METU
DEPARTMENT OF MATHEMATICS
Math 112 Discrete Mathematics
Answers and Solutions to Exercises (1 – 275)

1) Remove the third book. Now you can arbitrarily arrange the 6 books in 6! ways.

2) Taking a subset of three elements out a set of 15 elements can be done in $C(15,3)$ ways.

3) First, we choose the 3 workmen. This can be done in $C(20,3)$ ways. Then we choose the 2 employees. This can be done in $C(10,2)$ ways. The committee can be assembled in $C(20,3) \cdot C(10,2)$ ways.

4) First consider 5 cards, with exactly 2 aces. For the two aces, we have $C(4,2)$ possibilities and for the three other cards we have $C(48,3)$ possibilities. Then, 5 cards with two aces can be chosen in $C(4,2) \cdot C(48,3)$ ways. Analogously, 5 cards, with exactly 3 aces can be chosen in $C(4,3) \cdot C(48,2)$ ways. 5 cards, with exactly 4 aces can be chosen in $C(4,4) \cdot C(48,1)$ ways. So, there is a total of $C(4,2) \cdot C(48,3) + C(4,3) \cdot C(48,2) + C(4,4) \cdot C(48,1)$ ways.

5) It is sufficient to choose 3 persons to split the group. This can be done in $C(13,3)$ ways.

6) From each angular point we can count $(n-3)$ diagonals and since there are n points, we have counted $n \cdot (n-3)$ diagonals. But now we have counted twice each diagonal. Hence, there are $\frac{n(n-3)}{2}$ diagonals.
Alternative solution: There are $\binom{n}{2}$ ways to choose a pair of vertices. A pair of vertices describe either a diagonal or a side of the polygon. Since the number of sides is n, the remaining $\binom{n}{2} - n = \frac{n(n-3)}{2}$ pairs correspond to diagonals.

7) There are 4 possibilities for the first figure of the number. There are 5 possibilities for the second figure of the number. There are 5 possibilities for the third figure of the number. So, there is a total of $4 \cdot 5 \cdot 5$ possibilities.

8) The number of subsets with k elements is equal to the number of subsets with $15-k$ elements. So, the number of subsets with less than 7 elements is half of total number of subsets. Total possibilities = 2^{14}.

9) Applying the binomial theorem, we find $C(10,3) \cdot \frac{1}{2^7}$.

10) By removing one stone from each pile, this is the number of ways you can arrange $m-k$ identical stones into k (possibly empty) piles. Now, view the k piles as a numbered set. Write on each stone the number of a chosen pile and order the stones accordingly. The numbered stones constitute a combination with repetition of k elements (the numbers) choose $m-k$ (the stones). This can be done in $C(m-1, m-k)$ ways.

11) This is the same problem as 'In how many ways can you arrange 100 identical stones into 3 piles so that each pile has at least 1 stone in it'. From previous problem the answer is $C(99,97) = 4851$ ways.

12) All terms can be written as $A \cdot a^p b^q c^r$ with $p+q+r = 20$. The number of terms is the number of solutions of the equation $p + q + r = 20$ with p, q, r as positive integer unknowns. Now regard (p, q, r) as three ordered elements. Point 20 times one of these elements, and order these elements in the same order as the given elements. This corresponds with one solution of $p + q + r = 20$ and it is a combination with repetition of 3 elements choose 20. The number of terms is the number of such combinations: $C(22,2) = 231$.

13) a) Choosing 3 people out of 10 people: $\binom{10}{3}$. b) Consider the cases where there are 2 males and 1 female and 1 male and 2 females separately: $\binom{7}{2}\binom{3}{1} + \binom{7}{1}\binom{3}{2} = 21 \times 3 + 7 \times 3 = 84$.

14) Separate into two cases: Bekir choosing a mathematics book or a psychology book. Use product rule in each case. $11\,(10 \times 9 \times 7 + 7 \times 10 \times 6) = 11{,}550$ (or $11 \times (10 \times 7) \times 15 = 11{,}550$).

15) a) Choose the places for the 3 a's and then choose rest of the letters: $\binom{8}{3} \times 25^5$. b) Separate into two cases: Three or four vowels. For the three vowels, first choose the place of vowels, then choose 3 vowels and 21 consonants. Same for 4 vowel case. $\binom{8}{3}5^3 21^5 + \binom{8}{4}5^4 21^4$. c)Permute 8 letters out of 26. $P(26,8)$. d)The letter can contain 0, 2, 4, 6, or 8 e's. Check for each case. $25^8 + \binom{8}{2}25^6 + \binom{8}{4}25^4 + \binom{8}{6}25^2 + 10$

16) a) Choose 2 mathematicians out of 3, one physician out of 4 and 1 other from biologists and chemists. $\binom{3}{2}\binom{4}{1}\binom{11}{1} = 132$ b) Similar with a). $\binom{3}{1}\binom{4}{1}\binom{5}{1}\binom{6}{1} = 360$ c) From total number of committees remove those in which all scientists are from same field. $\binom{18}{4} - \left(\binom{4}{4} + \binom{5}{4} + \binom{6}{4}\right) = 3039$ d) Separate into cases where there are at most 2 biologists. $\binom{13}{4}\binom{5}{0} + \binom{13}{3}\binom{5}{1} + \binom{13}{2}\binom{5}{2} = 2925$

17) a) Permutation of 26 letters: 26! b) First permute the consonants, then place vowels in between consonants and permute the vowels in their own. $C(22,5) \cdot 5! \cdot 21!$ c) Permutation of 26 letters in a circle. 25! d) Same with b but instead of 22 places for vowels to be placed in there are 20 and the permutation of 21 consonants are in a circle. $C(20,5) \cdot 5! \cdot 20!$

18) a) Choose 6 people out of 12 to sit at one table, but consider that tables don't matter so divide by 2. Lastly, permute each table in its own. $\binom{12}{6}\frac{(5!)^2}{2}$ b) From all, remove the cases in which no faculty member sitting at a table. $\binom{12}{6}\frac{(5!)^2}{2} - \binom{8}{2}\frac{(5!)^2}{2}$

19) a) Each homework is given to a student or not so there are 2^{20} ways of distributing homework to a single student. Thus, total number of ways is $(2^{20})^{10}$. b) For each student choose 2 homework out of 20. $\binom{20}{2}^{10}$

20) a) Permute 9 numbers. 9! b) Permutation with repetition where 1, 3, 5 and 7 is used twice. $\frac{9!}{2!^4}$ c) Permutation with some repeated numbers. $\frac{9!}{3!^3}$ d) Permutation with some repeated numbers. $\frac{9!}{2!5!}$

21) a) Since scoops are identical, the problem is same as distributing 16 different objects to 8 boxes. $\binom{16+8-1}{16-1} = \binom{23}{15}$ b) Choose 8 flavors out of 16. $\binom{16}{8}$ c) The flavors are already chosen. Consider permuting these flavors so that first kid gets first two flavors, second kid next two flavors etc. However, the order between 1st and 2nd place does not matter since both goes to first kid. Similarly, each two places don't need to be ordered. $\frac{8!}{2^4}$ d) Distribute one scoop to each child at the start. So, must find the number of ways of distributing 4 identical scoops to 4 children. $\binom{4+4-1}{4-1} = \binom{7}{3}$ =35 e) If order matters, choose a flavor for each scoop. $16^3 = 4{,}096$; otherwise, it is combination with repetition of 16 flavors into 3 scoops. $\binom{18}{3} = 816$

22) a) Choose 3 digits out of 10. $\binom{10}{3}$ b) Use the stars and bars technique. There will be three stars which will represent chosen digits and seven bars that will represent digits that are not chosen. Distribute three stars and seven bars with at least one bar between each star. $\binom{8}{3}$

23) a) Choose one from each. $5 \cdot 3 \cdot 2 = 30$ b) Permute three books from ten books. $10 \cdot 9 \cdot 8 = 720$ c) Choose three books as in a) and permute them. $30 \cdot 3 \cdot 2 = 180$ d) Remove one subject or three subject cases from total. $P(10,3) - 30 \cdot 3! - 3! - C(5,3) \cdot 3! = 474$ e) They must be calculus or linear algebra books. $3! + C(5,3) \cdot 3! = 66$

24) When 5 is in each decimal place, there are 10^4 possible numbers. So, since there are 5 decimal places that 5 can be put in, there are $5 \cdot 10^4 = 50000$ such numbers.

25) When first five letters are chosen independently and differently, rest of the word is chosen as well. So, the question is same as choosing and permuting five different letters out of 29. $29 \times 28 \times 27 \times 26 \times 25$

26) First choose the man in n ways, then choose the woman among the women without the wife of the chosen man in $n-1$ ways. $n(n-1)$

27) a) It is the number of bridges from A to B times number of bridges from B to C by product rule. 12

b) Again, by product rule it is number of trips from A to C times number of trips from C to A. 144

c) Do the same in b) but for the return trip assume bridge numbers are reduced by one between each city. 72

28) a) Permute each type of vehicle in its own group. (4!)(5!)(6!) b) Do the same in a) but this time permute the types of the vehicles as well. (3!)(4!)(5!)(6!)

29) First permute the six boys around the circular fountain in 5! Ways and then on the 6 places between boys permute the six girls. $5!\,6! = 86{,}400$

30) a) Choose the rooms to be used and permute the children on those rooms. $C(11,9) \cdot 9!$

b) This is the permutation of 2 red, 3 white, 4 blue telephones and 2 no telephone at all. $\frac{11!}{(2!)(2!)(3!)(4!)}$

31) Use product rule since choosing men and women are independent of each other. $\binom{4}{2}\binom{9}{3} = 504$

32) First choose the r people out of n people to be seated then permute then in a round table. $\binom{n}{r}(r-1)!$

33) Choose 8 people to be seated around the round table and this way 6 people to be seated on a bench are selected as well. Then permute each group in their own way. $\binom{14}{8}\, 7!\ 6!$

34) Permute the letters other than S's in $\frac{7!}{(4!)(2!)}$ ways. Then choose four places out of eight places to put the S's in. $\binom{8}{4}\ \frac{7!}{(4!)(2!)}$

35) Choose the other letter to appear, then permute them all considering the repeated letters. $\binom{23}{1}\frac{10!}{(4!)(3!)(2!)}$

36) a) Choose 6 students from each major by product rule. $\binom{10}{6}\binom{12}{6}$ b) Find the number of selections for each case where there are 7, 8, 9, 10, 11 or 12 computer science majors. $\binom{12}{7}\binom{10}{5}+\binom{12}{8}\binom{10}{4}+\binom{12}{9}\binom{10}{3}+\binom{12}{10}\binom{10}{2}+\binom{12}{11}\binom{10}{1}+\binom{12}{12}\binom{10}{10}$

37) Permute 7 letters out of 26 letters. $P(26,7)$ b) Choose the first letter, then there are 25 options for the second letter and similarly 25 options for third letter etc. Hence, other than first letter, all other letter selections have 25 options. $26 \cdot 25^6$ c) Choose the letter to be selected 3 times. Then choose 4 other letters out of 25. Permute them considering a letter is used 3 times. $26\binom{25}{4}\frac{7!}{3!}$
d) If the first letter is a vowel there are 5 options for the first letter and 21 options for the last letter. Other selections are from 26 letters. If the first letter is a consonant there are 21 options for the first letter and 26 for each other. $5 \cdot 21 \cdot 26^5 + 21 \cdot 26^6 = 651 \cdot 26^5$ e) If the string starts with consonant there are $26^4 5^3$ strings, if it starts with a vowel there are $26^3 5^4$ strings. $26^4 5^3 + 26^3 5^4$
f) There can be at most 3 consonants. Break into cases, 0, 1, 2, or 3consonants. If there are no consonants all will be vowels, 5^7 such strings. If there are 1 consonant, choose the consonant, place it in six places it can be placed in and choose rest of the vowels. Similar for 2 and 3 consonants.
$5^7 + 6 \cdot 5^6 \cdot 21 + 10 \cdot 5^5 \cdot 21^2 + 4 \cdot 5^4 \cdot 21^3$ g) Separate into cases, there can be 0, 1, 2 or 3 letters used twice. For example, for 2 letters being used twice, choose 2 letters that will be used twice, choose 3 letters that will be used once and permute them considering some are used more than once.
$7!\,[\binom{26}{7}+3\binom{26}{6}+\frac{5}{2}\binom{26}{5}+\frac{1}{2}\binom{26}{4}]$

