METU

DEPARTMENT OF MATHEMATICS

Math 112 Discrete Mathematics

**Exercises**

**Week 1 (Fundamental Principles of Counting)**

**1)** In how many ways can you arrange 7 different books, so that a specific book is on the third place?

**2)** In how many ways can you take 3 marbles out of a box with 15 different marbles?

**3)** In a firm there are 20 workmen and 10 employees. In how many ways can you have a committee with 3 workmen and 2 employees?

**4)** In how many ways can you take 5 cards, with at least 2 aces, out of a deck of 52 cards?

**5)** In how many ways can you split a group of 13 students in 3 students and 10 students?

**6)** How many diagonals are there in a convex $n$-polygon?

**7)** How many 3-digit integers can be written with the digits 0,1 , 2, 3, 4?

**8)** Let $A$ be a set with 15 elements. How many subsets of $A$ has 7 or less elements?

**9)** Calculate the term with $x^2$ in the expansion of $\left(x^3+\frac{1}{2x}\right)^{10}$.

**10)** In how many ways can you share $m$ identical stones into $k$ boxes so that each pile has at least one stone in it. Assume that boxes labeled with integers $1,2,\ldots,k$.

**11)** How many strictly positive integer solutions $(x,y,z)$ are there, such that $x+y+z = 100$.

**12)** How many terms are there in $(a+b+c)^{20}$?

**13)** A city police department has ten detectives: seven males and three females. In how many ways can a team of three detectives can be chosen to work on a case if

a) there are no restrictions?

b) the team must have at least one male and one female?

**14)** Given 11 different mathematics and seven different psychology books. How many ways are there for Ahmet to choose a mathematics book, and then for Bekir to choose a mathematics book or a psychology book, and then for Mustafa to chose both a mathematics book and a psychology book.

**15)** How many eight-letter sequences can be constructed using the 26 letters of the English alphabet which

a) contain exactly three a's,

b) contain three or four vowels (a,e,i,o,u),

c) have no repeated letters,

d) contain an even number of e's.

**16)** In a group of scientists there are three mathematicians, four physicians, five biologists and six chemists. A committee of four scientists is to be formed. In how many different ways such a committee can be formed if it must contain

a) exactly two mathematicians and one physicians.

b) one scientist of each field.

c) not all scientists from the same field.

d) at least two scientists that are not biologists.

**17)** In how many ways can the 26 letters of the English alphabet be placed

a) in a row?

b) in a row so that no two vowels are consecutive?

c) in a circle?

d) in a circle so that no two vowels are consecutive?

**18)** Eight students and four faculty members must be seated at two (identical) six-person round tables for lunch.

a) In how many ways can this be done?

b) In how many ways can this be done with at least one faculty member at each table?

**19)** How many ways are there to distribute 20 different homework problems to ten students if

a) there are no restrictions?

b) each student gets two problems?

**20)** How many different positive integers can be formed using the digits

a) 1,2,3,4,5,6,7,8,9
b) 1,1,3,3,5,5,7,7,9
c) 3,3,3,6,6,6,9,9,9
d) 8,8,9,9,9,9,9,9,9

**21)** A local dairy offers 16 flavors of ice cream.

a) How many ways are there to purchase eight scoops of ice cream?
b) How many ways are there to purchase eight scoops of ice cream of eight different flavors?
c) How many ways are there to distribute the eight scoops of part b) to four students so that each student gets two scoops?
d) How many ways are there to distribute eight scoops of chocolate ice cream to four students so that each student gets at least one scoop?
e) If Zeynep wants to purchase a triple-scoop ice cream cone, how many choices does she have? (Does the order of flavors on the cone matter? Answer the question for both cases.)

**22)** Consider choosing three digits from {0,1,2,3,4,5,6,7,8,9}.

a) In how many ways can this be done?
b) In how many ways can this be done if no two of the digits chosen are consecutive?

**23)** Given five calculus books, three linear algebra books, and two number theory books (all distinct), how many ways are there to

a) select three books, one in each subject?
b) make a row of three books?
c) make a row of three books, with one book in each subject?
d) make a row of three books, with exactly two of the subjects represented.
e) make a row of three books, all in the same subject?

**24)** If the numbers from 1 to 100,000 are listed, how many times does the digit 5 appear?

**25)** A sequence of characters is called a palindrome if it reads the same way forward or backward. For example 5FTTF5 is a six-character palindrome, and H8G8H is a five-character palindrome. Some other instances of palindromes: NEVER ODD OR EVEN, ANASTAS MUM SATSANA, RUHU BULUR ULU BUHUR, TOO HOT TO HOOT. Find the number of nine-character palindromes that can be formed using the 29 letters of the Turkish alphabet such that no letter appears more than twice in each of them.

**26)** There are $n$ married couples in a group. Find the number of ways of selecting a woman and a man who is not her husband.

**27)** There are three bridges connecting two towns, $A$ and $B$. Between towns $B$ and $C$ there are four bridges. A salesperson has to travel from $A$ to $C$ via $B$. Find

a) The number of possible choices of bridges from $A$ to $C$.
b) The number of choices for a round-trip travel from $A$ to $C$.
c) The number of choices for a round-trip travel if no bridge is repeated.

**28)** Four station wagons, five sedans, and six vans are to be parked in a row of 15 parking spots. Find the number of ways of parking these vehicles such that

a) The station wagons are parked at the beginning, then the sedans, and then the vans.
b) Vehicles of the same type are parked en bloc.

**29)** Six girls and six boys are to be assigned to stand around a circular fountain. Find the number of such assignments if on either side of a boy there is a girl.

**30)** Find the number of ways of

a) Assigning 9 students to 11 rooms (numbered serially from 100 to 110) in a dormitory so that each room has at most one occupant.
b) Installing nine color telephones (two red, three white, and four blue) in these rooms so that each room has at most one telephone.

**31)** There are four women and nine men in the mathematics department of a university. Find the number of ways of forming a hiring committee consisting of 2 women and 3 men from the department.

**32)** Find the number of ways of seating $r$ people from a group of $n$ people around a round table.

**33)** Find the number of ways of seating 14 people such that 8 of them are around a round table and the rest are on a bench.

**34)** Find the number of ways in which the letters that appear in MISSISSIPPI can be rearranged so that no two S's are adjacent.

**35)** Let $X$ be the set of all words of length 10 in which the letter P appears 2 times, Q appears 3 times, and R appears 4 times. Find the cardinality of $X$. (Consider the English alphabet.)

**36)** A class consists of 10 mathematics majors and 12 computer science majors. A team of 12 has to be selected from this class. Find the number of ways of selecting a team if

a) The team has 6 from each discipline.

b) The team has a majority of computer science majors.

## Week 2 (Basic tools of counting)

**37)** Using the letters of English alphabet, in how many different ways is it possible to write a 7 letter string so that

a) no two letters are the same,
b) no two consecutive letters are the same,
c) three letters are the same, the remaining four letters are all different,
d) if the first letter is a vowel, the last letters is a consonant,
e) letters are alternatingly consonant / vowel
f) a consonant is always followed by a vowel,
g) a letter is used at most twice.

**38)** In the following pile of letters we start from one of S's and at each step we move to one of the adjacent letters to trace the word STRAMBOŞE. In how many different ways can this be done?

```
                S
              S T S
            S T R T S
          S T R A R T S
        S T R A M A R T S
      S T R A M B M A R T S
    S T R A M B O B M A R T S
  S T R A M B O Ş O B M A R T S
S T R A M B O Ş E Ş O B M A R T S
```

**39)** All digits of the integer 6105293748 are distinct and the difference (by absolute value) of the last and first digits is 2. Find the number of all such 10-digit positive integers.

