^{n} k^2 \binom{n}{k}$.

2. Compute the following sums by converting the given sum to a double sum:

a) $S_n = \sum_{k=1}^{n} k$,

b) $S_n = \sum_{k=1}^{n} k^3$,

c) $S_n = \sum_{k=0}^{n} kr^k$.

3. Show that $\sum_{k=1}^{n}(-1)^k k^2 = (-1)^n \frac{n(n+1)}{2}$.

4. Compute the following sums

a) $\sum_{k=0}^{n}(-1)^k \binom{n}{k} k^n$,

b) $\sum_{k=1}^{n} 2^{n-k} k^2$,

c) $\sum_{k=1}^{n} \frac{k}{(k+1)!}$.

Appendix B

RANDOM MAPPINGS AND PERMUTATIONS

The coupon collector's problem is one of the most popular topics in discrete probability, as it is simple and useful. It is known since 1708, when the problem is first seen in the literature in 'De Mensura Sortis' (On the Measurement of Chance) written by A. De Moivre. In 1938 the problem appeared in 'A Problem in Cartophily', written by F. G. Maunsell. In 1950, it was introduced in the book 'An Introduction to Probability Theory and Its Applications', written by Willam Feller. From then on, the coupon collector appears in many textbooks. The coupon collector's problem has many applications, including cryptography, electrical engineering and biology. In cryptography the problem is important for its relation with the random mappings. In electrical engineering it is related to the cache fault problem; in biology, the problem can be used to estimate the number of species of animals.

COUPON COLLECTOR'S PROBLEM

We assume a bag which contains m distinct balls. We pick a ball from the bag randomly in the manner that each ball is equally likely and the choices are independent. After replacing the picked ball to the bag, we repeat the same experiment until we see all of the balls at least once. Let T_m be the random variable defined to be the number of trials required for each ball being picked at least once. We wish to compute the expected value $E(T_m)$ of T_m, that is, the expected number of times to pick an object until seeing each object (ball) at least once.

Example 1. Consider a football fan who wants to collect a complete set of 10 football cards. Cards are available in a completely random fashion, one per package of candy, which the fan buys one package a day. How long, on the average, will it take the fan to get a complete set?

If the set of cards for $m = 10$ is represented by `a,b,c,d,e,f,g,h,i,j` then a possible sequence of cards bought on each day could be

$$\boxed{\texttt{c}}\ \boxed{\texttt{g}}\ \boxed{\texttt{h}}\ \texttt{g}\ \boxed{\texttt{d}}\ \texttt{h}\ \boxed{\texttt{a}}\ \boxed{\texttt{j}}\ \texttt{c}\ \texttt{g}\ \boxed{\texttt{f}}\ \boxed{\texttt{b}}\ \texttt{d}\ \texttt{c}\ \texttt{a}\ \boxed{\texttt{e}}\ \texttt{d}\ \texttt{c}\ \texttt{j}\ \texttt{g}\ \texttt{e}\ \texttt{f}\ \texttt{a}\ \boxed{\texttt{i}}$$

A frame around a term of the sequence indicates the first occurrence of the card `a,…,j`. In that sequence we see that the collection is completed (for the first time) at 24-th package of candy so that the fan had to buy 24 packages to complete the set of the 10 distinct cards. After buying the 8th package, there are six different cards in the collection :

$$\boxed{\texttt{c}}\ \boxed{\texttt{g}}\ \boxed{\texttt{h}}\ \texttt{g}\ \boxed{\texttt{d}}\ \texttt{h}\ \boxed{\texttt{a}}\ \boxed{\texttt{j}}.$$

As the example suggests, we may expect to collect the first cards very quickly with a small number of repetitions. But when we get down to the last few items in the collection, it seems to take much longer to obtain those pieces.

THEOREM B.1. *From a set of m distinct objects, the expected number of objects to be picked randomly, with replacement, until completing the collection is*

$$E(T_m) = mH_m.$$

Proof. If there is only one ball, that is if $m = 1$, when we buy the first package, the collection will be completed, $E(T_1) = 1$.

If there are just two balls a and b, the first time we pick a ball, we will naturally have a new ball, say a. After having the first piece of the collection we find how many additional balls we are expected to pick for finding the other ball. So we focus on a new experiment, namely, obtaining the ball 'b' starting from that point. In the first attempt of new experiment, the resulting ball can be a or b. The set will be completed if the first ball is a 'b'. Thus probability of completing the set at the first attempt is $1/2$. If the first ball is an 'a' (with pobability $1/2$) we have to buy a second ball and probability of seeing a 'b' is $1/2$. So completing the collection after the second ball is $1/4$. Continuing in this manner we see that completing the collection at k-th ball is $1/2^k$. Then after having the first ball, the expected number of balls required to be picked for seeing ball 'b' is

$$1 \cdot \frac{1}{2} + 2 \cdot \frac{1}{4} + 3 \cdot \frac{1}{8} + 4 \cdot \frac{1}{16} + \cdots + k \cdot \frac{1}{2^k} + \cdots.$$

To compute this sum recall that if $f(x) = \frac{1}{1-x} = \sum_{t=0}^{\infty} x^t$, then $f'(x) = \frac{1}{(1-x)^2} = \sum_{t=0}^{\infty} tx^{t-1}$ and $\sum_{t=1}^{\infty} tx^t = \frac{x}{(1-x)^2}$. By substituting $x = 1/2$ we see that above sum is equal to 2. To obtain the first ball we have to buy one ball, and for the second ball we are expected to buy 2 more balls. Then $E(T_2) = 3$.

Now we return to the general case of m balls. We consider the stage of the experiment when the number of distinct balls in the collection is increased to k. Assume that in the next $t-1$ tries we could not acquire a new ball and after picking the t-th ball we have found a new ball. Probability of this event is $\frac{k^{t-1}}{m^{t-1}} \cdot \frac{(m-k)}{m}$. Then, starting from that stage, the expected number of balls we have to pick until acquiring a new ball is

$$\sum_{t=0}^{\infty} t \cdot \frac{k^{t-1}(m-k)}{m^t} = \frac{(m-k)}{m} \sum_{t=0}^{\infty} t \left(\frac{k}{m}\right)^{t-1} = \frac{m}{m-k}$$

After picking the first ball, we have a new ball. To see a second new ball we expect $\frac{m}{m-1}$ additional tries. After seeing two distinct balls, we will expectedly try $\frac{m}{m-2}$ more times to see the third one. Then, to complete the entire collection, the expected number of drawings is $E(T_m) = \frac{m}{m-0} + \frac{m}{m-1} + \cdots + \frac{m}{m-(m-2)} + \frac{m}{m-(m-1)}$. But this expression is $E(T_m) = m(\frac{1}{m} + \frac{1}{m-1} + \cdots + \frac{1}{2} + 1)$, that is $E(T_m) = mH_m$. ■

Example 1. (Continued) We return back to the Example 1. Since $H_{10} = 2.928\cdots$ we have $E(T_{10}) = 29.28\cdots$. This means that, the fan is expected to buy about 30 packages of candies for completing the collection.

In general, after repeating the experiment for r times, the number of distinct balls we have collected so far will be called the collection size and will be denoted by C_r.

Now we focus on two questions.

When balls are picked for n times, what is the expected number $E(C_n)$ of distinct balls in the collection?

When n balls are picked, what is the probability $\Pr(C_n = m)$ of having a complete collection?

THEOREM B.2. *From a set of m distinct balls, if n balls are picked randomly, with replacement, then the expected number of distinct balls is*

$$E(C_n) = m(1 - (1 - 1/m)^n).$$

Proof. In an experiment of picking a ball n times from the bag, call each ball which has seen at least once as a revealed ball. If we have picked a ball n times, the number of all possible collections (respecting order of the balls we pick them) is m^n. The number of collections for which a certain ball, say the ball labeled '1', has never seen is $(m-1)^n$. Then, '1' is counted as a revealed ball in exactly $m^n - (m-1)^n$ collections. Since this is the same for all m balls as well, we count a total of $m(m^n - (m-1)^n)$ revealed balls in all possible collections. It follows that the average number of revealed balls per collection is, obtained by dividing this quantity by m^n. Hence we obtain $E(C_n) = m(m^n - (m-1)^n)/m^n$.