38) Consider starting at an S letter. From S to E you can move in two directions in a shape similar to a grid. So, for S which is m above E and n near E, there are $C(m+n,n)$ many ways to do it where $m+n$ is 8 in our case. So, from top to right corner there are 2^8 ways to reach from S to E and from top to left corner there are 2^8 ways as well. At the end, the top is counted twice, it should be removed. $2^9 - 1$

39) Check cases where difference of last digit and first digit is 2, multiply number of such cases by the number of permutations of other 8 numbers. $15 \cdot 8!$

40) Count for each case, 0, 2, 4, …, 20 blue pencils. 121

41) Similarly, to 40) count for each case. 72

42) From total number of distributions, remove cases in which a color is distributed more than 2n times. Total number can be found by the number of distributing 3n pencils into three colors which can be done in $\binom{3n+3-1}{3-1}$ ways. If a color is distributed more than 2n times to a single child, then the rest of the n-1 colors for that child can be counted as $\binom{n-1+3-1}{3-1}$ and also consider each 3 colors can be distributed more than 2n times. $\binom{3n+2}{2} - 3\binom{n+1}{2} = 3n^2 + 3n + 1$

43) a) Share the balls as pairs, so distribute 10 pairs to 6 children. $\binom{10+6-1}{6-1} = \binom{15}{5}$
b) Give one ball to each child to begin with and distribute rest as pairs. $\binom{7+6-1}{6-1} = \binom{12}{5}$

44) On each column in the grid any point can be reached. There are 8 options on each column and there are eight columns on which the path will decide a point to go on. Once all these 8 points are chosen in 8 columns, the path is decided. 8^8

45) For any element x in the set, there are equally many subsets that x is in and x is not in. Hence, each element is counted 2^{n-1} times when adding number of elements of all subsets. Since there are n elements total counting is $n2^{n-1}$.

46) a) Each twin will be in a specific group. Separate the first twins, this can be done in 1 way as order of groups doesn't matter. Then choose 5 students from each 5 twins which can be done in 2^5 ways. 32
b) For the first group choose 4 twins out of 6 and then choose a student from each 4 twins which can be done in $\binom{6}{4}2^4$ ways. Next choose two twins left from the first group and two from other left twins that can be chosen in $\binom{4}{2}2^2$. Lastly, since order groups don't matter, divide by 6. 960

47) Yes. $8!\,(4!)^8 = 4{,}438{,}236{,}667{,}576{,}320$ exams can be prepared.

48) Break down to cases: He can choose 4 questions from one part and 2 from another or he can choose 3 questions from both. $\binom{5}{4}\binom{5}{2}2 + \binom{5}{3}^2$ =200

49) There are 5! strings starting with letter A. 5! starting with İ, 5! starting with K, 4! starting with NA, 2! starting with NİA, 2! starting with NİKA and NİKR each. Adding all these together and adding 1 more we get the rank of NİKSAR as 395.

50) Similar to the question 49) rank of MALZEME is 755.

51) First line the boys in a line in 15! ways. Then choose places for the girls in those 16 places so that none of them are near each other. Then let the girls to sit these places in 6! ways. $15!\binom{16}{6}6!$

52) If the first two rows are chosen, last row is decided as well. We know that each row must have a sum of 15. So, the total number of choosing the numbers in first two rows is distributing 17 numbers in three cells each which can be done in $\binom{17}{2}^2$ ways. However, there are cases that fail such as the sum exceeding in a column within first two rows. These situations must be subtracted. To do this, consider the first two cell in the first column exceeds 15. This happens only if the rest of the four cells in the first two rows gets less than 15 which can happen in $\binom{14+5-1}{5-1}$ ways and each three column can exceed 15. $\binom{17}{2}^2 - 3\binom{18}{4}$

53) Assume there is k many 1's. There must be at least one 0 between each 1. Hence k-1 0's are placed so that there is a 0 between each 1. Now, distribute rest of the 0's between 1's as pairs. This has a counting of $\binom{\left\lfloor \frac{n}{2} \right\rfloor - k + k + 1 - 1}{k+1-1} = \binom{\left\lfloor \frac{n}{2} \right\rfloor}{k}$. Since k can be chosen from 0 to $\left\lfloor \frac{n}{2} \right\rfloor$, there are $2^{\left\lfloor \frac{n}{2} \right\rfloor}$ such strings.

54) There will be 6 changes from 0 to 1 or from 1 to 0 in the string. Hence place those 6 changes in between numbers of the sequence so that no two change is consecutive. Then multiply by 2 since the string can start with either 1 or 0. $2 \cdot \binom{12}{6}$

55) a) First permute the numbers on a line in $\frac{10!}{32}$ ways. Then see that every sequence produces the same result as a circle when it is shifted by 1, 2, … or 11 and each sequence is different from after the shifted version, so divide by 11. There are $\frac{10!}{32}$ such circular sequences. b) Again, start as in a) and permute the numbers on a line. However, consider that the sequence ABCDEFABCDEF becomes itself again when it is shifted by 6. So, it is wrong to divide the number of permutations in line by 12. The only sequences that come to themselves before 12 shifts is the sequences in which first six terms are in same order with the last six terms, there are 6! such sequences, so we must add the lost linear cases. $\frac{11!}{2^6} + 6!\frac{1}{2}$ c) Follow the same procedure in b). $\frac{5!}{3!3!} + \frac{2}{3} = 4$.

56) a) Distribute 5 fruits (identical objects) to three types (non-identical objects). $\binom{5+3-1}{3-1} = \binom{7}{2} = 21$
b) From total counting, remove the cases in which 6 or 7 apples are picked. $\binom{9}{2} - 2 - 1 = 33$

57) The only thing to decide is which values are in the range of the function because once those values are known the order is determined, multiply by 2 since the function can be increasing or decreasing. $2\binom{m}{n}$

58) This time choose values in the range with repetition, so match n values in the domain into different values in the range. $\binom{m+n-1}{n}$

59) For the function to be idempotent, each element must be mapped to a fixed element. For each number of fixed elements from 1 to n, choose fixed elements and map the rest of the elements to those fixed ones. $\sum_{k=1}^{n}\binom{n}{k}k^{n-k}$

60) By principle of inclusion-exclusion, from all integers remove those that are divisible by 3, remove integers divisible by 5 and then add integers that are divisible by both 3 and 5. 767

61) a) Choose the digit that appears at least 3 times and distribute other two using combination with repetition. $5\binom{5+2-1}{5-1} = 75$ b) Remove the solution in a) from all combinations. 51

62) Assume each letter is used once. Then distribute n-5 choices into 5 letters. $\binom{n-5+5-1}{5-1} = \binom{n-1}{4}$

63) Use principle of inclusion-exclusion on failed courses. $100 - 25 \cdot 5 + 12 \cdot 10 - 9 \cdot 10 + 7 \cdot 5 - 5 = 35$

64) Use principle of inclusion-exclusion on M's, A's and T's being adjacent. $\frac{11!}{(2!)^3} - 3\frac{10!}{(2!)^2} + 3\frac{9!}{2!} - 8!$

65) Use principle of inclusion-exclusion on couples sitting together. From all permutations remove those with one couple sitting together, add those with two couples sitting together, remove those with three couples sitting together and lastly, add those in which all couples sit together. 13824. For the one with the circular arrangement, repeat the same process with circular permutation calculations. 1284.

66) Apply principle of inclusion-exclusion to divisibility by 7, 10, and 15. 721

67) Apply principle of inclusion-exclusion to each x_i being greater than 4. $\binom{20}{5} - 6\binom{15}{5} + 15\binom{10}{5} - 20\binom{5}{5}$

68) Apply principle of inclusion-exclusion to SAYI, DOĞRU and ŞEKİL being read directly. $29! - 28 \cdot 25! + 45 \cdot 21! + 18!$

69) For the path to cross from the missing square it must pass through the specific point. From all paths, remove those that must pass through that point. $\binom{17}{10} - \binom{8}{4}\binom{9}{6}$

70) In the first one, from all paths remove those that pass through the first missing part and the second passing part separately since the path cannot pass through both missing parts. $\binom{17}{10} - \binom{8}{4}\binom{8}{6} - \binom{10}{7}\binom{6}{2}$ / Apply the same procedure as in the first grid but this time, the path can pass through both missing parts, so by principle of inclusion-exclusion after removing paths that pass through those missing parts separately add the paths that pass through both missing parts. $\binom{17}{10} - \binom{5}{3}\binom{11}{4} - \binom{12}{7}\binom{4}{2} + \binom{5}{3}\binom{6}{4}\binom{4}{2}$

71) Apply principle of inclusion-exclusion two first two children getting more than four candies. $\binom{24}{4} - 2\binom{19}{4} - \binom{14}{4}$ / Apply principle of inclusion-exclusion to each child getting more than six candies. $\binom{24}{4} - 5\binom{17}{4} + 10\binom{10}{4}$

72) a) Apply principle of inclusion-exclusion to points being fixed. From all permutations remove those with one fixed points, add those with two fixed points, remove those with three fixed points, … $D_n = n!\sum_{k=1}^{n}\frac{(-1)^k}{k!}$ b) Choose the fixed point in n ways and derange the rest of the integers. nD_{n-1} c) Choose the k fixed points and derange the remaining n-k integers. $\binom{n}{k}D_{n-k}$

73) 2, 3, and 5 are the prime divisors of 360. Apply principle of inclusion-exclusion to integers from 1 to 360 being divisible by 2, 3, and 5. $180 + 120 + 72 - 60 - 36 - 24 + 12 = 264$

74) From total number of functions remove those with one unmapped element in Y, add those with two unmapped elements in Y, … Result follows.

75) Choose k elements in Y, then apply the formula found in 75). $\binom{n}{k}\sum_{i=0}^{k}(-1)^i\binom{k}{i}(k-i)^m$.

76) This is the same problem as distributing 7 identical objects to 4 different boxes. $\binom{10}{3}$

77) Use principle of inclusion-exclusion on each condition specified on the children. 3,736

78) Distribute n candies to n^th^ child at start. Then find the number of distributing remaining candies. $\binom{38}{6}$

79) Separate to cases, first two children getting 0, 1, 2, …, 15 candies each. In each case find the number of distributing rest of the candies to rest of the children. $\binom{34}{4} + \binom{32}{4} + \binom{30}{4} + \cdots + \binom{4}{4}$

80) For each student, each project can be given or not give. So, for each student there is 2^{30} ways of distributing the projects. Consider this for all students. $(2^{30})^7 = 2^{210}$

81) Each project is assigned to exactly one student and each project can be assigned in 7 ways. 7^30

82) For each student there are 2^{30} ways of assigning projects and only one of them is the case in which no project is assigned. Remove it and consider all students. $(2^{30} - 1)^7$

83) Same with question 83) with project and student role reversed. $(2^7 - 1)^{30}$

84) This is the same question as finding the number of onto functions from the set of projects onto the set of students. Question 75) provides the formula. $\sum_{i=0}^{7}(-1)^i\binom{7}{i}(2^{7-i} - 1)^{30}$

85) For each project there are 8 cases in which a project is assigned to 6 or 7 students. Remove them and consider 30 projects are going to be assigned. $(2^7 - 8)^{30} = 120^{30}$

86) This is the same question as finding number of onto relations from the set of projects onto the set of students in which every element in the domain must choose two elements in the codomain to be paired with. Adjust the formula in Question 75). $\sum_{i=0}^{5}(-1)^i\binom{7}{i}\binom{7-i}{2}^{30}$