**40)** There are blue, yellow and white pencils. In how many different ways is it possible to choose 20 pencils if the number of blue pencils has to be even.

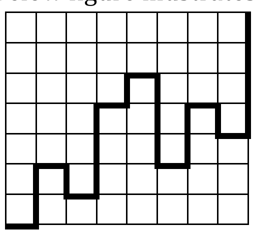
**41)** There are blue, yellow and white pencils. In how many different ways is it possible to choose 15 pencils if the number of blue pencils has to be even.

**42)** In how many different ways can two children share $2n$ blue, $2n$ yellow and $2n$ white pencils so that each one receives $3n$ pencils?

**43)** In how many different ways can 6 children share 20 balls so that each one receives

a) an even number of balls,
b) an odd number of balls.

**44)** On a $p \times q$ rectangular grid, a 'right-up-down path' is a path which joins the lower left corner to the upper right corner and at each vertex which moves towards right or up or down. Find the number of right-up-down paths. Below figure illustrates such a path on a $7 \times 8$ grid.



**45)** A set with $n$ elements has $2^n$ subsets. Find the sum of cardinalities of all these subsets.

**46)** A group of 12 students consists of 6 pairs of twins. If a student is not allowed to be in the same group with her twin, in how many different ways can they be partitioned into

a) two equal sized groups?
b) three equal sized groups?

**47)** A multiple choice consists of 8 questions, with 4 choices for each question. By re-ordering the questions and choices, is it possible to design a different exam paper for each of 1.000.000 students?

**48)** A question paper consists of 10 questions divided into two parts A and B. Each part contains five questions. A candidate is required to attempt six questions in all of which at least 2 should be from part A and at least 2 from part B. In how many ways can the candidate select the questions if he can answer all questions equally well?

**49)** The letters of the word NİKSAR are written in all possible orders and these words are sorted as in a dictionary. Find the rank of the word NİKSAR?

**50)** The letters of the word MALZEME are written in all possible orders and these words are sorted as in a dictionary. Find the rank of the word MALZEME?

**51)** 15 boys and 6 girls have to stand in a line such that no two girls are next to each other. How many possible ways?

**52)** Find the number of ways of filling the cells of a $3 \times 3$ table with non-negative integers such that the sum of each row and each column is 15. Below, three examples are given.

| | | |
|---|---|---|
| 2 | 9 | 4 |
| 7 | 5 | 3 |
| 6 | 1 | 8 |

| | | |
|---|---|---|
| 15 | 0 | 0 |
| 0 | 4 | 11 |
| 0 | 11 | 4 |

| | | |
|---|---|---|
| 3 | 9 | 3 |
| 11 | 1 | 3 |
| 1 | 5 | 9 |

**53)** Find the number of binary sequences of length $n$ if each 1 is followed by an odd number of 0's.

**54)** The binary string 0011010001110 of length 13 has 7 runs: 00 11 0 1 000 111 0. Find the number of all such strings (i.e. all binary strings of length 13 with 7 runs).

**55)** Find the number of ways of arranging the following letters around a circle

a) A B B C C D D E E F F

b) A A B B C C D D E E F F

c) A A A B B B

**56)** I have 5 identical apples, 8 identical oranges and 13 identical bananas. How many different non-empty baskets can I make consisting of

a) 5 fruits?

b) 7 fruits?

**57)** Let $X = \{1,2,\dots,n\}$ and $Y = \{1,2,\dots,m\}$. Find the number of monotonic (either increasing or decreasing) functions $X \to Y$.

**58)** Let $X = \{1,2,\dots,n\}$ and $Y = \{1,2,\dots,m\}$. Find the number of non-decreasing functions $X \to Y$.

**59)** Let $X = \{1,2,\dots,n\}$. Find the number of idempotent functions $X \to X$.

(A function $f$ is said to be idempotent if $f \circ f = f$.)

## Week 3 (Principle of Inclusion-Exclusion)

**60)** Find the number of positive integers not larger than 1000 which are divisible by 3 or 5.

**61)** Find the number of five-digit combinations from the set $\{1,2,3,4,5\}$ in which

a) Some digit appears at least three times.

b) No digit appears more than twice.

**62)** Find the number of collections of $n$ letters chosen from the set {t, w, x, y, z} if each element appears at least once.

**63)** 100 students are examined in 5 different courses. In each course, 25 students failed. For any two courses, the number of students failed in both of them is 12. For any three courses, the number of students failed in all three of them is 9. For any four courses, the number of students failed in all four of them is 7. Five students failed in all of the courses. How many students passed all courses?

**64)** In how many different ways is it possible to rearrange the letters of MATHEMATICS so that no two adjacent letters are the same?

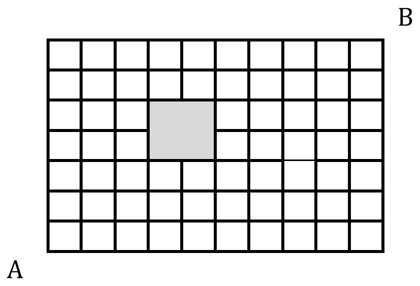
**65)** Four couples are sitting in a row. Find the number of arrangements in which no person is sitting next to his or her partner. What if they are sitting in a circle?

**66)** Find the number of all positive integers less than or equal to 1000 that are not divisible by 7, 10, or 15.

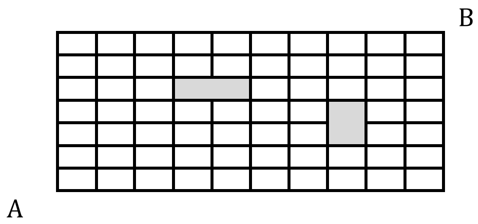
**67)** Find number of non-negative integer solutions of $x_1 + x_2 + x_3 + x_4 + x_5 + x_6 = 15$ such that $x_i \leq 4$.

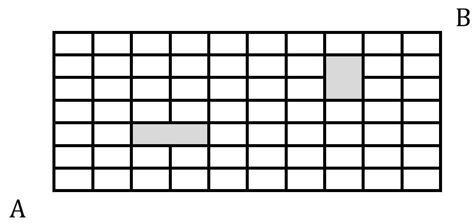
**68)** In the permutation BMTGJSAYIVÇFDOĞRUZHNÜŞEKİLCÖP of Turkish alphabet, the words SAYI, ŞEKİL and DOĞRU can be read directly: BMTGJSAYIVÇFDOĞRUZHNÜŞEKİLCÖP. Find the number of permutations of the alphabet in which none of these words can be read directly.

**69)** For the following $7 \times 10$ grid, find the number of 'right-up' paths which join the corner $A$ to the corner $B$.




**70)** For each of the following $7 \times 10$ grids, find the number of 'right-up' paths which join the corner $A$ to the corner $B$.




**71)** Find the number of ways of distributing 20 candies to 5 children so that
a) each of the first two children receives no more than 4 candies,
b) no child receives more than 6 candies.

**72)** In a permutation $\sigma_1\sigma_2 \cdots \sigma_n$ of the integers $1,2,\ldots,n$, the term $\sigma_i$ is called a fixed point if $\sigma_i = i$. For example, the permutation 4135762 has two fixed points $41\underline{3}57\underline{6}2$. A permutation without any fixed point is called a derangement. The number of derangements is denoted by $D_n$. Find the number of
a) derangements,
b) permutations with exactly one fixed point,
c) permutations with exactly $k$ fixed points.