Alternative proof. When we pick the $(k+1)$-st ball, we can have a ball which is already in the collection with a probability $E(C_k)/m$ or a ball which appears for the first time with probability $1 - E(C_k)/m$. Then

$$\begin{aligned} E(C_{k+1}) &= (E(C_k)+1)\left(1 - \frac{E(C_k)}{m}\right) + E(C_k)\cdot\frac{E(C_k)}{m} \\ &= \left(1 - \frac{1}{m}\right)E(C_k) + 1. \end{aligned}$$

Multiply both sides with x^{k+1} and sum over $k = 1,2,\ldots$:

$$\sum_{k=1}^{\infty} E(C_{k+1})x^{k+1} = \left(1 - \frac{1}{m}\right)\sum_{k=1}^{\infty} E(C_k)x^{k+1} + \sum_{k=1}^{\infty} x^{k+1}$$

If we let $\mathcal{F}(x)$ to be the generating function of $E(C_0), E(C_1), \ldots$ we can write

$$\begin{aligned} \mathcal{F}(x) &= \frac{x}{\left(1 - \left(1 - \frac{1}{m}\right)x\right)(1-x)} \\ &= \frac{m}{1-x} - \frac{m}{\left(1 - \left(1 - \frac{1}{m}\right)x\right)} \\ &= m\sum_{n=0}^{\infty} x^n - m\sum_{n=0}^{\infty}\left(1 - \frac{1}{m}\right)^n x^n \\ &= m\sum_{n=0}^{\infty}\left(1 - \left(1 - \frac{1}{m}\right)^n\right)x^n. \end{aligned}$$

Now, the coefficient of x^n gives $E(C_n)$. ■

Since $\lim_{m\to\infty}(1 - 1/m)^m = e^{-1}$, for large values of m, $(1-1/m)^m$ can be approximated with e^{-1}. Now, writing the expected collection size given as $m\left(1 - [(1-1/m)^m]^{n/m}\right)$, for large values of m we can write the approximation

$$E(C_n) \approx m\left(1 - e^{-n/m}\right).$$

Example 1. (Continued) We return back to the problem where the fan has stopped buying new packages after he has bought 15 packages. In this case, for $m = 10$, the expected size of his collection is $E(C_{15}) \approx 10(1 - e^{-1.5}) = 7.76\cdots$. He has expectedly completed %78 of the entire collection.

For $m = n$, the expected value is

$$E(C_n) = n\left(1 - \left(1 - \frac{1}{n}\right)^n\right) \approx n\left(1 - \frac{1}{e}\right) = 0.6321\, n$$

THEOREM B.3. *From a set of m distinct balls, if n times a ball is picked randomly, with replacement, then the probability of having c ≤ m distinct balls is*

$$\Pr(C_n = c) = \frac{c!}{m^n} \left\{ {n \atop c} \right\} \binom{m}{c}.$$

Proof. If the collection size is c, the balls in the collection can be chosen in $\binom{m}{c}$ ways. After choosing the balls, assume that we have ordered them so that it is determined which will be the first ball, which will be the second ball, and so on. But this ordering describes a partition of the ordering numbers $1,2,\dots,n$, into c classes. So the number of such orderings is $\left\{ {n \atop c} \right\}$. Finally the classes can be permuted in $c!$ different ways. Then, the number of ways of having a collection of size c after picking n packages is $c!\left\{ {n \atop c} \right\}\binom{m}{c}$. And we obtain $\Pr(C_n = c) = \frac{c!}{m^n}\left\{ {n \atop c} \right\}\binom{m}{c}$. ■

COROLLARY B.4. *From a set of m distinct balls, if n ≥ m balls are picked randomly, with replacement, then the probability of having a complete collection of m distinct balls is*

$$\Pr(C_n = m) = \frac{m!}{m^n} \left\{ {n \atop m} \right\}.$$

Proof. Just take $c = m$ in Theorem 3. ■

After computing $E(C_n)$, we could obtain $E(T_m)$ alternatively as follows. It is clear that the sequence $E(C_1), \dots, E(C_k), \dots$ is an increasing sequence with $\lim_{k \to \infty} E(C_k) = m$. This means that by choosing k sufficiently large, $E(C_k)$ can be made as closer to m as we wish.

Example 1. (Continued) In the below table we see the number of packages bought and the corresponding expected collection size for $m = 10$.

n	$E(C_n)$	n	$E(C_n)$	n	$E(C_n)$
1	1.00	16	8.15	31	9.62
2	1.90	17	8.33	32	9.66
3	2.71	18	8.50	33	9.69
4	3.44	19	8.65	34	9.72
5	4.10	20	8.78	35	9.75
6	4.69	21	8.91	36	9.77
7	5.22	22	9.02	37	9.80
8	5.70	23	9.11	38	9.82
9	6.13	24	9.20	39	9.84
10	6.51	25	9.28	40	9.85
11	6.86	26	9.35	41	9.87
12	7.18	27	9.42	42	9.88
13	7.46	28	9.48	43	9.89
14	7.71	29	9.53	44	9.90
15	7.94	30	9.58	45	9.91

for $n \geq 22$ we have $m - E(C_n) \leq 1$. If $n \geq 29$, then $m - E(C_n) \leq 1/2$ and so on.

For any finite k, we can think of $E(T_m)$ to be the smallest integer k such that $E(C_k) > m - \varepsilon$. Here ε measures how the expected value is close to m. Inequality $m\left(1 - (1 - 1/m)^k\right) > m - \varepsilon$ can be written as $k > \frac{\ln m - \ln \varepsilon}{\ln m - \ln(m-1)}$. If we take $\varepsilon = 1$, by the approximation $\ln m - \ln(m - 1) \approx \frac{1}{m}$ for large values of m, we get $k > m \ln m$. We can conclude that

$$E(T_m) \approx m \ln m.$$

Example 1. (Continued) We again return back to Example 1 to consider the case where the fan has stopped buying new packages after he sees a card which has seen for the second time. Now what is the expected number of cards he has collected until seeing the first repeating card?

Now we define a new random variable R_m to be the number of the balls picked until the first repeating card.

Consider the sequence $E(C_1), \dots, E(C_k), \dots$. Necessarily, $E(C_1) = 1$ and $E(C_k) < k$ for $= 2,3, \dots$. In the first few terms $E(C_k)$ will be quite close to k. As k gets larger, $E(C_k)$ will get farther from k. When the $E(C_k) < k - 1$ the expected image size is smaller than the cards picked. In such a case it is natural to expect repetitions of the balls. In the table above, we see that for 6 packages, expected collection size is less than 5 cards in which case a repetition is not a surprise.

In general, we can expect a repetition when the difference $k - E(C_k)$ exceeds a certain bound, say $1/2$. Let R_m be the number of packages bought until the first repetition. Then we can think of $E(R_m)$ to be the smallest integer k such that $k - E(C_k) > 1/2$.

THEOREM B.5. *From a set of m distinct cards, the expected number of cards to be picked randomly, with replacement, until the first repetition is*

$$E(R_m) \approx \sqrt{m}.$$

Proof. After some simplifications, the inequality $m\big(1 - (1 - 1/m)^k\big) < k - \frac{1}{2}$ can be written as

$$1 - \left(1 - \frac{1}{m}\right)^k < \frac{k}{m} - \frac{1}{2m}$$

$$1 - \left(1 - \binom{k}{1}\frac{1}{m} + \binom{k}{2}\frac{1}{m^2} - \binom{k}{3}\frac{1}{m^3} + \cdots\right) < \frac{k}{m} - \frac{1}{2m}$$

$$\frac{k(k-1)}{m^2} - 2\binom{k}{3}\frac{1}{m^3} + \cdots < -\frac{1}{m}$$

$$-\frac{k^2}{m^2} - \frac{k}{m^2} + \mathcal{O}\left(\left(\frac{k}{m}\right)^3\right) < -\frac{1}{m}$$

As k is quite small compared to m, we can asymptotically write $k \geq \sqrt{m}$. We conclude that $E(R_m) \approx \sqrt{m}$. ■

Birthday Paradox

Birthday paradox (or birthday problem) considers the probability that, in a set of n people, at least one pair to have the same birthday. The result of the problem is quite surprising and far away from the intuitive answer. For this reason, the problem is known as a paradox and is a classic of counting and probability.