87) Consider the men lined up and women will be deranged on this line up. D_n

88) k elements will be chosen from numbers 1 to n, assume the chosen numbers will be represented with stars and rest with bars. The number of such subsets is equal to the number of permuting k stars and n-k bars for which between every star there must be a bar. Thus, the number of distributing n-k-(k-1) bars into k+1 places is the solution. $\binom{n-k-k+1+k+1-1}{k+1-1} = \binom{n-k+1}{k}$

89) a) 1 can be mapped to one element in Y, 2 can be mapped to two elements in Y, … , 5 can be mapped to five elements in Y. 5! b) 1 must be mapped to 1. Since f is 1-1, 2 must be mapped to 2, … , 5 must be mapped to 5. 1

90) a) There are 4 choices for every element in X to be mapped to. 4^5 b) This question is same as the derangement of five elements. $D_5 = 44$

91) There can be no b, 1 b, 2b's or 3b's. If there is only 1 b, consider "bcc" as a block and choose a place for it and choose the rest of the letters. If there are 2 b's choose places for them and choose rest of the 4 letters. If there are 3 b's choose places for them and choose the remaining letter. $2^{10} + \binom{8}{1}2^7 + \binom{6}{2}2^4 + \binom{4}{3}2 = 2296$

92) a) For boys, permute the boys so that the groups will be separated as first two in the line, second two in the line,… Divide by 2, five times since the ordering in each group does not matter and there are five groups with two boys in it. Then divide that result by 5! since ordering of the group is not important as well. Since the procedure is same for girls, take the square of it. $\left(\frac{10!}{5!2!^5}\right)^2$ b) Choose a girl for each boy which can be done with permutation of 10 girls. 10! c) Choose the boys and girls to be mixed, arrange them. Then distribute rest as two girls, two boys. $\binom{10}{6}^2 6!\frac{\binom{4}{2}\binom{4}{2}}{2!2!} = 9\binom{10}{6}^2 6!$

93) a) There are pq squares and in each square, there is a checker or not. 2^{pq} b) Choose n squares out of pq squares and again in each square there is a checker or not. $2\binom{pq}{n}$ c) For each row there are q options and there are p rows. q^p d) There are q options for first checker on first row, q-1 options for second checker,… $P(q,p)$ e) This is the same question as n d) with q=p. $q! = n!$

f) Use principle of inclusion-exclusion on columns having no checkers. From total, remove cases in which one column has no checkers, add those with two columns with no checkers,…
$\sum_{i=0}^{q-1}(-1)^i \binom{q}{i}(q-i)^p$ g) On each row, choose two out of q. $\binom{q}{2}^p$

94) a) From students registered in one course, remove students registered in two courses, add students registered in three courses and remove students registered in all courses. $160 - 96 + 27 - 4 = 87$
b) From students who are registered in at least one course, subtract the number of students who are registered in at least 2 courses using principle of inclusion-exclusion. $87 - (96 - 2 \cdot 27 + 3 \cdot 4) = 33$
c) From students who are registered in at least one course, subtract the number of students who are registered to exactly one course and at least three courses. $87 - 33 - (27 - 3 \cdot 4) = 39$
d) From students registered to Math 307 subtract students who are registered to Math 307 and another class, add students who are registered to Math 307 and two other classes, subtract students who are registered to all courses. $40 - 49 + 20 - 4 = 7$

95) a) Choose 3 balls out of 9. $\binom{9}{3} = 84$ b) For three yellow balls choose 3 out of 5, for a yellow and two blue balls choose 1 out of 5 and 2 out of 4, for two yellow balls and a blue ball choose 2 out of 5 and 1 out of 4, for three blue balls choose 3 out of 4. 10, 30, 40, 4 c) It would not change anything since the order in which the balls are picked does not matter.

96) a) All configurations are BBB, BBY, BYY, YYY. 4 b) Each can be done in one way as shown in part a) as order does not matter. 1, 1, 1, 1

97) Permute the elements in X and assume that first 3 in the permutation goes to y=1, second three goes to y=2, … last three goes to y=10. Since orders of elements that are mapped into the same element in Y does not matter divide by 3! 10 times. $\frac{30!}{3!^{10}}$

98) a) Everything is related to itself already. There are $n(n-1)$ pairings more to decide whether they are in the relation or not. $2^{n(n-1)}$ b) For the first element there are 2^n choices. For the second element there are 2^{n-1} choices, … , for the last element there is 2^0 choices. $2^{\frac{n(n+1)}{2}}$
c) For the first element there are 2^{n-1} choices. For the second element there are 2^{n-2} choices, … , for the last element there is 2^0 choices $2^{\frac{n(n-1)}{2}}$ d) For each pair of elements a and b, there are three strict possible ways they can show up in the relation. (a, b) is in the relation or (b, a) is in the relation or neither of them are in the relation. Consider this for each pair of elements. $3^{\frac{n(n-1)}{2}}$

99) Similar to 98). Permute the elements of X so that first element is mapped to 1 element in Y, next two elements are mapped to 2 elements, next three elements are mapped to 3 elements, next four elements are mapped to 4 elements and last five elements are mapped to 5 elements. By dividing, remove the number of unnecessary orderings. $\frac{15!}{5!4!3!2!}$

100) a) Initially, put one yellow ball between blue balls so that no blue balls are near each other. Find the number of distributing remaining 6 yellow balls in 6 places. $\binom{6+6-1}{6-1} = \binom{11}{5}$ b) Similarly, put two yellow balls between blue balls so that there are two yellow balls between each blue ball. Then count the number of ways of distributing remaining 2 yellow balls in 6 places. $\binom{2+6-1}{6-1} = \binom{7}{5}$

101) a) This is the same problem as in 101), the only difference is that instead of 10 yellow balls, there are 5 yellow and 5 white balls. So, for each ordering in the previous question, choose 5 balls that decides which of the 10 balls are going to be white and yellow. $\binom{11}{5}\binom{10}{5}$ b) $\binom{7}{5}\binom{10}{5}$

102) Orange candies can be distributed in 6 ways to 2 children and same for lemon candies. Since these are independent of each other use product rule. 36

103) Initially, distribute 1 orange candy and 1 lemon candy to each child. Then find the number of ways of distributing 3 orange candies and 3 lemon candies to 2 children. Consider that the orange candies can be distributed in 4 ways and same for lemon candies. 16

104) The only way to create 4 different sized groups is to create groups of sizes 1, 2, 3 and 5. Then choose five students for the first group, three students for the second group with remaining students and two students for the third group. $\frac{11!}{5!3!2!}$

105) Proceed as in question 105), the only difference is that there can be more cases. There are three such group distributions, 1, 2, 3 and 7 sized groups or 1, 2, 4 and 6 sized groups or 1, 3, 4 and 5 sized groups.
$\frac{13!}{2!3!7!} + \frac{13!}{2!4!6!} + \frac{13!}{3!4!5!}$

106) First, choose the room for the boy, then choose the room in which a single girl will stay, then distribute remaining 4 girls to the remaining rooms. $4 \cdot 3 \cdot \frac{5!}{2!2!} = 720$

107) a) Permute 30 students on a line and put first three to the first room, second three to the second room, … but consider that the ordering of the first three does not affect the outcome as all three of them will go into first room without any order. This holds for any three students that go into the same room. $\frac{30!}{3^{10}}$

b) This time, break into two cases: The four-capacity room will be shared by four students and one other room will be used by two students or each room will be used by 3 students. Then proceed as in the previous example. $\frac{30!}{3^{10}} + 9 \cdot \frac{30!}{2!3!4!}$

108) First find the number of distributing lemon drink and ayran to four students which can be done in $4 \cdot 3$ ways. Then give one orange drink to students who did not get lemon drink or ayran so that each student has at least one drink. Lastly, find the number of distributing remaining 8 orange drinks to 4 students and multiply it with the number of ways ayran and lemon drink can be given. $12\binom{11}{3}$

109) First choose the 3 children that receives odd number of candies in $\binom{6}{3}$ ways. Then at start, give one candy to each of these 3 children. Then, find the number of distributing remaining 22 candies so that 2 candies are given at a time which converts to distributing 11 candy pairs to 6 children. $\binom{6}{3}\binom{16}{5}$

110) There are two cases: The 4-letter word contains two A letters or nor. If it contains two A letters, choose the other two letters and permute them. If there are no 2 A's choose 4 letters to use in the word and permute them. $\binom{7}{2}\frac{4!}{2!} + \binom{8}{4}4!$

111) Every triple of vertices constructs a triangle since no triple of vertices are linear. Hence there are $\binom{8}{3}$ ways of choosing those vertices. / The only way to get an equilateral is to choose two vertices which are on the opposite sides of a surface of the cube and choose the last vertex so that none of these vertices share a line. 8

112) Assume there are 2 empty seats between each people because if there were three or more spaces between two people then it would be possible to put more people into the circle without making him sit next to anyone else. Considering there are 2 empty seats between each people there must be at least 20 people to sit.

113) Consider that everyone is counted as boys and girls are different among themselves. a) There are 12 people who will stand around a circle. Permute them. 11! b) From all cases 7 people can be chosen from 12, remove those that includes at most 1 girl. If there are no girls the choosing is among 8 boys, if there is one girl, choose the girl and 6 boys from 8 boys. $\binom{12}{7} - \binom{8}{7} - 4\binom{8}{6}$

114) In 29 parts, there are $\frac{29+1}{2} \cdot 29 = 435$ numbers so the next number is 436 / For the sum, take the summation of the numbers from 435 to 464. 13515

115) Each triple of vertices creates a triangle except those which are linear. There are 8 linear triples of vertices. $\binom{9}{3} - 8 = 76$ / Count all non-congruent triangles that can be drawn in a 3x3 grid. 8

116) Consider a partition of the integer n into parts so that the largest part is k. Imagine this constructs the shape of blocks in which there are k blocks at the bottom, and at each level there are number of blocks equal to the parts of the partition. If this shape is rotated 90 degrees, again there is a pile of blocks, there are k lines of blocks which represents a partition of the integer n into k parts. Hence, there is a 1-1 correspondence between partitions with largest part k and partitions into k parts.

117) Two points are needed to choose a chord. Since no three chords intersect at a common point, to create a triangle, 6 points are needed. Also, for any 6 points chosen, there can be only one triangle to be constructed. Hence, there are $\binom{n}{6}$ such triangles.

118) A line can cut through a polygon at most two times. Hence, any side of a polygon can intersect the other polygon at most 2 times. Thus, the smaller polygon cuts the largest polygon at most 2 times its number of sides. $2 \cdot \min\{n_1, n_2\}$

119) First find the number of permutations of oak and maple trees and them multiply it with the number of distributing 5 birch trees in 8 places between the permuted oak and maple trees so that at most one birch tree is put between other types of trees (choose 5 places to put birch threes out of 8). $\binom{7}{4}\binom{8}{5}$

120) Assume two sets of lines are drawn on two sides of the triangle. The first set creates n+1 spaces and each of these spaces is cut by each of the lines drawn on the second side. Hence, from the sets of lines of two sides $(n+1)(n+1) = n^2 + 2n + 1$ regions are created. Now consider the lines that are drawn from the third side of the triangle. Each line drawn from that side divides through 2n+1 regions so that by the third set of lines $n(2n+1) = 2n^2 + n$ regions are created. In total there are $3n^2 + 3n + 1$ regions.