**73)** How many integers in the set $\{1, 2, 3, 4, \ldots, 360\}$ have at least one prime divisor in common with 360?

**74)** Let X be a set with m elements and Y be a set with n elements. Show that the number of onto functions from X to Y is

$$\sum_{k=0}^{n} (-1)^k \binom{n}{k} (n-k)^m.$$

**75)** Let $X$ be a set with $m$ elements and $Y$ be a set with $n$ elements. Find the number of functions $f: X \to Y$ such that $|f(x)| = k$.

## Week 4 (Basic Counting Models 1)

**76)** Find the number of ways of forming a bouquet of 7 flowers, using roses, cloves, tulips and chrysanthemums.

**77)** Find the number of ways of distributing 15 candies to 7 children so that the first child receives at most one candy, the second child receives at most two candies,... and the seventh child receives at most seven candies.

**78)** Find the number of ways of distributing 60 candies to 7 children so that the first child receives at least one candy, the second child receives at least two candies,... and the seventh child receives at least seven candies.

**79)** Find the number of ways of distributing 30 candies to 7 children so that the first and the second children receive the same number of candies.

**80)** Find the number of ways of assigning 30 project titles to 7 students.

**81)** Find the number of ways of assigning 30 project titles to 7 students so that each project is assigned to exactly one student.

**82)** Find the number of ways of assigning 30 project titles to 7 students so that each student receives at least one project.

**83)** Find the number of ways of assigning 30 project titles to 7 students so that each project is assigned to at least one student.

**84)** Find the number of ways of assigning 30 project titles to 7 students so that each project is assigned to at least one student and each student receives at least one project.

**85)** Find the number of ways of assigning 30 project titles to 7 students so that each project is assigned to at most 5 students.

**86)** Find the number of ways of assigning 30 project titles to 7 students so that each project is assigned to exactly two students and each student receives at least one project.

**87)** How many ways can $n$ married couples be paired up to form $n$ couples so that each couple consists of a man and a woman and so that no couple is one of the original married couples?

**88)** Find the number of $k-$subsets of $\{1,2,\ldots,n\}$ which do not contain any pair of consecutive integers.

**89)** Let $X=\{1,2,3,4,5\}$ and $Y=\{1,2,3,4,5,6,7,8\}$. Find the number of functions $f: X\to Y$ if

a) $f(i)\leq i$ for all $i\in X$,

b) $f$ is one-to-one and $f(i)\leq i$ for all $i\in X$.

**90)** Let $X=\{1,2,3,4,5\}$ and $Y=\{1,2,3,4,5\}$. Find the number of functions $f: X\to Y$ if

a) $f(i)\neq i$ for all $i\in X$,

b) $f$ is one-to-one and $f(i)\neq i$ for all $i\in X$.

**91)** Find the number of words of length 10, consisting of letters a, b, c where a b is followed by two c's.

**92)** A class consists of 10 boys and 10 girls. Find the number of ways of splitting them into 10 equal sized groups

a) if each group consists of either two girls or two boys,

b) if each group consists of a boy and a girl,

c) if there are exactly 6 mixed (consisting of a boy and a girl) groups.

**93)** Consider a chess board of dimension $p\times q$ ($p$ rows, $q$ columns). Find the number of placing $n$ checkers on the board such that each cell (unit square) contains at most one checker.

a) If $n$ is not known.

b) If $n$ is fixed.

c) If $n=p$ and on each row there is exactly one checker.

d) If $n=p$, $p\leq q$ and on each row there is exactly one, on each column there is at most one checker.

e) If $n=p=q$ and on each row and on each column, there is exactly one checker.

f) If $n=p$, $p\geq q$ and on each row there is exactly one, on each column there is at least one checker.

g) If $n=2p$ and on each row there are exactly two checkers.

**94)** In the spring semester, four elective courses are offered: Math 307, Math 369, Math 404 and Math 427. Each course completed the registration period with full capacity of 40 students. It is observed that 4 students are registered to all of the four courses. 6 students registered to 307, 404 and 427; and there are 7 students registered to any other triple of these courses. 17 students registered to 307 and 427; 15 students are registered to 369 and 404, and for any other pair of these courses there are 16 students in common.

a) Find the number of students which registered to at least one of these courses.

b) Find the number of students which registered to exactly one of these courses.

c) Find the number of students which registered to exactly two of these courses.

d) Find the number of students which registered to only Math 307.

### Week 5 (Basic Counting Models 2)

**95)** In a bag there are 5 yellow balls labeled $y_1,\ y_2,\ y_3,\ y_4,\ y_5$ and 4 blue balls labeled $b_1, b_2, b_3, b_4$. First a ball is drawn randomly and put in a box, then a second ball and then a third ball is drawn and put in the same box.

a) For three balls in the box, how many diffferent configurations are there?

b) In how many configurations there are three yellow balls? A yellow and two blue balls? Two yellow balls and a blue ball? Three blue balls?

c) If three balls are drawn at once, instead of being picked one by one, how would you answer the above parts?

**96)** In a bag there are 5 yellow and 4 blue balls. All the balls, except their colors, are identical. Three balls are drawn and put in a box.

a) For three balls in the box, how many diffferent configurations are there?

b) In how many configurations there are three yellow balls? A yellow and two blue balls? Two yellow balls and a blue ball? Three blue balls?

**97)** Let $X = \{1,2,3, \dots ,30\}$ and $Y = \{1,2,3, \dots ,10\}$. Find the number of functions $f: X \to Y$ such that for any $y \in Y,\ |\{x \in X | f(x) = y\}| = 3$.

**98)** Let $A$ be a finite set with $|A| = n$.

a) Find the number of reflexive relations defined on $A$.

b) Find the number of symmetric relations defined on $A$.

c) Find the number of reflexive and symmetric relations defined on $A$.

d) Find the number of anti-symmetric relations defined on $A$.

**99)** Let $X = \{1,2,3, \dots ,15\}$ and $Y = \{1,2,3,4,5\}$. Find the number of functions $f: X \to Y$ such that for any $y \in Y,\ |\{x \in X | f(x) = y\}| = y$.

**100)** Find the number of ways of arranging 5 blue and 10 yellow balls in a row so that

a) there is at least one ball between any two blue balls,

b) there are at least two balls between any two blue balls.

**101)** Find the number of ways of arranging 5 blue, 5 yellow and 5 white balls in a row so that

a) there is at least one ball between any two blue balls,

b) there are at least two balls between any two blue balls.

**102)** Find the number of distributing 5 orange candies and 5 lemon candies to 2 children.

**103)** Find the number of distributing 5 orange candies and 5 lemon candies to 2 children so that each child receives at least one candy of each kind.

**104)** In how many different ways can 11 schoolboys be separated in 4 groups of different sizes?

**105)** In how many different ways can 13 schoolboys be separated in 4 groups of different sizes?

### Week 6 (Basic Counting Models 3)

**106)** Five girls travel with one boy to a math contest. They have four hotel rooms, numbered 1 through 4. Each room can hold up to two people, and the boy has to have a room to himself. How many different ways are there to assign the students to the rooms?

**107)** 30 students has to share 10 hotel rooms numbered 1501 through 1510. Find the number of ways of assigning the students to the rooms if

a) each room can hold at most three people,

b) room 1501 can hold 4 people and each of the remaining rooms can hold three people.

**108)** Determine the number of ways to distribute 10 orange drinks, 1 lemon drink, and 1 ayran to four thirsty students such so that each student gets at least one drink, and the lemon drink and ayran go to different students.

**109)** Find the number of ways of distributing 25 candies to 6 children such exactly half of the boys receive an even number of candies each.