The main question asks how many people are required to have a 50-50 chance that two of them will share a birthday.

In the general form, we wish to know the minimum number k of selections, among n different equally likely items so that the probability of at least one match has probability at least p_0.

THEOREM B.6. *From a set of m distinct balls, if n balls are picked randomly, with replacement, then the probability that at least two balls are the same is*

$$p(n,k) = 1 - \frac{n!}{(n-k)!\,n^k}.$$

Proof. Let $p(n,k)$ denote the probability that at least two balls are the same for the experiment of choosing k balls, out of n equally likely balls, allowing repetitions. We first compute the complementary probability $q(n,k)$ that all selected balls are distinct. Start with an arbitrary ball, then that the probability that the second ball is different is $\frac{n-1}{n}$, that the third ball is different from the first two is $\frac{n-1}{n} \cdot \frac{n-2}{n}$ and so on, up through the k-th ball. Explicitly,

$$q(n,k) = \frac{n}{n} \cdot \frac{n-1}{n} \cdot \frac{n-2}{n} \cdots \frac{n-k+1}{n}$$
$$= \frac{n!}{(n-k)!\,n^k}$$

and the complementary probability is what we try to compute. ■

For the birthday problem we have to find the smallest value of k such that

$$p(365,k) = 1 - \frac{365!}{(365-k)!\,365^k} > \frac{1}{2}.$$

In the table below we see the values of $p(365,k)$ for $k = 1, \ldots, 60$:

k	$p(365,k)$	k	$p(365,k)$	k	$p(365,k)$	k	$p(365,k)$	k	$p(365,k)$	k	$p(365,k)$
1	0,000	11	0,141	21	0,444	31	0,730	41	0,903	51	0,974
2	0,003	12	0,167	22	0,476	32	0,753	42	0,914	52	0,978
3	0,008	13	0,194	23	0,507	33	0,775	43	0,924	53	0,981
4	0,016	14	0,223	24	0,538	34	0,795	44	0,933	54	0,984
5	0,027	15	0,253	25	0,569	35	0,814	45	0,941	55	0,986
6	0,040	16	0,284	26	0,598	36	0,832	46	0,948	56	0,988
7	0,056	17	0,315	27	0,627	37	0,849	47	0,955	57	0,990
8	0,074	18	0,347	28	0,654	38	0,864	48	0,961	58	0,992
9	0,095	19	0,379	29	0,681	39	0,878	49	0,966	59	0,993
10	0,117	20	0,411	30	0,706	40	0,891	50	0,970	60	0,994

We see that the smallest value of k for which $p(365,k) > 1/2$ is $k = 23$. We conclude that if there are 23 people, then the probability that at least two of them have the same birthday is larger than 0.5. For 41 people the same probability is larger than 0.9 and for $= 57$, by probability 0.99 at least two people in the group have the same birthday.

Since the formula $p(n,k) = 1 - \frac{n!}{(n-k)!n^k}$ is not practical for large values of n, we develop some approximations. First note that if $\frac{m}{n}$ is sufficiently small, then in the expansion

$$e^{-m/n} = 1 - \frac{m}{n} + \frac{m^2}{n^2} - \frac{m^3}{n^3} + \cdots$$

the terms $\frac{m^2}{n^2}, \frac{m^3}{n^3}, \cdots$ can be neglected and we can write

$$e^{-m/n} \approx 1 - \frac{m}{n}.$$

Now write $p(n,k)$ in the form

$$p(n,k) = 1 - \frac{n}{n} \cdot \left(1 - \frac{1}{n}\right) \cdot \left(1 - \frac{2}{n}\right) \cdots \left(1 - \frac{k-1}{n}\right)$$

and use the above approximation to have

$$\begin{aligned} p(n,k) &\approx 1 - e^{-1/n} e^{-2/n} \cdots e^{-\frac{k-1}{n}} \\ &= 1 - e^{-\frac{1+2+\cdots+(k-1)}{n}} \\ &= 1 - e^{-\frac{k(k-1)}{2n}}. \end{aligned}$$

Using the approximation $e^{-m/n} = 1 - m/n$ once more we obtain

$$\begin{aligned} p(n,k) &\approx \frac{k(k-1)}{2n} \\ &= \frac{k^2}{2n} - \frac{k}{2n}. \end{aligned}$$

As k is very small, compared to n, we can neglect $k/2n$ and reach to a coarser but practical approximation

$$p(n,k) \approx \frac{k^2}{2n}.$$

To find the smallest k for which $p(n,k) > p_0$ we write $k^2 > 2np_0$ or $k > \sqrt{2np_0}$. We conclude that, when n is large enough, if at least

$$k = \sqrt{2np_0}$$

items are chosen (out of n distinct items), then at least two of them will be the same with probability at least p_0.

Random Mappings

By $\mathcal{F}_{n,m}$ we denote the collection of all functions f from a finite domain X of size n to a finite range Y of size m. We assume the random mapping model where every function from $\mathcal{F}_{n,m}$ is chosen equally likely. This model is equivalent to the model where $f: X \to Y$ assigns each input $x \in X$ independently to an image point $y \in Y$, that is $\Pr(f(x) = y) = 1/m$ for all $x \in X$ and $y \in Y$. The number of all functions $X \to Y$ is m^n, in other words $|\mathcal{F}_{n,m}| = m^n$. A random mapping $f \in \mathcal{F}_{n,m}$ is equivalent to the experiment of buying n candies to collect the set of m cards. The properties we have obtained in coupon collector's problem translates in the language of functions as follows.

Image size of a random function $f \in \mathcal{F}_{n,m}$, is k with probability

$$\Pr(|f(X)| = k) = \frac{k!}{m^n}\binom{m}{k}\left\{{n \atop k}\right\}.$$

If $n \geq m$, then Probability of a random function $f \in \mathcal{F}_{n,m}$ to be an onto function is

$$\Pr(|f(X)| = m) = \frac{m!}{m^n}\left\{{n \atop m}\right\}.$$

The expected image size of a random function $f \in \mathcal{F}_{n,m}$ is

$$E(|f(X)|) = m\left(1 - \left(1 - \frac{1}{m}\right)^n\right).$$

From the last statement it follows that

$$\sum_{k=1}^{m} k\frac{k!}{m^n}\binom{m}{k}\left\{{n \atop k}\right\} = m\left(1 - \left(1 - \frac{1}{m}\right)^n\right).$$

RANDOM PERMUTATIONS

A random permutation of n-objects is a permutation which is chosen among all permutations where each of the $n!$ possible permutations are equally likely. Since a random permutation is a random ordering of a set of objects, the use of them is fundamental to fields that use randomized algorithms such as coding theory, cryptography, and simulation. A good example of a random permutation is the shuffling of a deck of cards.

There are $n!$ permutations of n elements. That is too many to generate them by numbering them all and choosing one at random. Random permutations are quite useful in randomized algorithms. So it is helpful to have efficient algorithms for generating them. An equivalent way to generate a random permutation σ of $\{1, \dots, \mathrm{n}\}$ is to put n balls labeled 1 to n in a box and then at each step drawing a ball randomly and without replacement from the box to determine the values of $\sigma(1), \sigma(2), \dots, \sigma(n)$ sequentally.

The statistics of random permutations, such as the cycle structure, fixed points are of fundamental importance in the analysis of algorithms. We compute certain characteristics of random permutations.