121) Choose places for 3's in $\binom{10}{3}$ many ways and then choose rest of the numbers in 2^7 ways. $\binom{10}{3}2^7$ / Put a 2 at the end since the number must be even. For the remaining 6 numbers their sum should be 7 or 10 so that the number is divisible by 3 as well. Break into cases in which the sum of remaining 6 numbers to be chosen is 7 or 10. $\binom{9}{3}(6 + \binom{6}{2})$

122) For each quadruple of vertices chosen, there are two diagonals which join opposite vertices and these two diagonals intersect at a point and any intersection point is the intersection of two diagonals only. Hence, counting the number of intersections of diagonals is same as counting quadruples of vertices. $\binom{n}{4}$

123) Use principle of inclusion-exclusion on numbers being a square or a cube of a number. $1000 - 31 - 10 + 4 = 963$

124) Be warned that no consecutive vertices are chosen from a circular arrangement not a linear arrangement. Consider two cases: A_1 is chosen or not. If A_1 is not chosen, the question is same as choosing k integers from n-1 integers so that no two are consecutive. If A_1 is chosen, then A_n and A_2 cannot be chosen. Then the question becomes choosing k-1 integers from n-3 integers so that no two are consecutive. Counting of non-consecutive integers is done in question 89). $\binom{n-k}{k} + \binom{n-k-2}{k-1}$

125) All three vertices of the triangle could be from the vertices of the polygon in which case there will be $\binom{n}{3}$ such triples. Two vertices could be from the polygon and one vertex could be intersection of two diagonals in the interior of the polygon for which four vertices of the polygon are required and from these four vertices there are four triangles which adds $4\binom{n}{4}$ many triangles. If two of the vertices are intersections of diagonals and one vertex is the vertex of the polygon, five vertices must be chosen and from each such vertices, five triangles can be drawn, there are $5\binom{n}{5}$ such triangles. Lastly, there are cases where all vertices of the triangle are on the interior of the polygon which is counted in question 118). $\binom{n}{3} + 4\binom{n}{4} + 5\binom{n}{5} + \binom{n}{6}$

126) There will be 10 numbers on the top row and 10 numbers on the bottom row. The problem is to place numbers into either top row or bottom row in order one by one so that the number of filled boxes in the bottom row will always be greater than in the top row. This is the definition of the sequence of Catalan numbers and since there are 10 columns, there are $C_{10} = \frac{1}{11}\binom{20}{10}$ many ways to fill the table which is the 10^{th} Catalan number.

127) $\binom{15}{7}$ locks, $8\binom{15}{7}$ keys, $\binom{14}{7}$ keys to each member

128) Using the hint, we get $(n+1)! - n! = n \cdot n!$. Then

$$\begin{aligned} a_1 &= a_0 + 2! - 1! \\ a_2 &= a_1 + 3! - 2! \\ a_3 &= a_2 + 4! - 3! \\ &\vdots \\ a_{n-1} &= a_{n-2} + n! - (n-1)! \\ a_n &= a_{n-1} + (n+1)! - n! \end{aligned}$$

If we add all the expressions side by side, we get $a_n = (n+1)! - a_0 + 1! = (n+1)! + 1$. Then $a_{112} = 113! + 1$.

129) First ten terms are $1, 2, 4, \mathbf{2}, \mathbf{0}, \mathbf{-2}, -4, \mathbf{2}, \mathbf{0}, \mathbf{-2}$. The pattern $2, 0, -2$ is repeated. Since the order of recursion is 3, we conclude that after the fourth term (a_3), the sequence has period 4. Then, $a_{112} = a_4 = 0$.

130)

$a_2 = a_1$
$a_3 = a_2 + a_1 = 2a_1$
$a_4 = a_3 + 2a_2 = 4a_1$
$a_5 = a_4 + 3a_3 = 10a_1$
$a_6 = a_5 + 4a_4 = 26a_1$
$a_7 = a_6 + 5a_5 = 76a_1$

Then $a_1 = \frac{1}{2}$ and $a_5 = 5$.

131) Put the general terms in the recurrence relation.

a) $1 = 6 - 11 + 6$
b) $6 \cdot 2 - 11 \cdot 2 + 6 \cdot 2 = 2$
c) $6 \cdot 2^{n+2} - 11 \cdot 2^{n+1} + 6 \cdot 2^n = (24 - 22 + 6)2^n = 2^{n+3}$
d) $6 \cdot 3^{n+2} - 11 \cdot 3^{n+1} + 6 \cdot 3^n = (54 - 33 + 6)3^n = 3^{n+3}$
e) $6 \cdot (2^{n+3} + 3^{n+4} + 3) - 11 \cdot (2^{n+2} + 3^{n+3} + 3) + 6 \cdot (2^{n+1} + 3^{n+2} + 3) = 2^n + 3^{n+1} + 3$

132) For any choice of the real numbers A, B and C, the sequence with the general term $A2^n + B3^n + C$ satisfies the given recursion.

133) Put the general term in the recurrence relations.

a) $3^n = 3 \cdot 3^{n-1}$.

b) $5 \cdot 3^{n-1} - 6 \cdot 3^{n-2} = 5 \cdot 3^{n-1} - 2 \cdot 3^{n-1} = 3 \cdot 3^{n-1} = 3^n$.
c) $4 \cdot 3^{n-1} - 3 \cdot 3^{n-2} = 4 \cdot 3^{n-1} - 3^{n-1} = 3 \cdot 3^{n-1} = 3^n$.
d) $3^{n-1} + 6 \cdot 3^{n-2} = 3^{n-1} + 2 \cdot 3^{n-1} = 3 \cdot 3^{n-1} = 3^n$.
e) $-3^{n-1} + 9 \cdot 3^{n-2} + 9 \cdot 3^{n-3} = -9 \cdot 3^{n-3} + 3^n + 9 \cdot 3^{n-3} = 3^n$.
f) $3^{n-3} + 15 \cdot 3^{n-3} - 6 \cdot 3^{n-3} = 27 \cdot 3^{n-3} = 3^n$.

134) For any $\lambda \in \mathbb{R}$, the sequence $\{3^n\}$ satisfies the relation $a_{n+2} = (3-\lambda)a_{n+1} + 3\lambda a_n$ for $n \geq 0$.

135) Put the general terms in the recurrence relations.

a) $(n-1) + 1 = n$.
b) $2(n-1) - (n-2) = n$.
c) $3(2^{n-1} - 1) - 2(2^{n-2} - 1) = 6 \cdot 2^{n-2} - 2 \cdot 2^{n-2} - 1 = 2^n - 1$.
d) $3[2(n-1) - 1] - 3[2(n-2) - 1] + [2(n-3) - 1] = 3[2n-3] - 3[2n-5] + [2n-7] = 2n - 1$.
e) $3(2n-2) - 3(2n-4) + (2n-6) = 2n$.
f) $3(3n-3) - 3(3n-6) + (3n-9) = 3n$.
g) $3(n-1)^2 - 3(n-2)^2 + (n-3)^2 = 3(n^2 - 2n + 1) - 3(n^2 - 4n + 4) + (n^2 - 6n + 9) = n^2$.

136) Find the roots of the characteristic polynomials and generate the general terms.

a) $a_n = A + B3^n + C4^n$.
b) $a_n = A2^n + Bn2^n + C7^n$.
c) $a_n = A2^n + Bn2^n + Cn^2 2^n$.
d) $a_n = A2^n + B(1-i)^n + C(1+i)^n$.

137) Find the roots of the characteristic polynomials, generate the general terms and put the initial conditions into the general terms to find the coefficients.

a) $a_n = \frac{1}{20}(5 + 2^{n+4} - (-3)^n)$.
b) $a_n = 2^{n+1} + n2^{n-1} - 3^n$.
c) $a_n = 3^n - n3^{n-1}$.

138)

a) Write the recursion equation for a_{n-1} as well and from the initial recursion equation, subtract this equation. $a_0 = a_1 = 1,\ a_2 = 3$ and $a_n = 2a_{n-1} - a_{n-3}$ for $n \geq 3$.
b) Write the recursion equation for a_{n-1}, subtract it from the initial recursion equation. Then, we have $a_n = 3a_{n-1} - 3a_{n-2} + a_{n-3} + 1$. Then apply the same procedure to this recursion. Find suitable initial conditions. $a_0 = a_1 = a_2 = a_3 = 1$ and $a_n = 4a_{n-1} - 6a_{n-2} + 4a_{n-3} - a_{n-4}$ for $n \geq 4$.
c) Follow the same procedure as in b) but this time you will subtract the recursion from itself three times, first to get rid of n^2, then n and lastly, the constant term. $a_0 = 2,\ a_1 = 1,\ a_2 = 8,\ a_3 = 34,\ a_4 = 113$ and $a_n = 6a_{n-1} - 13a_{n-2} + 13a_{n-3} - 6a_{n-4} + a_{n-5}$ for $n \geq 5$.
d) Follow the same procedure as in part b). Only initial conditions are different. $a_0 = 2,\ a_1 = 7,\ a_2 = 12,\ a_3 = 17$ and $a_n = 4a_{n-1} - 6a_{n-2} + 4a_{n-3} - a_{n-4}$ for $n \geq 4$.
e) Write the recursion equation for a_{n-1}, multiply that equation by 7 and subtract it from the initial form to get rid of the non-homogeneous term. Find suitable initial conditions. $a_0 = 3,\ a_1 = 4,\ a_2 = 54$, and $a_n = 9a_{n-1} - 15a_{n-2} + 7a_{n-3}$ for $n \geq 3$.
f) First do the procedure at part e) to get rid of the term 3^n, then follow the procedure in c) to get rid of the other non-homogeneous term. $a_0 = 1,\ a = 1$ and $a_n = a_{n-1} - 3a_{n-2} + 3^n - n^2$ for $n \geq 2$,
g) Write the recursion equation for a_{n-1} as well, multiply it by 2, and from the initial recursion equation, subtract this equation and the only non-homogeneous term becomes 2^n. Remove it using the procedure in part e). $a_0 = 1,\ a = 1$ and $a_n = 2a_{n-1} + 4a_{n-2} + n2^n$ for $n \geq 2$.

139) We have to show that $a_{2(n+2)} = (2\beta + \alpha^2)a_{2(n+1)} - \beta^2 a_{2n}$.

$$\begin{aligned} a_{2n+4} &= \alpha a_{2n+3} + \beta a_{2n+2} \\ &= \alpha[\alpha a_{2n+2} + \beta a_{2n+1}] + \beta a_{2n+2} \\ &= (\alpha^2 + \beta)a_{2n+2} + \alpha\beta a_{2n+1} \\ &= (\alpha^2 + \beta)a_{2n+2} + \beta(a_{2n+2} - \beta a_{2n}) \\ &= (\alpha^2 + 2\beta)a_{2n+2} - \beta^2 a_{2n}. \end{aligned}$$

140) Let $a_{n+2} = \alpha a_{n+1} + \beta a_n$ and $b_{n+2} = \gamma b_{n+1} + \delta b_n$ for some $\alpha, \beta, \gamma, \delta \epsilon \mathbb{R}$ and $n \geq 0$. We first assume that characteristic equations $r^2 - \alpha r - \beta = 0$ and $s^2 - \gamma s - \delta = 0$ both have two distinct roots say r_1, r_2 and $s_1,\ s_2$. Then general terms of the sequences are $a_n = Ar_1^n + Br_2^n$ and $b_n = Cs_1^n + Ds_2^n$ for some $A, B, C, D \in \mathbb{R}$. It follows that the general term of $\{a_n b_n\}$ is $AC(r_1 s_1)^n + AD(r_1 s_2)^n + BC(r_2 s_1)^n + BD(r_2 s_2)^n$. It is obvious that the sequence $\{a_n b_n\}$ satisfies a constant coefficient linear homogeneous recursive relation of order whose characteristic equation is $(t-r_1s_1)(t-r_1s_2)(t-r_2s_1)(t-r_2s_2)=0$. We conclude that the linear complexity of $\{a_nb_n\}$ is at most 4.

The case of the multiple roots can be shown similarly.

141) Characteristic equation of the common recursion is $(r-2)^2(r^2+1)(r-1)=r^5-5r^4+9r^3-9r^2+8r-4=0$. Then $a_{n+5}=5a_{n+4}-9a_{n+3}+9a_{n+2}-8a_{n+1}+4a_n$ and in particular $a_5=5a_4-9a_3+9a_2-8a_1+4a_0=9$.

142) By putting $b_{n-1}=\frac{1}{\beta}(a_n-\alpha a_{n-1})$ in $b_n=\gamma a_{n-1}+\delta b_{n-1}$, we get $a_{n+1}-\alpha a_n=\beta\gamma a_{n-1}+\delta a_n+\delta\alpha a_{n-1}$ which simplifies into $a_{n+1}=(\alpha+\delta)a_n+(\gamma\beta-\alpha\delta)a_{n-1}$. This proves that $\{a_n\}$ satisfies the given relation. In a similar way it can be shown that $\{b_n\}$ satisfies the same relation.