**110)** Using the letters of , KASTAMONU how many 4-letter words can be written?

**111)** Given a cube with vertices $A_1, A_2, \dots, A_8$. How many triangles are there with each vertex being a vertex of the cube? How many of these triangles are equilateral?

**112)** A circular table has exactly 60 chairs around it. There are $N$ people seated around this table in such a way that the next person to be seated must sit next to someone. What is smallest possible value of $N$?

**113)** 4 girls and 8 boys are standing together.

a) In how many ways can they stand around a circle?

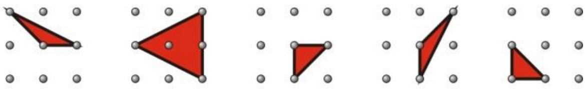
b) In how many ways can a 7-person committee be selected from the group if at least 2 girls must be included?

**114)** Positive integers are partitioned as follows.

(1) (2 3) (4 5 6) (7 8 9 10) (11 12 13 14 15) (16 17 18 19 20 21) (22 23 24 25 26 27 28)

Each part contains one more integer than the preceeding part. For example, there are 6 integers in part 6 and their sum is 111. What is the smallest integer in part 30? What is the sum of integers in part 30?

**115)** 9 points are arranged to form a $3 \times 3$ square lattice. How many different triangles can be drawn with each vertex at one of the lattice points? The figure shows 5 such triangles.



How many non-congruent triangles can be drawn? In the above figure, the first (counting from left) and the fourth triangles are congruent, third and fifth triangles are also congruent. Thus, the figure contains 3 non-congruent triangles.

**116)** There are five partitions of 8 with the largest part 4:

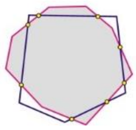
$$4+4, \quad 4+3+1, \quad 4+2+2, \quad 4+2+1+1, \quad 4+1+1+1+1$$

The number of partitions into 4 parts is also five:

$$2+2+2+2, \quad 3+2+2+1, \quad 3+3+1+1, \quad 4+2+1+1, \quad 5+1+1+1$$

Show that for any integer $n$, the number of partitions with $k$ parts is equal to the number of partitions with the largest part $k$.

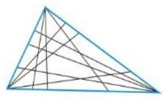
**117)** Given a circle with $n$ points marked on it. Each pair of points are joined by a chord such that no three chords intersect in a common point. Find the number of triangles formed in the circle.



**118)** A convex polygon with $n_1$ sides overlaps with another polygon with $n_2$ sides. What is the largest possible number of the intersection points of their boundaries?

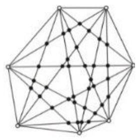
**119)** You must arrange 3 oak trees, 4 maple trees and 5 birch trees in a line such that no two of the birch trees are adjacent to one another. In how many ways can you arrange the trees? (Assume that two oak trees are indistinguishable, as are two maples and two birches.

**120)** On each edge of a triangle $n$ points are marked and each point is joined to the opposite vertex by a stright line. If no three points intersect at a point, how many regions are formed in the triangle?



**121)** How many different 10-digit numbers can be formed from the digits 1, 2, 3 where digit 3 in each number is found exactly three times ? How many of those numbers can be divided by 6 ?

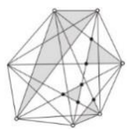
**122)** No three diagonals of a convex $n$ −gon intersect at a point. Find the number of intersection points of these diagonals in the $n$ −gon.



**123)** Find the number of positive integers not exceeding 1000 that are neither the square nor the cube of a positive integer.

**124)** Given a convex $n$ −gon $A_1, A_2, \dots, A_n$. Find the number of ways of chosing $k$ of these vertices which define a convex $k$ −gons with no side common with the $n$ −gon.

**125)** No three diagonals of a convex $n$ −gon intersect at a point. Find the number of all triangles with vertices at the intersection points of the diagonals.



**126)** Find the number of ways of arranging the integers $1,2,3,\dots,20$ in a $2 \times 10$ array such that each row is in ascending order and in each column, integer in the first row is larger than the integer in the second row. An example of such an array is given below.

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| 4 | 5 | 6 | 8 | 10 | 13 | 14 | 16 | 18 | 20 |
| 1 | 2 | 3 | 7 | 9 | 11 | 12 | 15 | 17 | 19 |

**127)** There are 15 members of a club. To open a room at least 8 members must be present to unlock the room. No group of 7 or fewer can open the room. Each lock has a different key, but you can make several copies of the same key to distribute to club members. What is the fewest number of locks and keys that you will need and how would the keys be distributed among members?

**Week 7 (Solving Recurrence Relations)**

**128)** The sequence $\{a_n\}$ is defined by $a_n = a_{n-1} + n \cdot n!$ for $n \geq 1$. If $a_0 = 0$, find $a_{112}$.
[Hint. Simplify $(n+1)! - n!$ .]

**129)** The sequence $\{a_n\}$ is defined by $a_{n+3} = a_{n+2} \cdot a_n - a_{n+1}$ for $n \geq 0$ and $a_0 = 1$, $a_1 = 2$ and $a_2 = 4$. Find $a_{112}$. [Hint. Compute first ten terms.]

**130)** A sequence $\{a_n\}$ is defined by $a_{n+2} = a_{n+1} + na_n$ for $n \geq 0$. If $a_0 = 0$ and $a_7 = 38$, find $a_5$.

**131)** Given the recurrence relation $a_{n+3} = 6a_{n+2} - 11a_{n+1} + 6a_n$. Show that each of the following sequences satisfy this relation.

a) $1, 1, 1, 1, \ldots$
b) $2, 2, 2, 2, \ldots$
c) $1, 2, 4, 8, \ldots, 2^n, \ldots$
d) $1, 3, 9, 27, \ldots, 3^n, \ldots$
e) $7, 14, 34, 92, \ldots, 2^n + 3^{n+1} + 3, \ldots$

**132)** Find (other than the ones mentioned in the previous question) a sequence which satisfies the recurrence relation $a_{n+3} = 6a_{n+2} - 11a_{n+1} + 6a_n$ .

**133)** Show that the sequence $1, 3, 9, 27, \ldots, 3^n, \ldots$ satisfies all of the following recurrence relations.

a) $a_n = 3a_{n-1}$,
b) $a_n = 5a_{n-1} - 6a_{n-2}$,
c) $a_n = 4a_{n-1} - 3a_{n-2}$,
d) $a_n = a_{n-1} + 6a_{n-2}$,
e) $a_n = -a_{n-1} + 9a_{n-2} + 9a_{n-3}$,
f) $a_n = 2a_{n-1} + 5a_{n-2} - 6a_{n-3}$.

**134)** Find (other than the ones mentioned in the previous question) a second order recurrence relation which is satisfied by the sequence $1, 3, 9, 27, \ldots, 3^n, \ldots$ .

**135)** In each of the following, show that the given sequence satisfies the corresponding recurrence relation

| | | |
|---|---|---|
| a) $1, 2, 3, \ldots, n, \ldots$ | $a_n = a_{n-1} + 1$ | $n \geq 1$ |
| b) $1, 2, 3, \ldots, n, \ldots$ | $a_n = 2a_{n-1} - a_{n-2}$ | $n \geq 2$ |
| c) $0, 1, 3, 7, \ldots, 2^n - 1, \ldots$ | $a_n = 3a_{n-1} - 2a_{n-2}$ | $n \geq 2$ |
| d) $1, 3, 5, 7, \ldots, 2n - 1, \ldots$ | $a_n = 3a_{n-1} - 3a_{n-2} + a_{n-3}$ | $n \geq 3$ |
| e) $2, 4, 6, 8, \ldots, 2n, \ldots$ | $a_n = 3a_{n-1} - 3a_{n-2} + a_{n-3}$ | $n \geq 3$ |
| f) $3, 6, 9, 12, \ldots, 3n, \ldots$ | $a_n = 3a_{n-1} - 3a_{n-2} + a_{n-3}$ | $n \geq 3$ |
| g) $0, 1, 4, 9, \ldots, n^2, \ldots$ | $a_n = 3a_{n-1} - 3a_{n-2} + a_{n-3}$ | $n \geq 3$ |

**136)** Find the general solutions of the following recursions

a) $a_{n+3} = 8a_{n+2} - 19a_{n+1} + 12a_n$ ,
b) $a_{n+3} = 11a_{n+2} - 32a_{n+1} + 28a_n$ ,
c) $a_{n+3} = 6a_{n+2} - 12a_{n+1} + 8a_n$ ,
d) $a_{n+3} = 4a_{n+2} - 6a_{n+1} + 4a_n$ .