THEOREM B.7. *The expected number of k-cycles of a random permutation of n objects is $\frac{1}{k}$.*

Proof. Assume that that we have written down the cycle decomposition of all permutations of $\{1, \dots, n\}$. How many cycles of length k are there? Fix a cycle, say $[1,2, \dots, k]$, of k elements and arrange the remaining $n-k$ elements in all possible ways. In this way we obtain all $(n-k)!$ permutations in which $[1,2, \dots, k]$ appears in the cycle decomposition. Since these k elements can define $(k-1)!$ cycles and the k elements of the cycle can be picked in $\binom{n}{k}$ different ways, in the list of all permutations, there are $(n-k)!\binom{n}{k}(k-1)! = \frac{n!}{k}$ cycles of length k. Thus, on the average a random permutation has $\frac{1}{k}$ cycles of length k. ■

COROLLARY B.8. *The expected number of cycles of a random permutation of n objects is H_n.*

Proof. Follows immediately from the theorem.

Alternative proof. Recall that $x^{\overline{n}} = \sum_{k=0}^{n} \left[{n \atop k}\right] x^k$ so that $f(x) = x^{\overline{n}}$ is the generating function of the sequence $\left[{n \atop 0}\right], \left[{n \atop 1}\right], \left[{n \atop 2}\right], \dots, \left[{n \atop n}\right]$. Then the expected number of cycles is $f'(1)/f(1)$. But

$$f'(x) = x^{\overline{n}} \left(\frac{1}{x} + \frac{1}{x+1} + \cdots + \frac{1}{x+k-1} \right),$$

then $f'(1) = n!\, H_n$ and $f(1) = n!$. Expected number of cycles is H_n. ■

COROLLARY B.9. *For any positive integer* n,

$$\sum_{k=1}^{n} k \begin{bmatrix} n \\ k \end{bmatrix} = n!\, H_{n.}$$

Proof. Each side of equality counts all cycles in all permutations of n objects. ■

Following theorem shows that the probability of a fixed element to be in a cycle of length k does not depend on k.

THEOREM B.10. *Probability that a fixed element of* $\{1, \dots, n\}$ *is in a cycle of length* k *is* $1/n$.

Proof. A fixed element say '1' is in a cycle of length k if $\sigma(1), \sigma^2(1), \dots, \sigma^{k-1}(1)$ are pairwise distinct and they are all different from 1 , and $\sigma^k(1) = 1$. Probability of that event is $\frac{n-1}{n} \cdot \frac{n-2}{n-1} \cdots \frac{n-k}{n-k+1} \cdot \frac{1}{n-k} = \frac{1}{n}$. ■

THEOREM B.11. *The expected length of the cycle containing a fixed element is* $\frac{n+1}{2}$.

Proof. The length of the cycle containing a fixed element can take any value $1,2, \dots, n$ with equal probabilities. The average of these lengths is $\frac{n+1}{2}$. ■

THEOREM B.12. *Any fixed* m *elements of* $\{1, \dots, n\}$ *lie in the same cycle of a random permutation, with probabilty* $1/m$.

Proof. Assume that the fixed elements lie in some cycle of length $m + k$, then we need k more elements to form the cycle, which can be chosen in $\binom{n-m}{k}$ ways. Then the cycle can be formed in $(m + k - 1)!$ ways and the remaining $n - m - k$ elements can be arranged in one of $(n - m - k)!$ ways. As a result, the number of permutations in which the fixed m elements in the same cycle of length k is

$$\binom{n-m}{k} (m + k - 1)!\, (n - m - k)! = \frac{(n-m)!}{k!} (m + k - 1)!$$

If we sum over $k = 1, \dots, n - m$:

$$\begin{aligned} (n-m)! \sum_{k=0}^{n-m} \frac{(m+k-1)!}{k!} &= (n-m)!\,(m-1)! \sum_{k=0}^{n-m} \binom{m+k-1}{k} \\ &= (n-m)!\,(m-1)! \binom{n}{n-m} \\ &= \frac{n!}{m}. \end{aligned}$$

Then assertion follows. ■

A cycle which contains more than half of all elements is called a ***long cycle***.

THEOREM B.13. *As n gets larger, the probability of having a long cycle converges to* $\ln 2$.

Proof. The number of cycles of length k in all permutations is $\frac{n!}{k}$. If $k > n/2$, the permutation can not have more than one cycles of length k. Thus, the probability of having a cycle of length $k > n/2$ is $1/k$. Then summing on all values of k larger than $n/2$ we see that the probability of having a long cycle is $\frac{1}{\frac{n}{2}+1} + \cdots + \frac{1}{n} = H_n - H_{n/2}$. For large values of n we use the approximation $H_n \approx \ln n + \gamma$ which results in $H_n - H_{n/2} \approx \ln n - \ln(n/2) = \ln 2$. ■

The function $\frac{1}{1-x}$ is the exponential generating function of number of permutations of n objects: $\frac{1}{1-x} = \sum_{k=0}^{\infty} x^k = \sum_{k=0}^{\infty} k! \frac{x^k}{k!}$ so that the coefficient of $\frac{x^n}{n!}$ is $n!$, the number of permutations of n objects. We can write this function as $\frac{1}{1-x} = \exp(-\ln(1-x))$. But $-\ln(1-x) = \sum_{k=1}^{\infty} \frac{x^k}{k} = \sum_{k=1}^{\infty}(k-1)! \frac{x^k}{k!}$. It is seen that $-\ln(1-x)$ is the exponential generating function of number of cycles of length k. Then, expression

$$\exp(-\ln(1-x)) = \exp\left(x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} + \cdots\right)$$

for the exponential generating function of permutations, takes the cycle structure in account. For example

$$\exp\left(x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} + \frac{x^5}{5}\right)$$

is the exponential generating function of number of permutations with longest cycle length 5.

Example 2.

(Problème des Rencontres)

A derangement is a permutation with no fixed points, that is with no cycles of length 1. Then the exponential generating function of derangements is

$$\exp\left(\frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} + \cdots\right) = \exp(-x - \ln(1-x))$$
$$= \frac{e^{-x}}{1-x}.$$

e^{-x} is the exponential generating function of the sequence $\left\{\frac{(-1)^k}{k!}\right\}$ And $\frac{e^{-x}}{1-x}$ is the sequence of partial sums of that sequence. Then $D_n = n! \sum_{k=0}^{n} \frac{(-1)^k}{k!}$.

Example 3.

(Number of Involutions)

An involution is a permutation whose cycle decomposition consists of fixed points and 2-cycles. Then the exponential generating function of involutions is

$$\begin{aligned}
\exp\left(x+\frac{x^2}{2}\right) &= \sum_{k=0}^{\infty}\frac{1}{k!}\left(x+\frac{x^2}{2}\right)^k \\
&= \sum_{k=0}^{\infty}\frac{1}{k!}\sum_{i=0}^{k}\binom{k}{i}\frac{1}{2^i}x^{k+i} \\
&= \sum_{n=0}^{\infty}\left(\sum_{k=0}^{\lfloor n/2\rfloor}\frac{1}{2^k(n-k)!}\binom{n-k}{k}\right)x^n \\
&= \sum_{n=0}^{\infty}\left(\sum_{k=0}^{\lfloor n/2\rfloor}\frac{1}{2^k k!\,(n-2k)!}\right)x^n.
\end{aligned}$$

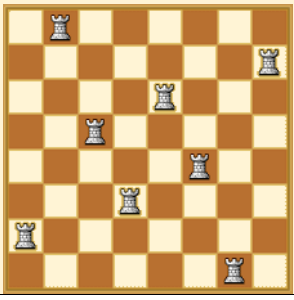
Then number of involutions of n objects is

$$\frac{n!}{2}\sum_{k=0}^{\lfloor n/2\rfloor}\frac{1}{2^k k!\,(n-2k)!}.$$

Appendix C

ROOK POLYNOMIALS

The theory of rook polynomials provides a way of counting permutations with restricted positions. Classical theory was developed by Kaplansky and Riordan in 1946 and has been researched and studied quite extensively since then. In chess, a rook can attack any square in its corresponding row or column of the 8×8 chessboard. Rook theory focuses on the placement of non-attacking rooks in a more general situation. The rook polynomial of a board counts the number of ways of placing non-attacking rooks on the board.