143) Since $S_n=S_{n-1}+a_n$ for any positive integer n, we can write

$$\beta\underbrace{(S_n-S_{n-1}-a_n)}_{=0}+\alpha\underbrace{(S_{n+1}-S_n-a_{n+1})}_{=0}-\underbrace{(S_{n+2}-S_{n+1}-a_{n+2})}_{=0}=0.$$

Then $-\beta S_{n-1}+(\beta-\alpha)S_n+(\alpha+1)S_{n+1}-S_{n+2}+\underbrace{(a_{n+2}-\alpha a_{n+1}-\beta a_n)}_{=0}=0$ and finally,

$$S_{n+2}=(\alpha+1)S_{n+1}+(\beta-\alpha)S_n-\beta S_{n-1}.$$

144) Let A_n be the number of ways Ayşe can climb a staircase with n stairs. At the first step if she climbs one stair, she can climb the remaining $n-1$ stairs in A_{n-1} ways. If the first step of Ayşe is 2 stairs, there are A_{n-2} ways to climb the remaining part of the staircase. It follows that $A_n=A_{n-1}+A_{n-2}$. Since $A_1=1$ and $A_2=2$ we observe that $A_n=F_n$.

145) Call a subset of $A=\{1,2,\dots,n\}$ which do not contain any pair of successive integers an n −stramboshe and let S_n be the number of n −stramboshes. If an n −stramboshe X does not contain n, it can be regarded as an $(n-1)$-stramboshe. If it contains n, it can be written as $B\cup\{n\}$ where B is an $(n-2)$-stramboshe. It follows that $S_n=S_{n-1}+S_{n-2}$. Since $S_1=2$ (1-stramboshes are $\emptyset$ and $\{1\}$), $S_2=3$ (2-stramboshes are $\emptyset,\{1\},\{2\}$) we see that S_n is the sequence $2,3,5,8,\dots$. Then, $S_n=F_{n+1}$.

146) Let B_n be the number of ways of tiling a $1\times n$ rectangular board using 1×2 and 1×1 pieces. If the first piece is 1×1, then the remaining part can be tiled in B_{n-1} ways. If the first part is 1×2, remaining part can be tiled in B_{n-2}. Then, $B_n=B_{n-1}+B_{n-2}$ for $n\ge 2$ and $B_1=1, B_2=2$. We conclude that $B_n=F_n$.

147) Let C_n be the number of ways of tiling a $2\times n$ rectangular board using 1×2 and 2×2 pieces. We can start the tiling in one of three ways: with a 'vertical' 1×2 piece, with a 2×2 piece or with two 'horizontal' 1×2 pieces (one on top of the other). In the first case, the remaining part can be tiled in C_{n-1} ways and in each of the other cases rest of the board can be tiled in C_{n-2} ways. Thus, $C_n=C_{n-1}+2C_{n-2}$. Characteristic equation of the recursion is $r^2-r-2r=(r-2)(r+1)=0$ whose roots are $r_1=-1$, $r_2=2$. Then $C_n=A2^n+B(-1)^n$. Above discussion gives that $C_1=1$ and $C_2=3$. Using these initial values, we have the system

$$2A-B=1$$
$$4A+B=3$$

whose solution is $A=\frac{2}{3}, B=\frac{1}{3}$. Then $C_n=\frac{1}{3}(2^{n+1}+(-1)^n)$.

148) Let T_n be number of messages that can be transmitted in n seconds. It can be shown that $T_n=T_{n-1}+2T_{n-2}$. As $T_1=1$ and $T_2=3$, from the previous problem it follows that $T_n=\frac{1}{3}(2^{n+1}+(-1)^n)$.

149) Let p_n be the number of such permutations. The last term of such a permutation can be either n or $n-1$. If the last term is n, there are p_{n-1} different arrangements for the first $n-1$ terms. If the last term is $n-1$, then the term next to the last one must be n. In this case first $n-2$ terms can be arranged in p_{n-2} distinct ways. It follows that $p_n=p_{n-1}+p_{n-2}$ for $n\ge 2$ and $p_1=1$, $p_2=2$. Then, $p_n=F_n$.

150) Let E_n be the number of strings of length n formed with letters A, B and C with an even number of A's. Then the number of strings with an odd number of A's is 3^n-E_n. For a string of length n with an even number of A's there are two possibilities:

- The first letter is B or C. Then the remaining letters can be arranged in E_{n-1} different ways. By the product rule, there are $2E_{n-1}$ such strings.
- The first letter is A. The remaining letters can be arranged in $3^{n-1}-E_{n-1}$.

Consequently, $E_n=E_{n-1}+3^{n-1}$. We can write $3E_{n-1}=3E_{n-2}+3^{n-1}$. From these two recursions we obtain a homogeneous recursion $E_n=4E_{n-1}-3E_{n-2}$ with characteristic equation is $(r-1)(r-3)=0$. It follows that $E_n=A+B3^n$ for some $A,B\in\mathbb{R}$. Using the initial conditions $E_1=2, E_2=5$ we find $A=B=\frac{1}{2}$. Thus $E_n=\frac{3^n+1}{2}$.

151) Let u_n be the number of strings of upper-case letters of length n that do not contain AA. By counting, $u_1 = 3$, $u_2 = 8$. Call a string valid if it does not contain AA. Consider a valid string of length n. There are two cases depending on whether the first letter is A.

Case 1. The first letter is not A (2 choices). Then, the remaining $n-1$ letters can be any valid string of length $n-1$. Since there u_{n-1} of these, by the Rule of Product there are $2u_{n-1}$ valid string in which the first letter is not A.

Case 2. The first letter is A (1 choice). Since the string is valid, the second letter is not A (2 choices), and then the remaining $n-2$ letters can be any valid string of length $n-2$. Since there u_{n-2} of these, by the product rule there are $2u_{n-2}$valid strings in which the first letter is A.

Therefore, by the addition rule $u_n = 2(u_{n-1} + u_{n-2})$.

Characteristic equation of the recurrence is $r^2 - 2r - 2 = 0$ whose roots are $r_{1,2} = 1 \pm \sqrt{3}$. Then general term is of the form $u_n = A(1+\sqrt{3})^n + B(1-\sqrt{3})^n$. Using the initial conditions, we compute $A = \frac{3-7\sqrt{3}}{6}$ and $B = \frac{3+7\sqrt{3}}{6}$. Then $u_n = \frac{3-7\sqrt{3}}{6} \cdot (1-\sqrt{3})^n + \frac{3+7\sqrt{3}}{6} \cdot (1+\sqrt{3})^n$.

152) Imitating the solution of problem 7, we have the relation $E_n = 24E_{n-1} + 26^{n-1}$ from which the homogeneous relation $E_n = 24E_{n-1} + 624E_{n-2}$ is obtained. Then $E_n = A \cdot 24^n + B \cdot 26^n$ for some $A, B \in \mathbb{R}$. Using the initial conditions $E_1 = 25,\ E_2 = 626$ we compute $A = B = \frac{1}{2}$. Then $E_n = \frac{24^n + 26^n}{2}$.

153) Following the solution of problem 8 we obtain the relation $u_n = 25(u_{n-1} + u_{n-2})$ for $n \geq 3$ and the initial terms $u_1 = 26$, $u_2 = 675$. General term is $u_n = \frac{145-27\sqrt{29}}{290} \cdot \left(\frac{25-5\sqrt{29}}{2}\right)^n + \frac{145+27\sqrt{29}}{290} \cdot \left(\frac{25+5\sqrt{29}}{2}\right)^n$.

154) a) Let L_n be the largest possible number of regions that can be defined by n straight lines in plane. $L_1 = 2$ and L_n satisfies $L_n = L_{n-1} + n$ for $n \geq 2$. We obtain the homogeneous relation $L_n = 3L_{n-1} - 3L_{n-2} + L_{n-3}$ for $n \geq 3$ with the initial conditions $L_1 = 2,\ L_2 = 4,\ L_3 = 7$. General term is given by $L_n = \frac{1}{2}(n^2 + n + 1)$.

b) Let C_n be the largest possible number of regions that can be defined by n circles in plane. $C_1 = 2$ and C_n satisfies $C_n = C_{n-1} + 2(n-1)$ for $n \geq 2$. We obtain the homogeneous relation $C_n = 3C_{n-1} - 3C_{n-2} + C_{n-3}$ for $n \geq 3$ with the initial conditions $C_1 = 2,\ C_2 = 4,\ C_3 = 8$. General term is given by $C_n = n^2 - n + 2$.

c) Let T_n be the largest possible number of regions that can be defined by n triangles in plane. $T_1 = 2$ and T_n satisfies $T_n = T_{n-1} + 6(n-1)$ for $n \geq 1$. We obtain the homogeneous relation $T_n = 3T_{n-1} - 3T_{n-2} + T_{n-3}$ for $n \geq 2$ with the initial conditions $T_1 = 2, T_2 = 8,\ T_3 = 26$. General term is given by $T_n = 6n^2 - 12n + 8$.

d) Let R_n be the largest possible number of regions that can be defined by n rectangles in plane. $R_1 = 2$ and R_n satisfies $R_n = R_{n-1} + 8(n-1)$ for $n \geq 1$. We obtain the homogeneous relation $R_n = 3R_{n-1} - 3R_{n-2} + R_{n-3}$ for $n \geq 3$ with the initial conditions $R_1 = 2,\ R_2 = 10,\ R_3 = 34$. General term is given by $R_n = 8n^2 - 16n + 10$.

155) For the number u_n of ways of paintings we can derive the recursion $u_n = 3 \cdot 2^n - u_{n-1}$ for $n \geq 2$ with the initial condition $u_1 = 0$. This recursion reduces to the homogeneous relation $u_n = u_{n-1} + 2u_{n-2}$ subject to $u_1 = 0$ and $u_2 = 6$. Roots of the characteristic equation $r^2 - r - 2 = (r-2)(r+1) = 0$ are $r_1 = -1$ and $r_2 = 2$. Then we get $u_n = 2^n + 2 \cdot (-1)^n$.

156) **a**) For the first column we have six possibilities: BY, BW, YB, YW, WB, WY. For any choice, we have 3 possibilities for the next column (for example if the first column is YW, second column can be BY, WY or WB). It follows that the entire board can be painted in $6 \cdot 3^{n-1} = 2 \cdot 3^n$ different ways.

b) Let $A_n$be the number of ways of painting the board so that no two white squares are adjacent. Also let W_n be the number of those paintings which has a white cell in the first column and put $R_n = A_n - W_n$. Then we can write $W_n = 4(A_{n-1} - W_{n-1}) + 2W_{n-1}$. This recurrence is equivalent with $A_n - R_n = 4R_{n-1} + 2(A_{n-1} - R_{n-1})$. Observing that $R_n = 4A_{n-1}$, we obtain $A_n - 4A_{n-1} = 16A_{n-2} + 2A_{n-1} - 8A_{n-2}$ which simplifies into $A_n = 6A_{n-1} + 8A_{n-2}$. Characteristic equation of the relation is $r^2 - 6r - 8 = 0$ whose roots are $r_{1,2} = 3 \mp \sqrt{17}$. Initial terms are $A_1 = 8$ and $A_2 = 56$. General term is $A_n = \frac{17-5\sqrt{17}}{34} \cdot (3-\sqrt{17})^n + \frac{17+5\sqrt{17}}{34} \cdot (3+\sqrt{17})^n$.

157) a) Observe that there are only four pairs of integers in A that add up to 9, and that each integer exactly occurs in exactly one pair. These pairs are $1+8,\ 2+7,\ 3+6,\ 4+5$. Let the pigeons be the five integers selected from A, and let the pigeonholes be the pairs of integers that add up to 9. According to pigeonhole principle, since there are more pigeons (5) than pigeonholes (4), at least two pigeons must go to the same hole. Thus, two distinct integers are sent to the same pair. But that implies that those two integers are the two distinct elements of the pair, so their sum is 9.

b) The answer is no. For instance, consider the numbers: 1, 2, 3, 4.