**137)** Find the general term of the sequence $\{a_n\}$ if $a_0 = 1, a_1 = 2, a_2 = 3$ and for $n \geq 3$

a) $a_n = 7a_{n-2} - 6a_{n-3}$ ,
b) $a_n = 7a_{n-1} - 16a_{n-2} + 12a_{n-3}$ ,
c) $a_n = 9a_{n-1} - 27a_{n-2} + 27a_{n-3}$.

**138)** In each of the following, a sequence $\{a_n\}$ is defined recursively. Find an equivalent homogeneous recursive relation and provide sufficient initial conditions to define the same sequence

a) $a_0 = a_1 = 1$ and $a_n = a_{n-1} + a_{n-2} + 1$ for $n \geq 2$,
b) $a_0 = 1$, $a_1 = 1$ and $a_n = 2a_{n-1} - a_{n-2} + n$ for $n \geq 2$,
c) $a_0 = 2$, $a_1 = 1$ and $a_n = 3a_{n-1} - a_{n-2} + n^2 + 3$ for $n \geq 2$,
d) $a_0 = 2$, $a_1 = 7$ and $a_n = 2a_{n-1} - a_{n-2} + n$ for $n \geq 2$,

e) $a_0 = 3,\ a_1 = 4$ and $a_n = 2a_{n-1} - a_{n-2} + 7^n$ for $n \geq 2$,

f) $a_0 = 1,\ a = 1$ and $a_n = a_{n-1} - 3a_{n-2} + 3^n - n^2$ for $n \geq 2$,

g) $a_0 = 1,\ a = 1$ and $a_n = 2a_{n-1} + 4a_{n-2} + n2^n$ for $n \geq 2$.

**139)** If the sequence $a_0, a_1, a_2, a_3 \ldots$ satisfies the relation $u_{n+2} = \alpha u_{n+1} + \beta u_n$, show that the sequence $a_0, a_2, a_4, a_6, \ldots$ satisfies the relation $v_{n+2} = (2\beta + \alpha^2)v_{n+1} - \beta^2 v_n$.

**140)** If the linear complexities of the sequences $\{a_n\}$ and $\{b_n\}$ are 2, show that the linear complexity of the sequence $\{a_n b_n\}$ is at most 4. [Note. Linear complexity of a sequence is order of the smallest order constant coefficient linear homogeneous recursive relation satisfied by the sequence.]

**141)** It is given that the sequences $\{a_n\}$ and $\{b_n\}$ both satisfy the same constant coefficient linear homogeneous recursive relation of order 5. If $b_n = 3 \cdot 2^{n+1} + i^n + 1$ for any nonnegative integer $n$ and $a_0 = 0,\ a_1 = 1,\ a_2 = 3, a_3 = 5, a_4 = 7$, find $a_5$.

**142)** If the sequences $\{a_n\}$ and $\{b_n\}$satisfy the relations

$$a_n = \alpha a_{n-1} + \beta b_{n-1}$$
$$b_n = \gamma a_{n-1} + \delta b_{n-1}$$

show that they both satisfy the relation

$$u_n = (\alpha + \delta)u_{n-1} + (\gamma\beta - \alpha\delta)u_{n-2}.$$

**143)** Let the sequence $\{a_n\}$ be defined by the relation $a_{n+2} = \alpha a_{n+1} + \beta a_n$. Find a constant coefficient linear homogeneous recursive relation which is satisfied by the sequence of partial sums $\{S_n\}$.
[Note. $S_n = a_0 + a_1 + \cdots + a_n$.]

## Week 8 (Constructing Recurrence Relations)

**144)** When climbing a staircase, with each step she takes Ayşe, moves up either one stair or two stairs. Find the number of different ways Ayşe can climb the staircase which consists of $n$ stairs.

**145)** Find the number of subsets of $\{1,2,3, \ldots, n\}$ which do not contain any pair of successive integers.

**146)** Find the number of ways of tiling a $1 \times n$ rectangular board using $1 \times 2$ and $1 \times 1$ pieces.

**147)** Find the number of ways of tiling a $2 \times n$ rectangular board using $1 \times 2$ and $2 \times 2$ pieces.

**148)** How many different messages can be transmitted in $n$ microseconds using three different signals if one signal requires 1 microsecond for transmittal, the other two signals require 2 microseconds each for transmittal, and a signal in a message is followed immediately by the next signal?

**149)** Find the number of permutations of $\{1,2,3, \ldots, n\}$ in which no integer is moved more than one place from its original position.

**150)** Find the number of strings of length $n$ formed with letters A, B and C if the number of A's is even.

**151)** Find the number of strings of length $n$ formed with letters A, B and C that do not contain AA.

**152)** Find the number of strings of upper case letters of length $n$ which contain an even number of Z's.

**153)** Find the number of strings of upper case letters of length $n$ that do not contain ZZ.

**154)** Find the largest possible number of regions in the plane that can be defined by

a) $n$ straight lines in the plane.

b) $n$ circles in the plane.

c) $n$ triangles in the plane.

d) $n$ rectangles in the plane.

**155)** A circular disk is separated into $n$ sectors by $n$ radii. Find the number of ways of painting each sector in blue, red or white such that no two neighboring sectors are of the same color.

**156)** Each unit square of a $2 \times 20$ chessboard is to be painted in blue, yellow or red. Find the number of possible ways if

a) two adjacent squares cannot receive the same color,

b) two red squares are not allowed to be adjacent.

**Week 9 (Pigeonhole Principle)**

**157)** Let $A = \{1,2,3,4,5,6,7,8\}$.

a) If five integers are selected from A, must at least one pair of integers have a sum 9?

b) If four integers are selected from A, must at least one pair of integers have a sum 9?

**158)** Given a group of $n$ women and their husbands, how many people must be chosen from this group of $2n$ people in order to guarantee that the set of those selected contains a married couple

**159)** In a round-robin tournament (in which every player plays against every other player exactly once), suppose that each player wins at least once. Show that there are at least two players with the same number of wins

**160)** To guarantee that there are ten diplomats from the same continent at a party, how many diplomats must be invited if they are chosen from

a) 12 Australian, 14 African, 15 Asian, 16 European, 18 South American, and 20 North American diplomats.

b) 7 Australian, 14 African, 8 Asian, 16 European, 18 South American, and 20 North American diplomats.

**161)** Show that, in a group of 150 people, at least six must have the same last initial.

**162)** There are 42 students who are to share 12 computers. Each student uses exactly one computer and no computer is used by more than 6 students. Show that at least five computers are used by three or more students.

**163)** A bag contains exactly 6 red, 5 white, and 7 marbles. Find the least number of marbles to be picked which will ensure that either at least 3 red or at least 4 white or at least 5 blue marbles picked.