For positive integers n and m we define *an* $\boldsymbol{n \times m}$ ***board*** to be a rectangular chessboard consisting of n rows and m columns. A ***rook placement*** is an arrangement of a number of non-attacking rooks on some board. Note that since rooks attack squares in their row and column, a rook placement is, in fact, choosing a number of unit squares (cells) no two of which are on the same column or on the same row.

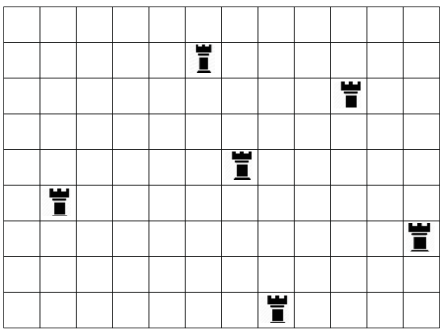
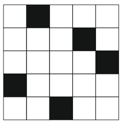


Figure 1. Six non-attacking rooks on a 9×12 board.

We use the ordered pair (i, j), $i = 1, \dots, n$, $j = 1, \dots, m$ to denote the cell that is on row i and column j of an $n \times m$ board. In the above figure, rooks are placed at the cells $(2,6), (3,10), (5,7), (6,2), (7,12), (9,8)$.

A placement of n rooks on an $n \times n$ square board can be associated to the permutation $\sigma = \sigma_1 \sigma_2 \cdots \sigma_n$ of $\{1, \dots, n\}$, by saying the placement has a rook on the cell (i, j) of the board if and only if $\sigma_i = j$. For example, the placement of 5 non-attacking rooks on a 5×5 board shown in below figure is the permutation 2 4 5 1 3.



In a similar manner, a placement of k rooks on an $n \times m$ board corresponds to a particular arrangement of k distinct objects, each of which is chosen from a set of n elements. In this way, we see that any theorem about rook placements is a generalization of a theorem about permutations.

We are interested in computing the number of ways of rook placements on a board. The k-th ***rook number*** $r_k(B)$, counts the number of ways to place k non-attacking rooks on a board B. We will often denote $r_k(B)$ as r_k when B is clear for the given context. The ***rook polynomial*** of a board B is the polynomial

$$R(B; x) = r_0 + r_1 x + r_2 x^2 + \cdots.$$

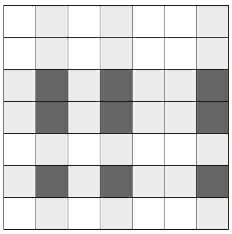
For any board, since there is only one way to place 0 rooks, we always have $r_0 = 1$, no matter which board is considered.

A single rook can be placed on any cell of a board with no other rook to attack it. It follows that r_1 is the number of unit squares of the board.

The largest possible number of non-attacking rooks on an $n \times m$ board is equal to the smallest of n and m, because we cannot allocate non-attacking rooks more than the number of columns or rows. Thus, $r_k = 0$ for $k > \min(m, n)$. For a 1×1 board, $r_0 = r_1 = 1$ and $r_k = 0$ for $k \geq 2$. Since we cannot allocate more than one rook on a $1 \times n$ or on an $n \times 1$ board, the rook polynomial of such a board is $1 + nx$.

THEOREM C.1. *The number of rook placements of k rooks on an n × m board is*

$$r_k = \binom{n}{k}\binom{m}{k} k!$$



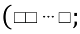
Proof. Consider an $n \times m$ board on which we wish to allocate k rooks. To place the rooks, first choose k columns and k rows arbitrarily. This can be done in $\binom{n}{k}\binom{m}{k}$ ways. Each chosen row intersects the chosen columns in k cells, hence for the first chosen row there are k positions to place a rook. For the second one there are $k - 1$ positions and continuing in this manner, we see that there are $k!$ ways to place k rooks on the chosen rows and columns. Hence we have a total of $\binom{n}{k}\binom{m}{k} k!$ ways to place the k rooks on an $n \times m$ board. ■

COROLLARY C.2. *Rook polynomial of an $n \times m$ board $B_{m,n}$ is*

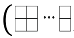
$$R(B_{m,n};x) = 1 + mnx + 2\binom{n}{2}\binom{m}{2}x^2 + \cdots + \binom{n}{k}\binom{m}{k}k!\,x^k + \cdots.$$

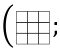
Proof. Claim is a direct consequence of the theorem. ■

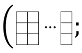
Example 1. Some obvious rook polynomials are

$R($  $;x) = 1 + nx,$

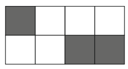
$R($  $;x) = 1 + 4x + 2x^2,$

$R($  $;x) = 1 + 2nx + n(n-1)x^2,$

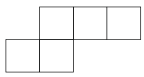
$R($  $;x) = 1 + 9x + 18x^2 + 6x^3,$

$R($  $;x) = 1 + 3nx + 3n(n-1)x^2 + n(n-1)(n-2)x^3.$

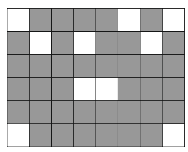
Now we generalize the problem in the sense that we consider boards where some of the squares are not allowed for placing a rook. For example, in the below figure we see a 2×4 board with 3 shaded squares which are forbidden to rook placement.



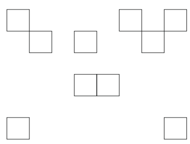
Such a subset of an $n \times n$ board will be called a ***generalized board***. A generalized board can be represented as a subset of a rectangular board by shading the forbidden cells or it can be drawn as it is, neglecting the forbidden cells:



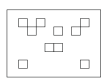
Example 2. We find the rook polynomial of the following generalized board $\mathfrak{B}$:



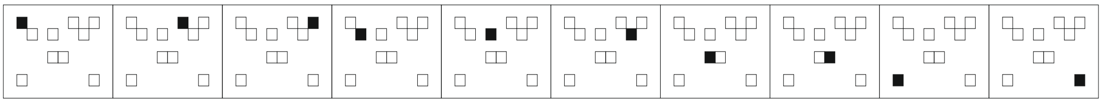
which can be represented as



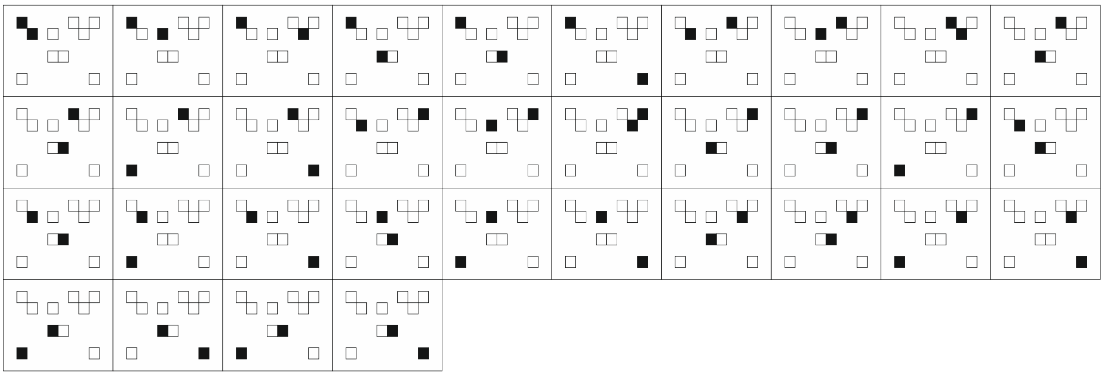
There is only one way to place 0 rooks on the board, hence $r_0(\mathfrak{B}) = 1$:



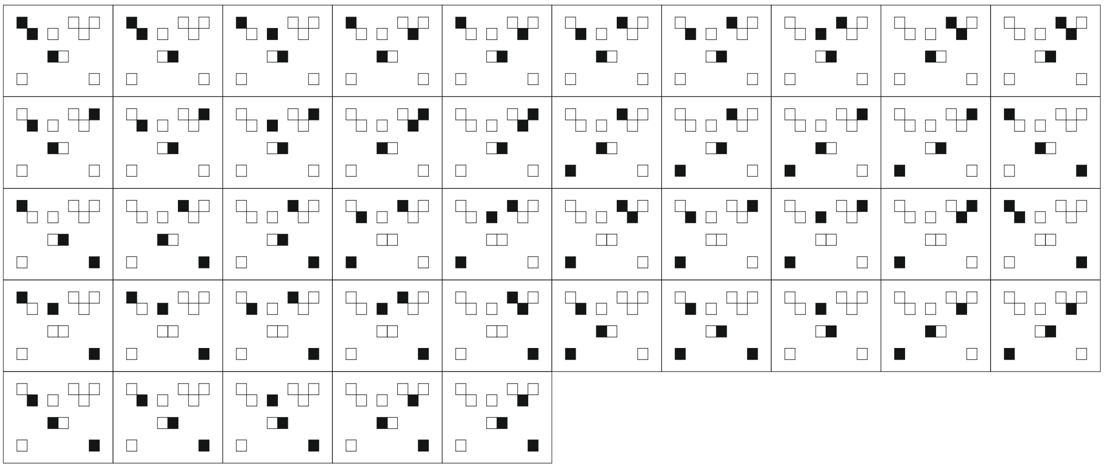
A single rook can be placed in any cell of $\mathfrak{B}$ with no other rooks to attack it. It follows that, as for any generalized board, r_1 is the number of available cells. In our example $r_1(\mathfrak{B}) = 10$:



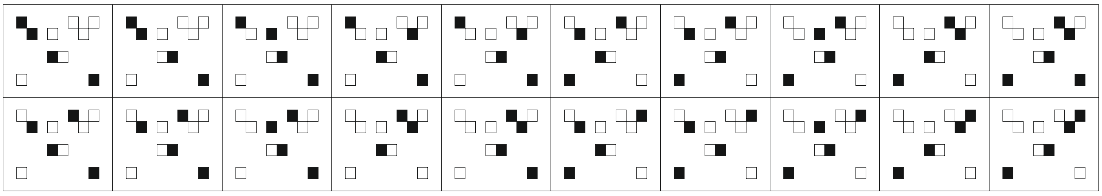
There are 34 ways to allocate 2 non-attacking rooks on the board, $r_2(\mathfrak{B}) = 34$:



The number of ways of placing 3 non-attacking rooks is 45, $r_3(\mathfrak{B}) = 45$:



Finally, the number of ways of placing 4 non-attacking rooks is 20, $r_4(\mathfrak{B}) = 20$:



For the board in this example we obtain

$$r_0(\mathfrak{B}) = 1, \qquad r_1(\mathfrak{B}) = 10, \qquad r_2(\mathfrak{B}) = 34, \qquad r_3(\mathfrak{B}) = 45, \qquad r_4(\mathfrak{B}) = 20.$$

Then, rook polynomial is

$$p(\mathfrak{B}; x) = 1 + 10x + 34x^2 + 45x^3 + 20x^4.$$

Equivalent Boards

To obtain the rook polynomial or rook numbers of a generalized board, it is not practical to count the ways of rook placements by explicitly considering all possible cases. In the following part, we obtain some shortcuts for obtaining rook numbers and the rook polynomial. We first list some trivial properties of rook numbers:

1. r_0 is always 1 because there is only one way to place 0 rooks on a generalized board.
2. r_1 is always the number of cells of the board because a single rook can be placed in any cell with no other rook to attack it.
3. Once we attain $r_k = 0$, we will always have $r_{k+1}, r_{k+2}, \ldots = 0$.
4. If the board is contained in an $n \times m$ square board and $k > \min(m, n)$, then $r_k = 0$. However, note that r_k could be equal to 0 for smaller values of k as well.

If the number of placements of k rooks on two boards are the same for all $k = 1, 2, \ldots$, we call the boards ***equivalent***. In other words, two boards are equivalent if and only if their rook polynomials are identical.

THEOREM C.3. *If a generalized board can be obtained from another one by permuting the rows and columns, then the boards are equivalent.*

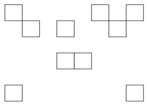
Proof. It is sufficient to prove the theorem for rectangular boards. Interchanging rows or columns of a rectangular board does not alter the number of ways to place k mutually non-attacking rooks. In fact, given a placement of k rooks, then permuting the rows or the columns of the board results in another rook placement. Moreover, the number of distinct placements is not affected by such permutations. ■

Converse of the above theorem is not true. As a counter example for the converse statement we can consider the following generalized boards.

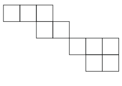
These boards cannot be obtained from each other by permuting the rows or columns. However, they both have the same rook polynomial $1 + 4x + 2x^2$, hence they are equivalent.

If there are some empty rows (or empty columns) of a generalized board, by permuting the rows (or columns) of the board, we can move these empty rows to the bottom (or empty columns to the rightmost) of the board so that the generalized board is squeezed to give an equivalent board without any empty rows or columns.

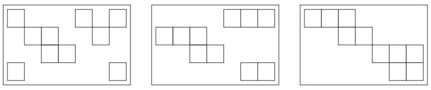
We return back to the board $\mathfrak{B}$ .The third and the fifth rows do not contain any cell of the generalized board $\mathfrak{B}$, similarly on the third column there is no cell of the generalized board.

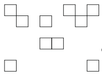
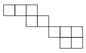


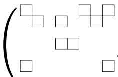
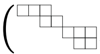
By squeezing these columns and rows, we obtain the following equivalent board:



Continuing on permuting the rows and columns we reach the following board:

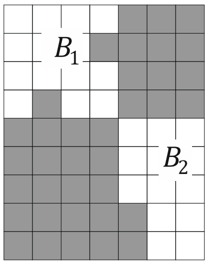


It follows that the boards  and  are equivalent:

$R\Big($  $;x\Big) = R\Big($  $;x\Big)$

Decomposition Rules

If a generalized board B is a disjoint union of two sub-boards B_1 and B_2 in which no cell of one subboard is in the same row or in the same column of any cell of the other subboard, then B_1 and B_2 are said to be ***disjoint***. The board B given in below figure is a disjoint union of the boards B_1 and B_2.



The next theorem shows that if a board is composed of two disjoint sub-boards, the rook polynomial of the board can be calculated in terms of the rook polynomials of the sub-boards.

THEOREM C.4 (*Board Decomposition*). *If a board B is union of two disjoint boards B_1 and B_2 then*

$$R(B;x) = R(B_1;x)R(B_2;x).$$

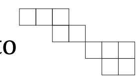
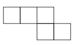
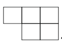
Proof. Rook placements in B_1 does not affect rook placements in B_2, except in terms of the total number of rooks being placed. Assume that we wish to place k rooks on board B. If we place m rooks on B_1, then $k-m$ rooks must be placed on B_2. The number of ways of placing m rooks on B_1 and $k-m$ rooks on B_2 is $r_m(B_1)r_{k-m}(B_2)$. Since m can take any value $k = 1, \dots, m$, the number of ways of placing k rooks on $B = B_1 \cup B_2$ is

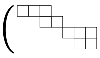
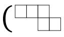
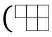
$$r_k(B) = r_0(B_1)r_k(B_2) + \cdots + r_m(B_1)r_{k-m}(B_2) + \cdots + r_k(B_1)r_0(B_2).$$

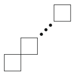
The right hand side of the equality is in fact the coefficient of x^k in the product

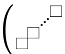
$$(r_0(B_1) + r_1(B_1)x + \cdots)(r_0(B_2) + r_1(B_2)x + \cdots) = R(B_1;x)R(B_2;x).$$

It follows that $R(B;x) = R(B_1;x)R(B_2;x)$. ■

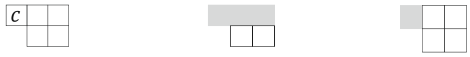
We have shown the board $\mathfrak{B}$ is equivalent to . Since the latter board is disjoint union of the boards  and , we can write

$R\Big($  $;x\Big) = R($  $;x)R($  $;x).$

Example 3. The staircase  consisting of n cells is a disjoint union of n single cells. Since the rook polynomial of a single cell is $1+x$, for the staircase we have

$R\Big($  $;x\Big) = (1+x)^n.$

Suppose that on a given board B, a cell c is selected and marked as a special cell. By $B\backslash c$ we denote the board obtained from B by deleting the row and column that contain the cell c, and we let $B - c$ to denote the board obtained from B by deleting only that special cell c.