158) Think of associating the married couples with boxes labeled 1 to n, so that whenever a member is selected from the set of $2n$ people, then that person is placed into his or her associated box. Thus, the question reduces to asking for the smallest number of members that can be placed in the n boxes in order that some box contains two members. Clearly n does not suffice; however, by the pigeonhole principle, $n+1$ works.

159) The number of wins for a player is 1 or 2 or 3 … or $n-1$. These $n-1$ numbers correspond to $n-1$ pigeonholes in which the pigeons (players) are to be housed. So at least two of them should be in the same pigeonhole and they have the same number of wins.

160) a) If there are 9 diplomats from each continent there will be 54 diplomats. Any more than 54 ensures that there will be 10 diplomats from at least one continent. 55 b) If 7 diplomats from Australia and 9 diplomats from other countries are invited there will be 51 diplomats. Since there are no more Australian diplomats left, any addition of diplomats will ensure that there will be 10 diplomats from a country. 52

161) In this example, the pigeons are the 150 people and the pigeonholes are the 29 possible last initials of their names (we consider the Turkish alphabet which consists of 29 letters). Note that $150 > 5 \times 28 = 140$. Then, the generalized pigeonhole principle states that some initial must be the image of at least six $(5+1)$ people. Thus at least six people have the same last initial.

162) Suppose that only four computers were used by three or more students. At most six students are allowed to share any computer, making a total of at most 24 students using these four computers. Since there are 42 students at all, that would leave at least 18 students to share the remaining eight computers with no more than two students per computer. But the generalized pigeonhole principle guarantees that if 18 students share eight computers, then at least three must share one of them. This is a contradiction. Thus, the supposition is false, and so at least five computers are used by three or more students.

163) The least number of marbles to be picked is $(3-1)+(4-1)+(5-1)+1=10$.

164) Let $x_1, x_2, \ldots, x_{1001}$ denote the 1001 integers chosen. We can express each x_i, $i = 1,2,\ldots,1001$ in the form $x_i = 2^{m_i} b_i$ where b_i is odd (for instance, $564 = 2^2 \cdot 141$, $1184 = 2^5 \cdot 37$, $512 = 2^9 9 \cdot 1$, $97 = 2^0 \cdot 97$). Now, consider the odd integers $b_1, b_2, \ldots, b_{1001}$. But there are exactly 1000 odd integers between 1 and 2000, hence by the pigeonhole principle, at least two of $b_1, b_2, \ldots, b_{1001}$ are equal. So $b_r = b_s$ for some r and s, where $r \neq s$. Then we have $x_r = 2^{m_r} b_r$ and $x_s = 2^{m_s} b_s$. Since $x_r \neq x_s$, we have $m_r \neq m_s$; without loss of generality we can assume $m_r < m_s$. But then $2m_r | 2m_s$, and consequently, $x_r | x_s$.

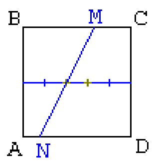
(Is it possible to choose 1000 integers from the list so that no number is divisible by any other?)

165) Number the locations $1,2,\ldots,100$ and let a_i be the number assigned to location i, $i = 1,2,\ldots,100$. There are 100 sums to consider: $a_1 + a_2 + a_3$, $a_2 + a_3 + a_4, \ldots, a_{98} + a_{99} + a_{100}$, $a_{99} + a_{100} + a_1$, $a_{100} + a_1 + a_2$ and each a_i appears in exactly three of the sums. Hence, the total of these sums is$3(a_1 + a_2 + \cdots + a_{100}) = 3 \cdot 5050 = 15150$. Thus, the average value of is $15150/100 = 151.5$, and so by the previous problem, one of the sums has value at least 152.

166) Number the days $1,2,\ldots,12$ and consider the subsets. $\{1,2\}, \{3,4\}, \{5,6\}, \{7,8\}, \{9,10\}, \{11,12\}$. Since these subsets partition the $12-$day period, the 110 hours of practice can be distributed among them. And since $110 > 18 \times 6 = 108$, the generalized pigeonhole principle implies that some consecutive $2-$day period contains at least 19 hours.

167) Split the triangle into four smaller ones by connecting midpoints of its sides. The largest possible distance between two points of one small triangle is ½. Now, we are given 4 triangles and 5 points. By the Pigeonhole Principle, at least one triangle contains at least two points. The distance between any two such points does not exceed ½.

168) None of the given lines may pass through two successive sides of the square because in this case we get a triangle and a pentagon and not two quadrilaterals. Assume one of them intersects sides BC and AD at points M and N, respectively. The quadrilaterals, $ABMN$ and $CDNM$, are both trapezoids with equal heights. Therefore, their areas are in the same ratio as their midlines. From here, MN intersects the midline of the square in ratio 2:3. This is true for any one of the nine lines. But there are only four points that divide the midlines of the square in the ratio 2:3. Therefore, by the Pigeonhole Principle, at least three of the lines pass through the same point.



169) The midpoint of the line joining two grid points (x_1, y_1) and (x_2, y_2) is located at $\left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right)$. The latter will be a grid point iff its coordinates are integers. The x-coordinate will be integer iff x_1 and x_2 have the same parity, i.e., iff they are either both even or both odd. Out of 5 points, at least three satisfy this condition. But the same is true of the y-coordinate. And out of the selected three points, at least two have y-coordinate with the same parity.

170) In the following we assume $f(x)$ is a polynomial with integral coefficients:

$$f(x) = c_n x^n + c_{n-1} x^{n-1} + \ldots + c_1 x + c_0$$

Lemma. For any two different integers p and q, the difference $f(p)–f(q)$ is divisible by $p–q$.

Proof. Indeed, $f(p)–f(q) = c_n(p^n–q^n) + c_{n-1}(p^{n-1}–q^{n-1}) + \ldots + c_1(p - q)$ and, since $(p–q) \mid (p^k–q^k)$ for every integer $k > 0$.

Now, assume that $f(a) = f(b) = f(c) = 2$ and $f(d) = 3$ with all a, b, c and d different. From Lemma we immediately obtain that

$$(d–a) \mid (f(d)–f(a)) = 3–2 = 1,$$
$$(d–b) \mid (f(d)–f(b)) = 3–2 = 1,$$
$$(d–c) \mid (f(d)–f(c)) = 3–2 = 1.$$

Thus differences $d–a, d–b, d–c$ all divide 1. But 1 has only two divisors: 1 and -1. Therefore, by the Pigeonhole Principle, two of the differences coincide. Which contradicts our assumption that the numbers a, b, c are all different.

171) There are 1997 remainders of division by 1997. Consider the sequence of powers $1, 3, 3^2, \ldots, 3^{1997}$. It contains 1998 members. Therefore, by the Pigeonhole principle, some two of them, say $3n$ and $3m$, $n > m$, have equal remainders when divided by 1997. Then their difference $(3n–3m)$ is divisible by 1997.

172) As in previous problem, let 3^n and 3^m $(n > m)$ have the same remainder when divided by 1000. Thus $3^n–3^m = 3^m(3^{n-m}–1)$ is divisible by 1000. Since 1000 and $3m$ have no common factors, 1000 is bound to divide the second factor $(3^{n-m}–1)$. This exactly means that 3^{n-m} ends with 001.

173) Form the n consisting of the i –th odd integer less than $2n$ together with its multiples by powers of 2:

$$A_1 = \{1, 2, 4, , \ldots, 512\},$$
$$A_3 = \{3, 6, 12, 24, \ldots, 768\},$$
$$A_5 = \{5, 10, 20, \ldots, 640\},$$
$$\vdots$$
$$A_{999} = \{999\}.$$

Then the union of these n sets contains $\{1, 2, \ldots, 2n\}$. Hence some two of the selected integers belongs to A_i, for some i; and so one of them divides the other.

174) Let a_i be the total number of aspirin consumed up to and including the i –th day, for $i = 1, \ldots, 30$. Combine these with the numbers $a1 + 14, \ldots, a30 + 14$, providing 60 numbers, all positive and less or equal $45 + 14 = 59$. Hence two of these 60 numbers are identical. Since all a_i's and, hence, $(a_i + 14)$'s are distinct (at least one aspirin a day consumed), then $a_j = a_i + 14$, for some $i < j$. Thus, on days $i + 1$ to j, the person consumes exactly 14 aspirin.

175) Five women each cast in 3 plays makes 15 woman's parts in the 7 plays. Since $\frac{15}{7} > 2$, some play has at least 3 women in its cast.

176) A person at the party can have 0 up to $n - 1$ friends at the party. However, if someone has 0 friends at the party, then no one at the party has $n - 1$ friends at the party, and if someone has $n - 1$ friends at the party, then no one has 0 friends at the party. Hence the number of possibilities for the number of friends the n people at the party have must be less than n. Hence two people at the party have the same number of friends at the party.

177) No. Consider the set {3,4,5,6,7,8}

178) Yes. Any subset with 7 or more elements must contain both members of at least one of the following pairs:

(3,13), (4,12), (5,11), (6,10), (7,9)

179) Consider the partition:

$$\underbrace{(7,93), (8,92), \ldots, (49,51)}_{43\ pairs}, \underbrace{(50), (94), (95), (96), (97)}_{5-singletons}$$

180) A has 64 subsets and largest possible sum is $7 + 8 + 9 + 10 + 11 + 12 = 57$.

181) See Problem 174).

182) Probability that A wins and C loses is equal to probability that A wins times probability that C loses. $\frac{3}{5} \cdot \frac{3}{7} = \frac{9}{35}$.

183) Sample space is the set of all possible choices of 3 cells out of 64 cells. Thus, size of the sample space is $C(64,3)$. In $C(8,3) \cdot P(8,3)$ different ways 6 checkers can be located such that no row or no column contains more than one checker. Then, the required probability is $\frac{C(8,3)\cdot P(8,3)}{C(64,3)} \approx 0.4516$.

184) Available odd numbers greater than 8 are 9 and 11. We can have the sum 9 with probability 4/36 and the sum with probability 2/36. Since the events of having a sum of 9 and 11 are disjoint, the probability of having an odd sum greater than 8 is 1/6.

185) Size of the sample space is $C(52,3)$ which consists of all possible choices of three cards out of 52 cards. We can have three cards none of which is a club in $C(39,3)$ different ways. Then, the probability of having at least one clubs is $1 - \frac{C(39,3)}{C(56,3)} = 0.4135$.

186) Probability that first toss is less than 4 and second toss is greater than 4 is equal to probability that first toss is less than 4 times probability that second toss is greater than 4. $\frac{3}{6} \cdot \frac{2}{6} = \frac{1}{6}$.

187) We have to consider two events:

The first card is a face card of diamonds and the second card is a clubs.

The first card is a face card of clubs and the second card is another clubs.

Probabilities of these events are $\frac{3}{52} \cdot \frac{13}{51} = \frac{39}{2652}$ and $\frac{3}{52} \cdot \frac{12}{51} = \frac{36}{2652}$, respectively. Then the answer is $\frac{75}{2652} = 0.02828\cdots$.

188) The product is a prime only when one die is a 1 and the other is a 2, 3 or 5. Then the probability is $\frac{6}{36} = \frac{1}{6}$.

189) Out of 6 permutations of DVDs only one of them is correct. $\frac{1}{6}$.

190) From 1, subtract the possibility that both vacant apartments are not on the top floor. $1 - \frac{48}{56} \cdot \frac{47}{55}$.

191) There are 16 winning cards (consider the counting of 10 of hearts two times). $\frac{16}{52}$ / From 1 probability, remove the probability of cases where there is no winning card. $1 - \frac{36}{52} \cdot \frac{35}{51} = \frac{116}{221} = 0.52488\cdots$.

192) There are 21! strings with MONKEY in it and 26! total strings. $\frac{21!}{26!} = 0.0000001267\cdots$.