**164)** In any set of 1001 integers chosen from $\{1,2,3,\dots,2000\}$, show that there must be two members such that one is divisible by the other.

**165)** Suppose that the numbers $1,2,\dots,100$ are randomly placed in 100 locations on a circle. Show that there exist three consecutive locations so that the sum of integers at these locations is at least 152.

**166)** A violinist practiced for a total of 110 hours over a period of 12 days. Show that he practiced at least 19 hours on some pair of consecutive days. Assume that he practiced a whole number of hours on each day.

**167)** Prove that among any five points selected inside an equilateral triangle with side equal to 1, there always exists a pair at the distance not greater than $\frac{1}{2}$.

**168)** Each of the given 9 lines cuts a given square into two quadrilaterals whose areas are in proportion $2:3$. Prove that at least three of these lines pass through the same point.

**169)** Five points are chosen at the nodes of a square lattice (grid). Why is it certain that at least one mid-point of a line joining a pair of chosen points, is also a lattice point?

**170)** Suppose $f(x)$ is a polynomial with integral coefficients. If $f(x) = 2$ for three different integers $a, b$, and $c$, prove that, for no integer, $f(x)$ can be equal to 3.

**171)** Prove that there exist two powers of 3 whose difference is divisible by 1997.

**172)** Prove that there exists a power of three that ends with 001.

**173)** If more than 500 integers from $\{1,2,\dots,1000\}$ are selected, then some two of the selected integers have the property that one divides the other.

**174)** A person takes at least one aspirin a day for 30 days. If he takes 45 aspirin altogether, in some sequence of consecutive days he takes exactly 14 aspirin.

**175)** A theater club gives 7 plays one season. Five women in the club are each cast in 3 of the plays. Then some play has at least 3 women in its cast.

**176)** Prove that at any party of $n$ people, some pair of people are friends with the same number of people at the party.

**177)** Let $S = \{3,4,5,6,7,8,9,10,11,12,13\}$. Suppose six integers are chosen from $S$. Must there be two integers whose sum is 16?

**178)** Let $S = \{3,4,5,6,7,8,9,10,11,12,13\}$. Suppose seven integers are chosen from $S$. Must there be two integers whose sum is 16?

**179)** Let $S = \{7, 8, 9, \dots, 97\}$. At least how many integers should be chosen from S to guarantee the existence of two integers whose sum is 100?

**180)** Let $A$ be a set of six positive (distinct) integers each of which is less then 13. Show that there must be two distinct subsets of $A$ sums of whose elements are same.

**181)** During a campaign, a politician visits 45 towns in 30 days. If he visits a positive whole number of towns each day, show that there must exist some period of consecutive days during which he visits exactly 14 towns.

**Week 10 (Discrete Probability 1)**

**182)** In a playoff series, the probability that Team A wins over Team B is 3/5 and the probability that Team C wins over Team D is 4/7. Find the probability that Team A wins and Team C loses.

**183)** Three checkers are located in the cells of a $8 \times 8$ checkerboard (at most one checker in each cell). Describe the sample space and compute the probability that no row or column contains more than one checker.

**184)** Two fair six-sided dice are rolled and the face values are added. Find the probability of obtaining an odd number greater than 8.

**185)** Three cards are pulled from a deck of 52 cards. Find the probability of obtaining at least one clubs.

**186)** If a fair six-sided die is tossed twice, find the probability that the first toss will be a number less than 4 and the second toss will be a number greater than 4.

**187)** Two cards are drawn without replacement from a deck of 52 cards. Find the probability of the first card being a red face card and the second card being a clubs.

**188)** Two dice are thrown. What is the probability that the product of the facing numbers is a prime?

**189)** Three different DVD's and their corresponding DVD cases are randomly strewn about on a shelf. If a child puts the DVD's in the cases at random, find the probability of correctly matching all DVD's and cases.

**190)** Derya is looking for a top-floor apartment. She hears about two vacant apartments in a building with 7 floors and 8 apartments per floor. What is the probability that there is a vacant apartment on the top floor?

**191)** You choose at random two cards from a standard deck of 52 cards. A card is called a winning card if it is a ten or/and a hearts. What is the probability of getting a winning card? If two cards are drawn at once, what is the probability of having at least one winning card?

**192)** A monkey at a typewriter types each of the 26 letters of the alphabet exactly once, the order being random. What is the probability that the word 'MONKEY' appears somewhere in the string of letters?

**193)** You draw at random five cards from a standard deck of 52 cards. What is the probability that there is an ace among the five cards and a king or queen?

**194)** A box contains 7 apples and 5 oranges. The fruits are taken out of the box, one at a time and in a random order. What is the probability that the bowl will be empty after the last apple is taken from the box?

**195)** A group of five people simultaneously enter an elevator at the ground floor. There are 10 upper floors. They choose their exit floors independently of each other.

a) Describe a sample space and determine the probability that they are all going to different floors.

b) How does the answer change when each person chooses with probability 1/2 the $10^{th}$ floor as the exit floor and the other floors remain equally likely as the exit floor with a probability of 1/18 each?

**196)** Three balls are randomly dropped into three boxes, where any ball is equally likely to fall into each box. Specify an appropriate sample space and determine the probability that exactly one box will be empty.

**197)** Three couples attend a dinner. Each of the six people chooses randomly a seat at a round table. What is the probability that no couple sits together?

**198)** Three friends and seven other people are randomly seated in a row. What is the probability that the three friends will sit next to each other?

**199)** In a group of $n$ boys and $n$ girls, each boy chooses at random a girl and each girl chooses at random a boy. The choices of the boys and girls are independent of each other. If a boy and a girl have chosen each other, they form a couple. What is the probability that no couple will be formed?

**200)** Six married couples participate in a tournament. The group of 12 people is randomly split into eight teams of three people each, where all possible splits are equally likely. What is the probability that none of the teams has a married couple?

**201)** In a high school class, 35% of the students take Spanish as a foreign language, 15% take French as a foreign language, and 40% take at least one of these languages. What is the probability that a randomly chosen student takes French given that the student takes Spanish?

**202)** A bowl contains four red and four blue balls. As part of drawing lots, you choose four times two balls at random from the bowl without replacement. What is the probability that one red and one blue ball are chosen each time?

**203)** In a restaurant, two of five people ordered coffee and the others ordered tea. The waiter forgot who ordered what and put the drinks in a random order. Describe the sample space and find the probability that each one gets the correct drink

**204)** A parking lot has 10 parking spaces arranged in a row. There are 7 cars parked. Assume that each car owner has picked at a random a parking place among the spaces available. Specify an appropriate sample space and determine the probability that the three empty places are adjacent to each other.

**205)** You and two of your friends are in a group of 10 people. The group is randomly split up into two groups of 5 people each. Specify an appropriate sample space and determine the probability that you and your two friends are in the same group.

**206)** Each student in a group of 10 has drawn an integer from $\{1, 2,3,4\}$. What is the probability that at least one of the integers is not picked by anyone?

**207)** There are three English teams among the eight teams that have reached the quarter-finals of the Champions League soccer. What is the probability that the three English teams will avoid each other in the draw if the teams are paired randomly?

**208)** A jar contains three white balls and two black balls. Each time you pick at random one ball from the jar. If it is a white ball, a black ball is inserted instead; otherwise, a white ball is inserted instead. You continue until all balls in the jar are black. What is the probability that you need no more than five picks to achieve this?

**209)** Each of hundred students chooses at random a number from $\{1,2,\ldots,9\}$, independently of the others. Next the chosen numbers are announced one by one. The first student who announces a number that has been announced before wins a bonus. Which person has the largest probability to win the bonus?