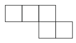
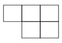


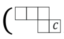
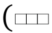
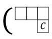
B $\qquad$ $B\backslash c$ $\qquad$ $B - c$

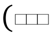
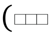
THEOREM C.5 (*Cell decomposition*). *For any cell c of a board B,*

$$R(B;x) = xR(B\backslash c;x) + R(B-c;x).$$

Proof. To find the value of $r_k(B)$, we observe that the ways of placing k non-attacking rooks on B can be divided into two classes, those that have rook in the special cell and those that do not have a rook in the special cell. The number of ways in the first class is equal to $r_{k-1}(B\backslash c)$, and the number of ways in the second class is equal to $r_k(B-c\ )$. We have then the relation $r_k(B) = r_{k-1}(B\backslash c) + r_k(B-c)$. Correspondingly, $R(B;x) = xR(B\backslash c;x) + R(B-c;x)$. ■

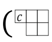
Example 4. As an example, we compute the rook polynomials of the boards  and :

$R($  $;x) = xR($  $;x) + R($  $;x)$

$= xR($  $;x) + xR($  $;x) + R($  $;x)$

$= (x+1)(1+3x) + x(1+2x)$

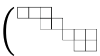
$= 1 + 5x + 5x^2.$

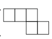
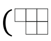
$R($  $;x) = xR($  $;x) + R\big($  $;x\big)$

$= x(1+2x) + 1 + 4x + 2x^2$

$= 1 + 5x + 4x^2.$

Now we can re-calculate the rook polynomial of the board $\mathfrak{B}$:

$R(\mathfrak{B};x) = R\Big($  $;x\Big)$

$= R($  $;x)R($  $;x)$

$= (1+5x+5x^2)(1+5x+4x^2)$

$= 1 + 10x + 34x^2 + 45x^3 + 20x^4.$

Complementary Boards

THEOREM C.6 (*Complementary Board Theorem*). *Let $\overline{B}$ be the complement of B in an $n \times m$ rectangular board. If $R(B;x) = \sum r_k x^k$ and $R(\overline{B};x) = \sum \overline{r}_k x^k$ are rook polynomials of B and $\overline{B}$, respectively, then*

$$r_k = \sum_{i=0}^{k} (-1)^i \binom{m-i}{k-i}\binom{n-i}{k-i}(k-i)!\,\overline{r}_i$$

where we let $\overline{r}_i = 0$ for i larger than the degree of $R(\overline{B};x)$.

Proof. We consider all possible placements of k non-attacking rooks on the $m \times n$ board and remove those where one or more rooks are placed on $\overline{B}$ using the Inclusion-Exclusion Principle. Assume that the k rooks are labeled in our counting process, which means we will be counting $k!\,r_k$ rather thanr_k. The total number of ways to place k labeled rooks on the $n \times n$ board is $\binom{m}{k}\binom{n}{k}k!^2$. Let A_i denote the set of rook placements where the i-th rook is on the board $\overline{B}$. We have to remove these placements from the set of all placements. There are $\overline{r}_1$ ways to place the i-th rook on $\overline{B}$ and $\binom{m-1}{k-1}\binom{n-1}{k-1}(k-1)!^2$ ways to place the remaining $k-1$ rooks in the other rows and columns. Hence there are $r_1\binom{m-1}{k-1}\binom{n-1}{k-1}(k-1)!^2$ placements in A_i and there are k many A_i's. Similarly, there are $\overline{r}_2\binom{m-2}{k-2}\binom{n-2}{k-2}(k-2)!^2$ elements in $A_i \cap A_j$ for $i \neq j$ and there are $\binom{k}{2}$ of these double intersections, and so on. Hence, using the principle of inclusion-exclusion, the number of ways to place k labeled rooks on B is

$$\sum_{i=0}^{k} (-1)^i \binom{k}{i} i! \binom{m-i}{k-i}\binom{n-i}{k-i}(k-i)!^2\,\overline{r}_i$$

which can be written as

$$\sum_{i=0}^{k} (-1)^i k! \binom{m-i}{k-i}\binom{n-i}{k-i}(k-i)!\,\overline{r}_i.$$

To discard the effect of labeling the rooks, we divide this quantity by $k!$ to achieve the result. ■

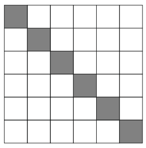
Finding the number of placements of n rooks on a board which is a subset of an $n \times n$ board is a quite common problem. For this particular case, the following corollary is very useful.

COROLLARY C.7. *Let B be a generalized board in an $n \times n$ rectangular board. If $\sum \overline{r}_k x^k$ is the rook polynomial of $\overline{B}$, then the number of ways of placing n non-attacking rooks on B is*

$$r_n = \sum_{i=0}^{n} (-1)^i (n-i)!\,\overline{r}_i.$$

Proof. Just put $k = m = n$ in the theorem. ■

Example 5 (Problème des Rencontres). We find the number of ways of placing on an $n \times n$ square board if the diagonal cells are forbidden. We have to compute r_n for the generalized board B which consists of white cells of the below board. The shaded cells constitute $\overline{B}$.



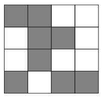
Note that the board B corresponds to a permutation $\sigma = \sigma_1\sigma_2 \cdots \sigma_n$ of the set $\{1, \dots n\}$ such that $\sigma_i \neq i$, which means a derangement. Hence r_n is the number of derangements of the set $\{1, \dots n\}$, that is $r_n = \mathcal{D}_n$. Now we obtain the number derangements by means of rook numbers.

$\overline{B}$ is a staircase with $R(\overline{B}) = (1+x)^n = \sum_{i=0}^{n} \binom{n}{i} x^i$, hence $\overline{r}_i = \binom{n}{i}$. Then from the corollary it follows that

$$\begin{aligned} r_n &= \sum_{i=0}^{n} (-1)^i (n-i)!\, \overline{r}_i \\ &= \sum_{i=0}^{n} (-1)^i (n-i)! \binom{n}{i} \\ &= n! \sum_{i=0}^{n} \frac{(-1)^i}{i!} \\ &= \mathcal{D}_n. \end{aligned}$$

EXERCISES

1. Find the rook polynomial of the following (unshaded) board.



2. At a university, seven freshmen, $F_1, F_2, \dots, F_7$, enter the same academic program. The chairman wants to assign each incoming freshman a mentor from among the upperclassmen of the program. Seven mentors $M_1, M_2, \dots M_7$, but there are some scheduling conflicts. M_1 cannot work with F_1 or F_3, M_2 cannot work with F_1 or F_5, M_4 cannot work with F_3 or F_6, M_5 cannot work with F_2 or F_7, and M_7 cannot work with F_4. In how many ways can the chairman assign the mentors so that each incoming freshman has a different mentor?