193) $\Pr(\text{No Ace}) = C(48,5)/C(52,5)$

$\Pr(\text{No K/Q}) = C(44,5)/C(52,5)$

$\Pr(\text{No Ace and No K/Q}) = C(40,5)/C(52,5)$

Then by principle of inclusion-exclusion the probability we are asked is $1 - \frac{C(48,5)+C(44,5)-C(40,5)}{C(52,5)} = 0.1764\cdots$.

194) Sample space consists of all possible orderings of picked fruits (such as aaaooaaoaaoo), hence the size of sample space is $C(12,5)$. The last one is an apple in $C(11,5)$ orderings. Hence the probability we are asked is $\frac{C(11,5)}{C(12,5)} = 0.5833\cdots$.

Alternative solution: The last fruit is an apple with probability $\frac{5}{12}$.

195) a) Five people can choose the floors they exit in 10^5 different ways. In $P(10,5)$ of these ways, choices are all distinct. The probability they all choose different floors is then $\frac{P(10,5)}{10^5} = 0.3024$.

Alternative solution: Each floor is chosen with probability 10^{-1}, then five distinct floors (in a specific order) can be chosen in 10^{-5} ways. Number of choices is $C(10,5)$ and number of orderings is 5!. Then the probability that each one chooses a different floor is $10^{-5} \cdot C(10,5) \cdot 5! = 0{,}3024$.

b) Floor 10 is chosen with probability 2^{-1} and each other floor is chosen with probability 18^{-1},

-If no one chooses 10th floor, the probability that each one chooses a different floor is $18^{-5} \cdot C(9,5) \cdot 5!$,

-If one chooses 10th floor, they choose different floors with the probability $2^{-1} \cdot 18^{-4} \cdot C(9,4) \cdot 5!$.

Then, the required probability is $2^{-5}9^{-4}5!\left(9^{-1}C(9,5) + C(9,4)\right) = 18^{-5} \cdot 5! \cdot 10 \cdot C(9,5) = \frac{525}{6561} = 0.0800\cdots$.

196) We consider all possible cases:

- All the balls go to a unique box with probability 1/9, (a ball is thrown to any of the boxes, and then each of the others go to the same box with probability 1/3),

- one ball goes to each box (with probability 2/9),
- two balls go to a box, a ball goes to another box and a box is left empty (with probability 6/9),

197) There are 5! total sittings. However, if couples are not sitting together arranging one gender is enough since the other sits at the opposite sides so there are 2! such arrangements. $\frac{4}{15}$

198) There are 10! sittings but if 3 friends are sitting together count them as one and permute them within themselves. $\frac{8!3!}{10!} = \frac{1}{15}$.

199) It is the number of derangements over all permutations. $\frac{D_n}{n!} \approx \frac{1}{e}$

200) Denote the people by $A_1,\ A_2,\ B_1,\ B_2,\ C_1,\ C_2,\ D_1,\ D_2,\ E_1,\ E_2,\ F_1,\ F_2$ and define the events
A: $A_1$and A_2 are in the same team,
B: $B_1$and B are in the same team,
C: $C_1$and C_2 are in the same team,
D: $D_1$and D_2 are in the same team,
E: $E_1$and E_2 are in the same team,
F: F_1 *and* F_2 are in the same team.
Then, the probability that none of the teams has a married couple is given by $p = \Pr(\bar{A} \cap \bar{B} \cap \bar{C} \cap \bar{D} \cap \bar{E} \cap \bar{F})$.
Now we compute the necessary probabilities:

$$\Pr(A) = \frac{1}{4}$$
$$\Pr(A \cap B) = \frac{3}{4} \cdot \frac{1}{4} \cdot \frac{1}{4} = \frac{3}{64}$$
$$\Pr(A \cap B \cap C) = \frac{3}{4} \cdot \frac{2}{4} \cdot \frac{1}{4} \cdot \frac{1}{4} \cdot \frac{1}{4} = \frac{3}{512}$$
$$\Pr(A \cap B \cap C \cap D) = \frac{3}{4} \cdot \frac{2}{4} \cdot \frac{1}{4} \cdot \frac{1}{4} \cdot \frac{1}{4} \cdot \frac{1}{4} \cdot \frac{1}{4} = \frac{3}{2048}$$

Then by principle of inclusion-exclusion:

$$p = 1 - \binom{6}{1}\frac{1}{4} + \binom{6}{2}\frac{3}{64} - \binom{6}{3}\frac{3}{512} + \binom{6}{4}\frac{3}{2048} = \frac{221}{2048} = 0.1079 \cdots.$$

201) 10% of the students take both Spanish and French. $\Pr(French \mid Spanish) = \frac{\Pr(French \cap Spanish)}{\Pr(Spanish)} = \frac{10}{35}$

202) $\left(1 - \frac{3}{7}\right) \cdot \left(1 - \frac{2}{5}\right) \cdot \left(1 - \frac{1}{3}\right) = \frac{4}{7} \cdot \frac{3}{5} \cdot \frac{2}{3} \cdot = \frac{8}{35}$

203) There are $\frac{10!}{2!3!}$ possible outcomes but only 1 correct situation. $\frac{1}{10}$

204) $\frac{1}{15}$ (compare to problem 17.)

205) There are $\frac{1}{2}C(10,5)$ situations as order of groups is not important. For the desired outcomes, consider the number of ways the two other friends could have been chosen. $\frac{C(7,2)}{\frac{1}{2}C(10,5)} = \frac{21}{126} = 0.16666 \cdots$.

206) A function from the set of students to the set $\{1,2,3,4\}$ is defined and we are asked to find the probability that the function is not onto. The number of onto functions is given by

$$A = \sum_{k=0}^{4} (-1)^k \binom{4}{k} (4-k)^{10} = 818520.$$

The required probability is $1 - \frac{818520}{4^{10}} = 0.2193 \cdots$.

207) There are $\frac{8!}{2!^4 4!} = 105$ ways to determine the pairs. If two fixed English teams are to be matched, matching the remaining teams can be completed in $\frac{6!}{2!^3 3!} = 15$ ways. Thus, in 45 different pairings, two English teams are in the same pair. Then, probability that no two English teams are matched is $1 - \frac{45}{105} = \frac{4}{7}$.

208) Let the pair (w, b) denote the number of white and black balls. From the situation we can pass to $(w-1, b+1)$ with probability $\frac{w}{5}$ or $(w+1, b-1)$ with probability $\frac{b}{5}$. Starting from (3,2), the situation (0,5) can be achieved in at most 5 steps in one of the following four ways:

$$(3,2) \xrightarrow{3/5} (2,3) \xrightarrow{2/5} (1,4) \xrightarrow{1/5} (0,5)$$
$$(3,2) \xrightarrow{2/5} (4,2) \xrightarrow{4/5} (3,2) \xrightarrow{3/5} (2,3) \xrightarrow{2/5} (1,4) \xrightarrow{1/5} (0,5)$$

$$(3,2) \xrightarrow{3/5} (2,3) \xrightarrow{3/5} (3,2) \xrightarrow{3/5} (2,3) \xrightarrow{2/5} (1,4) \xrightarrow{1/5} (0,5)$$
$$(3,2) \xrightarrow{3/5} (2,3) \xrightarrow{2/5} (1,4) \xrightarrow{4/5} (2,3) \xrightarrow{2/5} (1,4) \xrightarrow{1/5} (0,5)$$

Then the probability we are trying to compute is $\frac{6}{125} + \frac{48+54+48}{3125} = \frac{12}{125} = 0{,}096$.

209) First student cannot be the winner. Second one wins with the probability $\frac{9}{9} \cdot \frac{1}{9}$, third one has the probability $\frac{9}{9} \cdot \frac{8}{9} \cdot \frac{2}{9}$ to win, fourth one wins with the probability $\frac{9}{9} \cdot \frac{8}{9} \cdot \frac{7}{9} \cdot \frac{3}{9}, \ldots$. In general, k-th one wins with the probability $\frac{P(9,k-1)\cdot(k-1)}{9^k}$.

Computed values are as follows:

No	Probability to win
1	0,00000
2	0,11111
3	0,19753
4	0,23045
5	0,20485
6	0,14225
7	0,07587
8	0,02950
9	0,00749
10	0,00094

Fourth student has the largest probability to win the bonus.

210) a) Let `BWWWWWWWWW` denote the ordering of shuffled cards where the first card is black. X wins if the ordering is `BWWWWWWWWW` or `WWWBWWWWWW` or `WWWWWWBWWW`. Second player wins for the orderings `WBWWWWWWWW` or `WWWWBWWWWW` or `WWWWWWWBWW`. It follows that each player has the same probability $\frac{1}{3}$ to be the winner.

b) 2 black and 7 white cards can be arranged in $\frac{9!}{2!7!} = 36$ different ways. For the first player to win, the possible orderings are

`B [BWWWWWWW]`: 8 orderings

`WWWB [BWWWW]`: 5 orderings

`WWWWWWB [BW]`: 2 orderings

Winning orderings for the second are:

`WB [BWWWWWW]`: 7 orderings

`WWWWB [BWWW]`: 4 orderings

`WWWWWWWB [B]`: 1 ordering

Then, the probability that the first player wins is 15/36. Second player is the winner with probability 12/36 and consequently third one is the winner with probability 9/36.

211) Assume there are three states -1, 0, and 1. Of these states, -1 represents the state that Selim loses by 1 game at the moment, 0 means they are at tie and 1 means Selim is 1 game ahead. Let P(s), where s is a state, represent the probability that Selim wins from the state s. Then we can construct the linear system.

$$\Pr(0) = p \cdot \Pr(1) + (1-p) \cdot \Pr(-1)$$
$$\Pr(1) = 1 \cdot p + (1-p) \cdot \Pr(0)$$
$$\Pr(-1) = p \cdot \Pr(0) + 0 \cdot (1-p)$$

Solving this system of linear equations, we get that $\Pr(0) = \frac{p^2}{1-2p\cdot(1-p)}$ which is the probability that Selim wins given he is at the state where the game is at tie.

212) a) Choose which two will be woman and for each such combination find the probability. $\binom{10}{2}\left(\frac{1}{20}\right)^2\left(\frac{19}{20}\right)^8 = 0.0746\cdots$. b) Similar to a), but from 1, subtract the possibilities that no woman or 1 woman chosen. $1-\binom{10}{0}\left(\frac{1}{20}\right)^0\left(\frac{19}{20}\right)^{10}-\binom{10}{1}\left(\frac{1}{20}\right)^1\left(\frac{19}{20}\right)^9 = 0.0861\cdots$.

d) Similar to a). $\binom{10}{0}\left(\frac{1}{20}\right)^0\left(\frac{19}{20}\right)^{10} = 0.5987\cdots$.