**210)** In a game with three players X, Y, Z the winner is chosen by the following procedure. A deck consisting of 1 black and 8 white cards is shuffled and in the order X, Y, Z, X, Y... a card from the deck is dealt to each player until someone gets the black card. First player receiving the black card is the winner.

a) What are the chances of players to win?

b) What are the chances of players if the deck consists of 2 black and 7 white cards?

**211)** Selim and Baran play a series of games until one of the players has won two games more than the other player. Any game is won by Selim with probability $p$. The results of the games are independent of each other. Assuming that Selim is the first player, what is the probability that he wins the match?

**Week 11 (Discrete Probability 2)**

**212)** Five percent of the workers at a given factory are women. Suppose 10 workers are selected at random to be interviewed about quality of work conditions.

a) What is the probability that two of the workers will be women?
b) What is the probability that at least two of the workers will be women?
c) What is the probability that none will be women?

**213)** A die is rolled until a five appears,

a) what is the probability that the first five appears on the first trial?
b) what is the probability that the first five appears on the 2nd trial?

c) what is the probability that the first five appears before the 3rd trial?
d) what is the probability that the first five appears on or before the 10th trial?
e) what is the probability that the first five appears at the $k$-th trial?
f) what is the probability that the first five appears after the 5th trial?

**214)** If a five-letter word is formed at random (meaning that all sequences of five letters are equally likely), what is the probability that no letter occurs more than once?

**215)** A consumer organization estimates that over a 1-year period 17% of cars will need to be repaired once, 7% need repairs twice, and 4% will require three or more repairs. If you own two cars, what is the probability that
a) neither will need repair?
b) both will need repair?

**216)** A slot machine has three wheels that spin independently. Each has 10 equally likely symbols: 4 bars, 3 lemons, 2 cherries, and a bell. If you play, what is the probability
a) you get 3 lemons?
b) you get no fruit symbols?
c) you get 3 bells (the jackpot)?
d) you get no bells?
e) you get at least one bar (automatically lose)?

**217)** A company owns 400 laptops. Each laptop has an 8% probability of not working. You randomly select 20 laptops for your salespeople.
a) What is the likelihood that 5 will be broken?
b) What is the likelihood that they will all work?
c) What is the likelihood that they will all be broken?

**218)** A KONIA cell phone is made from 224 components. Each component has a .001 probability of being defective. What is the probability that a KONIA cell phone will not work perfectly?

**219)** A company manufactures toy robots. About 3 toy robots per 100 does not work. You purchase 35 toy robots. What is the probability that exactly 4 do not work?

**220)** The $\pi R$-50 Company manufactures tires. They claim that only .007 of $\pi R$-50 tires are defective. What is the probability of finding a defective tire in a set of 5 tires? What is the probability of finding at least four non-defective tires in a set of 5 tires?

**221)** A farmer plants 12 saplings. On average, 15% of saplings planted fail to survive their first winter. Find the probability that more than one of his saplings will die in that first winter.

**222)** A fair die is rolled six times. What is the probability that the largest number rolled is $r$ for $r = 1, \dots, 6$?

**223)** A door has 3 locks on it. The door is opened if at least 2 of these locks are unlocked. In a box there are 10 keys, three of which are the keys of the locks on the door. If 3 keys are picked at random, what is the probability of opening the door?

**224)** A box contains 1 black and 4 white balls. A boy picks a ball at random and if that ball is black he stops. Otherwise he returns the ball in the box and repeats the process. Find the probability that the boy stops until the $k$ th try.

**225)** A box contains 1 black and 2 white balls. A boy picks a ball at random and if that ball is black he stops. Otherwise he returns the ball together with a new white ball in the box and repeats the process. Find the probability that the boy stops until the $k$ th try.

**226)** Bill and Mark take turns picking a ball at random from a bag containing four red balls and seven white balls. The balls are drawn out of the bag without replacement and Mark is the first person to start. What is the probability that Bill is the first person to pick a red ball?

**227)** The probability of rain on any given day in April in Rize is 0.8. Assuming that the weather on each day is independent of the weather on other days, find the probability that it rains on at least 20 days in April.

**228)** At a certain intersection, the traffic light is red for 30 seconds, yellow for 5 seconds, and green for 45 seconds. Find the probability that out of the next eight cars that arrive randomly at the light, exactly three will be stopped by a red light.

**229)** A baker put 500 raisins into dough, mixed well, and made 100 cookies. You take a random cookie. What is the probability of finding at least 4 raisins in it?

**230)** A biased coin is flipped 6 times. Show that the probability of having three Heads and three tails cannot be equal to 1/3.

**231)** When a biased coin is flipped 4 times, the probability of two heads and two tails is 0.24. Compute the probability of having an equal number of heads and tails when the same coin is flipped 6 times.

**232)** You flip a coin either 10 times or 20 times and win the game if the number of heads is equal to the number of tails. Which one is better, flipping 10 times or 20 times?

### Week 12 (Discrete Probability 3)

**233)** A couple has two children and the older child is a boy. What is the probability that the couple has two boys?

**234)** A couple has two children, one of which is a boy. What is the probability that the couple has two boys?

**235)** Box $B_1$ contains 2 white and 3 red balls; box $B_2$ contains 6 white and 4 red balls. One of the boxes is chosen randomly and from this box a ball is picked, again randomly. If the ball is white, what is the probability that the chosen box is $B_1$.

**236)** Box $B_1$ contains 2 white and 3 red balls; box $B_2$ contains 6 white and 4 red balls. All the balls are put in a bag. Then, one of the balls is picked randomly. If the ball is white, what is the probability that it is a ball from box $B_1$.

**237)** A box contains 2 green, 4 yellow and 6 white balls; another box contains 15 green, 10 yellow and 5 white balls. One of the boxes is chosen randomly and from that a box a ball is picked. Denote the event of choosing the first and the second boxes by I and II, respectively and let $G, Y, W$ stand for the events of picking a green, a yellow and a white ball, respectively. Determine whether each of the events $G, Y$ and $B$ is independent with the event I or not.

**238)** Three fair dice are rolled. What is the probability that the sum of the three outcomes is 10 given that the three dice show different outcomes?

**239)** A bag contains four balls. One is blue, one is white and two are red. Someone draws together two balls at random from the bag. He looks at the balls and tells you that there is a red ball among the two balls drawn out. What is the probability the other ball drawn out is also red?

**240)** You pick an entry of a $5 \times 10$ matrix randomly. Let $A$ be the event that the entry comes from an odd-numbered row and $B$ be the event that the entry comes from the first five columns. Are the events $A$ and $B$ independent?

**241)** Your friend has chosen at random a card from a standard deck of 52 cards but keeps this card concealed. You have to guess what card it is. Before doing so, you can ask your friend either the question whether the chosen card is red or the question whether the card is the ace of spades. Your friend will answer truthfully. What question would you ask?

**242)** Suppose that the probability of living to be older than 70 is 0.6 and the probability of living to be older than 80 is 0.2. If a person reaches her 70$^{th}$ birthday, what is the probability that she will celebrate her 80$^{th}$?

**243)** A hat contains a number of cards, with 30% white on both sides, 50% black on one side and white on the other, 20% black on both sides. The cards are mixed up, then a single card is drawn at random and placed on the table. If the top side is black, what is the chance that the other side is white?

**244)** Experience shows that 95% of people buying airline tickets actually shows up for their flight. A plane has 100 seats and the airline sold 105 tickets. Find the probability that airline can accommodate all ticketed passenger who shows up. Assume that all passengers act independently.