3. Given a permutation $\sigma = \sigma_1\sigma_2 \cdots \sigma_n$ of $\{1, \dots, n\}$, consider the corresponding rook placement of n rooks on the $n \times n$ board and let $B = (b_{ij})$ be the matrix such that $b_{ij} = 1$ if the square (i, j) of the board is occupied by a rook and $b_{ij} = 0$ otherwise. Show that σ is an even permutation if and only if $\det(B) = 1$.

4. Which of the following polynomials can be the rook polynomial of a board? Give reasons, including examples of appropriate boards, where possible.

a) $1 + x$,

b) $(1 + x)^n$,

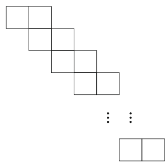
c) $1 + 5x + 6x^2$,

d) $1 + 5x + 7x^2$,

e) $(1 + 4x + 2x^2)^2$,

f) $1 + 7x - 6x^2 + 3x^3$,

5. Let S_n be the 'staircase' board illustrated below, consisting of n rows.

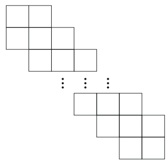


Show that S_n has the rook polynomial

$$R(S_n; x) = \sum_{k=0}^{n} \binom{2n-k+1}{k} x^k.$$

6. Let n be a positive even integer and consider an $n \times n$ chess-board in which the squares are coloured black and white in the usual chequered fashion. In how many ways can n non-attacking rooks be placed on the white squares?

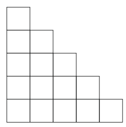
7. Find the rook polynomial of the 'staircase' board illustrated below, consisting of n rows.



8. Using the result of Problem 7, find the number of permutations $\sigma = \sigma_1 \cdots \sigma_n$ of $\{1, \dots, n\}$ such that $|\sigma_i - i| \leq 1$.

9. A club holds a weekly lottery for its n members, n being a positive even integer. Upon joining the club, each member is assigned a number, based on the order in which they have joined the club (the i-th member to join is assigned the integer i). In the weekly lottery drawing, each member is assigned randomly a number from 1 to n, such that each number is used once and only once, and winners are determined as follows: if Member i draws either i or $n + 1 - i$ for the week, he or she wins. In the case of multiple winners, the pot would be split among them. In how many ways can there be no winners the first week of the lottery?

10. **(Triangular Boards)** Consider the family T_n of 2-dimensional boards called the triangle boards. A triangle board of size n consists of the cells which are below the diagonal cells of a an $n \times n$ square board. The triangle board T_5 of size 5 is shown below.



Show that

$$r_k(T_n) = \left\{ {n+1 \atop n+1-k} \right\}.$$

11. **(Problème des ménages.)** This question concerns n couples. Making use of rook polynomials, find the number of ways of seating the n couples ($2n$ people) around a circular table such that men and women are sitting alternately and no woman is sitting next to her own partner.

Solutions to Exercises

1.a)

$$S_n = 1 - \binom{n+1}{m} + \sum_{k=m}^{n} \underbrace{\binom{k+1}{m}}_{\binom{k}{m}+\binom{k}{m-1}}$$

$$= 1 - \binom{n+1}{m} + \underbrace{\sum_{k=m}^{n} \binom{k}{m}}_{S_n} + \sum_{k=m}^{n} \binom{k}{m-1}$$

Thus we have

$$\sum_{k=m}^{n} \binom{k}{m-1} = 1 - \binom{n+1}{m} - 1$$

and replacing m with $m+1$

$$\sum_{k=m+1}^{n} \binom{k}{m} = \binom{n+1}{m+1} - 1$$

which gives

$$\sum_{k=m}^{n} \binom{k}{m} = \binom{n+1}{m+1}.$$

1.b) First define the sum $T_n = \sum_{k=1}^{n} kH_k$ and perturb this sum:

$$T_n = 1 - (n+1)H_{n+1} + \sum_{k=1}^{n} (k+1)H_{k+1}$$

$$= 1 - (n+1)H_{n+1} + \sum_{k=1}^{n} (k+1)\left(H_k + \frac{1}{k+1}\right)$$

$$= 1 - (n+1)H_{n+1} + \sum_{k=1}^{n} (k+1)H_k + n$$

$$= 1 - (n+1)H_{n+1} + \underbrace{\sum_{k=1}^{n} kH_k}_{T_n} + \underbrace{\sum_{k=1}^{n} H_k}_{S_n} + n$$

$$= (n+1)H_n + T_n + S_n + n$$

which simplifies as

$$\sum_{k=1}^{n} H_k = (n+1)H_n - n.$$

1.c) As in the previous example, we define a new sum whose perturbation leads to the desired sum. Let $U_n = \sum_{k=1}^{n} k^2 H_k$, then

$$U_n = 1 - (n+1)^2 H_{n+1} + \sum_{k=1}^{n} (k+1)^2 H_{k+1}$$
$$= 1 - (n+1)^2 H_{n+1} + \sum_{k=1}^{n} (k^2 + 2k + 1)\left(H_k + \frac{1}{k+1}\right)$$
$$= 1 - (n+1)^2 H_{n+1} + U_n + 2S_n + (n+1)H_n - n + \sum_{k=1}^{n} (k+1)$$

U_n vanishes and the equation takes the form

$$2S_n = (n+1)^2 H_{n+1} - 1 - (n+1)H_n + n - \frac{(n+1)(n+2)}{2} + 1$$
$$= (n+1)^2 H_{n+1} - (n+1)H_n - \frac{n^2+n+2}{2}$$
$$= (n+1)^2 H_{n+1} - (n+1)H_{n+1} + 1 - \frac{n^2+n+2}{2}$$
$$= n(n+1)H_{n+1} - \frac{n(n+1)}{2}$$

and finally

$$S_n = \frac{1}{2} n(n+1)\left(H_n - \frac{1}{2}\right).$$

2.a) We can write $k = \sum_{j=1}^{k} 1$ so that

$$S_n = \sum_{k=1}^{n} \sum_{j=1}^{n} 1$$
$$= \sum_{j=1}^{n} \sum_{k=j}^{n} 1$$
$$= \sum_{j=1}^{n} (n - j + 1)$$
$$= \underbrace{\sum_{j=1}^{n} (n+1)}_{n(n+1)} + \underbrace{\sum_{j=1}^{n} j}_{S_n}$$

Which results in

$$S_n = \frac{1}{2} n(n+1).$$

2.b) We can write $k^3 = \sum_{j=1}^{k} k^2$ so that

$$
\begin{aligned}
S_n &= \sum_{k=1}^{n} \sum_{j=1}^{k} k^2 \\
&= \sum_{j=1}^{n} \sum_{k=j}^{n} k^2 \\
&= \frac{1}{6}\left(n^2(n+1)(2n+1) - \sum_{j=1}^{n} j(j-1)(2j-1) \right) \\
&= \frac{1}{6}\left(n^2(n+1)(2n+1) - \sum_{j=1}^{n} (2j^3 - 3j^2 + j) \right) \quad .
\end{aligned}
$$

So we have

$$
\begin{aligned}
8S_n &= n^2(n+1)(2n+1) + \frac{1}{2}n(n+1)(2n+1) - \frac{n(n+1)}{2} \\
&= \frac{n(n+1)}{2}(2n(2n+1) + (2n+1) - 1) \\
&= \frac{1}{2}n(n+1)(4n^2 + 4n).
\end{aligned}
$$

Thus

$$S_n = \frac{1}{4}n^2(n+1)^2.$$

2.c) We can write $kr^k = \sum_{j=1}^{k} r^k$ so that

$$
\begin{aligned}
S_n &= \sum_{k=1}^{n} \sum_{j=1}^{k} r^k \\
&= \sum_{j=1}^{n} \sum_{k=j}^{n} r^k \\
&= \frac{1}{1-r} \sum_{j=1}^{n} (r^j - r^{n+1}) \\
&= \frac{r}{1-r}\left(\frac{(n+1)r^n - 1 - nr^{n+1}}{r-1} \right) \\
&= r\,\frac{nr^{n+1} - (n+1)r^n + 1}{(1-r)^2}.
\end{aligned}
$$