**245)** In a certain multiple choice question with 5 possible answers for each question, a student knows the answers of 23 questions out of the total 52 questions. Assume that you pick one of the questions randomly and the student answers it correctly. Find the probability that the student actually knows the answer.

**246)** A machine produces parts that are either good (90%), slightly defective (2%), or obviously defective (8%). Produced parts get passed through an automatic inspection machine, which is able to detect any part that is obviously defective and discard it.

a) What is the quality of the parts that make it through the inspection machine and get shipped?

b) A one-year warranty is given for the parts that are shipped to customers. Suppose that within the first year a good part fails with probability 0.01, while a slightly defective part fails with probability 0.1. What is the probability that a customer receives a part that fails within the first year and therefore is entitled to a warranty replacement?

**247)** In a city, one half percent of the population has a particular disease. A test is developed for the disease which gives a positive result for those who have and have not disease at 98% and %3 of the time, respectively.

a) What is the probability that a random person tests positive?

b) Asım just got the bad news that the test came back positive; what is the probability that Asım has the disease?

**248)** Consider the game of Let's Make a Deal in which there are three doors, one of which has a car behind it and two of which are empty. You initially select Door 1, then, before it is opened, the host opens one of the other doors, say Door 3, that is empty (selecting at random if both are empty). You are then given the option to switch your selection from Door 1 to the unopened Door 2. What is the probability that you will win the car if you switch your door selection to Door 2? Also, compute the probability that you will win the car if you do not switch. (What would you do?)

**249)** A factory is manufacturing bolts using three machines, $A, B$ and $C$. Of the total output, machine $A$ is responsible for 25%, machine $B$ for 35% and machine $C$ for the rest. It is known from previous experience with the machines that 5% of the output from machine $A$ is defective, 4% from machine $B$ and 2% from machine $C$. A bolt is chosen at random from the production line and found to be defective. What is the probability that it came from machine $A$.

**250)** A certain town has two taxi companies: Green Birds, whose cabs are green, and Night Owls, whose cabs are black. Green Birds has 15 taxis in its fleet, and Night Owls has 75. Late one night, there is a hit-and-run accident involving a taxi. The town's 90 taxis were all on the streets at the time of the accident. A witness saw the accident and claims that a green taxi was involved. At the request of the police, the witness undergoes a vision test under conditions similar to those on the night in question. Presented repeatedly with a green taxi and a black taxi, in random order, he shows he can successfully identify the color of the taxi 4 times out of 5. Which company is more likely to have been involved in the accident?

**251)** In front of you are two identical-looking coins. One is fair, and the other biased one comes up 75% Heads of the time. You choose one of the coins, and flip it three times, yielding the sequence HHT. What is the probability that the coin you've been flipping is the unfair one?

**252)** A blind frog sits on the second of three stones in a line. At each minute, it jumps either to left (with probability 0.4) or right. To the right of third stone there is a lake in which an alligator is waiting for the frog. To the left of the first stone there is a stork waiting for the frog. Find the probability that the stork eats the frog.

**253)** In a box you have 25 white marbles and 50 black marbles. You also have a lot of black marbles outside the box. Remove two marbles, randomly, from the box. If they are of different colors, put the white one back in the box. If they are the same color, take them out and put a black marble back in the box. Continue this until only one marble remains in the box. Find the probability that the last marble left in the box is white.

**254)** You are a prisoner sentenced to death. The Emperor offers you a chance to live by playing a simple game. He gives you 50 black marbles, 50 white marbles and 2 empty bowls. He then says, "Divide these 100 marbles into these 2 bowls. You can divide them any way you like as long as you use all the marbles. Then I will blindfold you and mix the bowls around. You then can choose one bowl and remove ONE marble. If the marble is WHITE you will live, but if the marble is BLACK... you will die." How do you divide the marbles up so that you have the greatest probability of choosing a WHITE marble?

**255)** A mathematician, his wife, and their son all play chess. One day when the son asked his father some pocket money for Saturday, his father replied, "Let's do it this way. Today is Wednesday. You will play a game of chess tonight, tomorrow, and a third on Friday. If you win two games in a row, you get the money." "Whom do I play first, you or mom?"

"You may have your choice," said the father.

The son knew that his father played a stronger game than his mother. To maximize his chance of winning two games in succession, should he play father-mother-father or mother-father-mother?

**256)** In a kingdom a man was found guilty and has been sentenced to death. According to law, two shots will to be taken at him from close range. Two bullets are placed into a six-chambered revolver in successive order. He will spin the chamber, close it, and take one shot. If he is still alive, he will then either take another shot, or spin the chamber again before shooting. If the first shot is blank, should he immediately pull the trigger for the second shot or spin the chamber before pulling the trigger?

**Week 13 (Graph Theory 1)**

**257)** For any even integer $n \geq 4$ show that there exists a $3-$regular simple graph.

**258)** Let $n$ and $r$ be positive integers such that $n \cdot r$ is even. Show that there exists a $r-$regular graph of order $n$.

**259)** Show that a sequence $\rho_1, \rho_2, \dots, \rho_n$ of non-negative integers is valency sequence of a (not necessarily simple) graph if and only if their sum is even.

**260)** Show that there exists no simple graph whose valency sequence is 2,3,3,4,5,6,7 or 1,3,3,4,5,6,6.

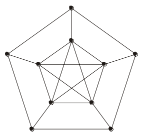
**261)** For any graph show that $\delta \leq \frac{2e}{v} \leq \Delta$.

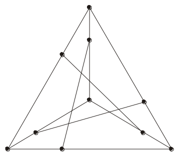
**262)** Prove that any simple graph has at least two vertices with the same valency.

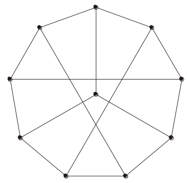
**263)** Determine all simple graphs of order 4 up to isomorphism.

**264)** Determine all simple regular graphs of order 6 up to isomorphism.

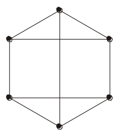
**265)** Let $G_1, G_2$ and $G_3$ be the plane graphs given below. Show that $G_1 \cong G_2 \cong G_3$.

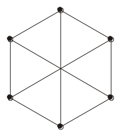

$G_1$


$G_2$


$G_3$

**266)** Let $G$ and $H$ be the plane graphs in the figure below. Show that $G \ncong H$.


$G$


$H$

**Week 14 (Graph Theory 2)**

**267)** Determine all simple graphs with valency sequence 1,2,2,3,3,3 up to isomorphism.

**268)** Show that $n-$cube is bipartite.

**269)** Show that if $G$ is a simple graph with $e > n^2/4$ where $e$ and $n$ are respectively the number of edges and vertices of $G$, then $G$ cannot be bipartite.

**270)** Let $G$ be a simple graph with $|V(G)| = n$ and $|E(G)| = e$. Let $T$ be the number of triangles in $G$. Show that

$$T \geq \frac{4e}{3n}\left(e - \frac{n^2}{4}\right).$$

**271)** Show that if there is a walk from $u$ to $v$ in $G$, then there is also a path from $u$ to $v$ in $G$.

**272)** Show that if $G$ is simple and $\delta \geq k$, then $G$ has a path of length k.

**273)** Show that if an edge $e$ is in a closed trail of $G$, then $e$ is in a cycle of $G$.

**274)** Show that if $G$ is simple and $\delta > \left[\frac{n}{2}\right] - 1$, then G is connected.

**275)** A mouse eats his way through a $3 \times 3 \times 3$ cube of cheese by tunneling through all of the 27 unit subcubes. If he starts at one corner and always moves on to an uneaten subcube, can he finish at the center of the cube?