213) a) It is the probability that a 5 comes up in one throw of dice. $\frac{1}{6}$

b) It is the probability that the first throw is not a 5 and second one is a 5. $\frac{5}{6}\cdot\frac{1}{6}=\frac{5}{36}$

c) Add the two conditions in part a) and part b). $\frac{1}{6}+\frac{5}{36}=\frac{11}{36}$

d) From 1, remove the possibility that 5 doesn't appear on 10 throws. $1-\left(\frac{5}{6}\right)^{10} = 0.8384\cdots$

e) Find the possibility that a 5 does not appear for k-1 throws and 5 appears on kth throw. $\left(\frac{5}{6}\right)^{k-1}\cdot\frac{1}{6}$

f) Add the probabilities that the first five appears on kth trial for k from 6 to infinity. $\sum_{k=6}^{\infty}\left(\frac{5}{6}\right)^{k-1}\frac{1}{6}=\left(\frac{5}{6}\right)^5 = 0.4018\cdots$

214) There are 5^{26} possible outcomes but only the permutations are the desired cases. $\frac{P(26,5)}{5^{26}} = 0.6643\cdots$

215) a) In total 28% needs repair. Hence, the probability that both cars not requiring repair is $(.72)^2 = 0.5184$ b) In this case both cars requires repair each car has a probability of 28% for needing a repair. $(.28)^2 = 0.0784$

216) a) The probability that a lemon comes up is $\frac{3}{10}$ and 3 spins are made. $\left(\frac{3}{10}\right)^3 = 0.027$ b) The probability of getting no fruit symbols is $\frac{5}{10}$ and there are 3 spins. $\left(\frac{5}{10}\right)^3 = 0.125$ c) The probability of getting a bell is $\frac{1}{10}$ and there are 3 spins. $\left(\frac{1}{10}\right)^3 = 0.001$

d) The probability of getting no bell in a single spin is $\frac{9}{10}$. $\left(\frac{9}{10}\right)^3 = 0.729$ e) From 1, subtract the possibility that there is no bar. $1-\left(\frac{4}{10}\right)^3 = 0.936$

217) a) Choose the ones that will be broken and multiply each case by the probability. $\binom{20}{5}(0.08)^5(0.92)^{15} = 0.0145\cdots$ b) Find the probability that all work $\binom{20}{0}(0.08)^0(0.92)^{20} = 0.1886\cdots$

c) Find the probability that all of them are broken. $\binom{20}{20}(0.08)^{20}(0.92)^0 = 1.15\cdot 10^{-22}$

218) Remove the probability that all parts work correctly. $1-(0.999)^{224} = 0.2007\cdots$

219) Choose four robots that do not work and in each case multiply by the probability of such a choosing. $\binom{35}{4}(0.03)^4(0.97)^{31} = 0.0164\cdots$

220) There can be 1, 2, 3, 4 or 5 defective tires. For each case calculate the probability as in the previous question. $\sum_{i=1}^{5}\binom{5}{i}(0.007)^i(0.993)^{5-i} = 0.0345\cdots$

Add the probabilities that there are no defective tires and there is only 1 defective tire. $\binom{5}{0}(0.007)^0(0.993)^5+\binom{5}{1}(0.007)^1(0.993)^4 = 0.9995\cdots$

221) From 1, remove the probabilities that no sapling will die and 1 sapling will die. $1-\binom{12}{0}(0.15)^0(0.85)^{12}-\binom{12}{1}(0.15)^1(0.85)^{11} = 0.5565\cdots$.

222) For any r=1, 2, 3, 4, 5, or 6, The probability that the largest number in 6 throws is r is the probability that a number less than or equal to r will come up for 6 throws minus the probability that a number less than or equal to r-1 will come up for 6 throws . $\left(\frac{r}{6}\right)^6-\left(\frac{r-1}{6}\right)^6=\frac{r^6}{46{,}656}$.

223) There are 3 correct and 7 fake keys in the box. One can pick two correct and one fake key in $\binom{3}{2}\binom{7}{1} = 21$ possible ways. Choosing 3 correct keys is possible only in 1 way. Thus, there are 22 possible choices which enable us to open the door. Since the number of all possible choices is $\binom{10}{3} = 120$, probability of opening the door is $\frac{22}{120}=\frac{11}{60}$.

224) First solution. Assume that the boy stops at the X th try. Then $Pr(X=k)=\left(\frac{4}{5}\right)^{k-1}\left(\frac{1}{5}\right)$ and

$$\Pr(X \le k) = \sum_{r=1}^{k} \left(\frac{4}{5}\right)^{r-1} \left(\frac{1}{5}\right) = \frac{1}{5}\sum_{r=0}^{k-1} \left(\frac{4}{5}\right)^{r} = 1 - \left(\frac{4}{5}\right)^{k}.$$

Second solution. The probability of drawing a white ball in each of the first k tries is $\left(\frac{4}{5}\right)^{k}$. Consequently, to have a black ball in one of the first k tries is $1-\left(\frac{4}{5}\right)^{k}$.

225) First Solution. Assume that the boy stops at the X th try. We have

$$\begin{aligned}
&\Pr(X=1) = \frac{1}{3}\\
&\Pr(X=2) = \frac{2}{3}\cdot\frac{1}{4} = \frac{2}{3\cdot 4}\\
&\Pr(X=3) = \frac{2}{3}\cdot\frac{3}{4}\cdot\frac{1}{5} = \frac{2}{4\cdot 5}\\
&\Pr(X=4) = \frac{2}{3}\cdot\frac{3}{4}\cdot\frac{4}{5}\cdot\frac{1}{6} = \frac{2}{5\cdot 6}\\
&\vdots\\
&\Pr(X=k) = \frac{2}{(k+1)(k+2)}
\end{aligned}$$

Then

$$\begin{aligned}
\Pr(X \le k) &= \sum_{r=1}^{k} \frac{2}{(r+1)(r+2)} = 2\left[\sum_{r=1}^{k}\frac{1}{r+1} - \sum_{r=1}^{k}\frac{1}{r+2}\right]\\
&= 2\left[\sum_{r=1}^{k}\frac{1}{r+1} - \sum_{r=2}^{k+1}\frac{1}{r+1}\right]\\
&= 2\left(\frac{1}{2} - \frac{1}{k+2}\right)\\
&= \frac{k}{k+2}.
\end{aligned}$$

Second solution. The probability of drawing a white ball in each of the first k tries is $\frac{2}{3}\cdot\frac{3}{4}\cdot\frac{3}{4}\cdots\frac{k+1}{k+2} = \frac{2}{k+2}$. Consequently, to have a white ball in one of the first k tries is $1-\frac{2}{k+2} = \frac{k}{k+2}$.

226) $\Pr(\text{Bill wins}) = \frac{7}{11}\cdot\frac{4}{10} + \frac{7}{11}\frac{6}{10}\frac{5}{9}\cdot\frac{4}{8} + \frac{7}{11}\frac{6}{10}\frac{5}{9}\frac{4}{8}\frac{3}{7}\cdot\frac{4}{6} + \frac{7}{11}\frac{6}{10}\frac{5}{9}\frac{4}{8}\frac{3}{7}\frac{2}{6}\frac{1}{5}\cdot\frac{4}{4} = 0.3939\cdots.$

227) Same problem as in 219). $\sum_{i=20}^{30}\binom{30}{i}(0.8)^i(0.2)^{30-i} = 0.9744\cdots$

228) Same problem as in 219). $\binom{8}{3}\left(\frac{3}{8}\right)^3\left(\frac{5}{8}\right)^5 = 0.2816\cdots.$

229) Same problem as in 219). $\sum_{i=4}^{500}\binom{500}{i}(0.01)^i(0.99)^{500-i} = 1-\sum_{i=0}^{3}\binom{500}{i}(0.01)^i(0.99)^{500-i} = 0.7363\cdots$

230) Let x be the probability of having heads in a single flipping. When the coin is flipped 6 times the probability of 3 heads and 3 tails is $P(x) = \binom{6}{3}x^3(1-x)^3$. We have $P'(x) = 60x^2(1-x^2)(1-2x)$. It follows that P attains its maximum value for $x = 1/2$ and this maximum value is $P\left(\frac{1}{2}\right) = 0.3125$. We conclude that, the probability of having 3 heads and 3 tails can never exceed 0.3125.

231) Let $P(n,k)$ denote the probability of having exactly k heads when the coin is flipped n times. Then

$$P(4,2) = \binom{4}{2}x^2(1-x)^2 = 6x^2(1-x)^2 = 0.24$$

From which we get $x(1-x) = 0.2$ and

$$\begin{aligned}
P(6,3) &= \binom{6}{3}x^3(1-x)^3 = 20x^3(1-x)^3\\
&= 20[x(1-x)]^3 = 0.16\,.
\end{aligned}$$

232) $P(10,5) = \binom{10}{5}\cdot 2^{-10} = 0.2460\cdots$ and $P(20,10) = \binom{20}{10}\cdot 2^{-20} = 0.1761\cdots.$

233) We already know that the older child is a boy. The probability of two boys is equivalent to the probability that the younger child is a boy, which is 1/2. In fact, given that the older child is a boy, sample space is {BB, BG}. In one of two equally probable cases the couple has two boys.

234) Given that one of two children is a boy, our sample space is {BB, BG, GB}. Since in only one of three equally probable cases the couple has two boys, the probability is 1/3.

235) $\Pr(white) = \Pr(white|B_1)\Pr(B1) + \Pr(white|B_2)\Pr(B_2) = \frac{2}{5}\cdot\frac{1}{2} + \frac{6}{10}\cdot\frac{1}{2} = \frac{1}{2}.$

$$\Pr(B_1|white) = \frac{\Pr(white|B_1)\Pr(B1)}{\Pr(white)} = \frac{1/5}{1/2} = \frac{2}{5}.$$

236) $\Pr(white) = \frac{8}{15}$, $\Pr(B_1) = \frac{5}{15}$, $\Pr(white|B_1) = \frac{2}{5}$

$$\Pr(B_1|white) = \frac{\Pr(white|B_1)\Pr(B1)}{\Pr(white)} = \frac{2/15}{8/15} = \frac{1}{4}.$$

237) We first compute the necessary probabilities;

$$\Pr(\mathrm{I}) = \Pr(\mathrm{II}) = \frac{1}{2},$$

$$\Pr(\mathrm{I}\cap G) = \frac{1}{2}\cdot\frac{2}{12} = \frac{1}{12},\quad \Pr(\mathrm{I}\cap Y) = \frac{1}{2}\cdot\frac{4}{12} = \frac{1}{6},\quad \Pr(\mathrm{I}\cap W) = \frac{1}{2}\cdot\frac{6}{12} = \frac{1}{4}$$

$$\Pr(\mathrm{II}\cap G) = \frac{1}{2}\cdot\frac{15}{30} = \frac{1}{4},\quad \Pr(\mathrm{II}\cap Y) = \frac{1}{2}\cdot\frac{10}{30} = \frac{1}{6},\quad \Pr(\mathrm{II}\cap W) = \frac{1}{2}\cdot\frac{5}{30} = \frac{1}{12}$$

From these, we obtain

$$Pr(G) = \Pr(\mathrm{I}\cap G) + \Pr(\mathrm{II}\cap G) = \frac{1}{3},\quad Pr(Y) = \Pr(\mathrm{I}\cap Y) + \Pr(\mathrm{II}\cap Y) = \frac{1}{3},$$

$$Pr(W) = \Pr(\mathrm{I}\cap W) + \Pr(\mathrm{II}\cap W) = \frac{1}{3},$$

We can observe that

$\Pr(I\cap G) = \frac{1}{12}$, and$\Pr(I)\cdot\Pr(G) = \frac{1}{6}$ so I and G are not independent events.

$\Pr(I\cap Y) = \frac{1}{6}$, and$\Pr(I)\cdot\Pr(Y) = \frac{1}{6}$ so I and Y are independent events.

$\Pr(I\cap W) = \frac{1}{4}$, and$\Pr(I)\cdot\Pr(W) = \frac{1}{6}$ so I and W are not independent events.

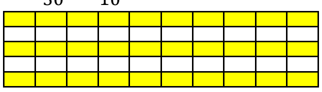
[We may interpret the situation as follows. Before choosing a box, the probability of picking a green ball is 1/3. If it is known that the first box is chosen the probability of picking a green ball reduces to 1/6. In a similar manner, probability of choosing a white ball increases from 1/3 to 1/2. Thus, the choice of the box has an effect on the probability of choosing a green or a white ball. On the other hand, the probability 1/3 of picking a white ball, before choosing a box, remains same even it is known that the first box is chosen. That is, the choice of the box has no effect on the probability of choosing a yellow ball.]

238) There are three possible cases of distinct face numbers with a sum 10: (1,3,6), (1,4,5), (2,3,5), then $\Pr(\text{sum is }10\ \cap\text{ all faces different}) = \frac{3!\cdot 3}{6^3} = \frac{1}{12}$. There are $\binom{6}{3}$ possibilities for all faces being distinct, then $\Pr(\text{all faces distinct}) = \frac{3!\cdot 20}{6^3} = \frac{5}{9}$. We conclude that $\Pr(\text{sum is }10\mid\text{all faces distinct}) = \frac{1/12}{5/9} = \frac{3}{20}$.

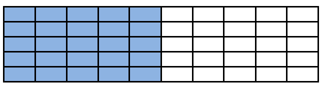
239) Let B, W, R and X stand for a blue ball, a white ball, a red ball and a ball of any color, respectively. We are asked to find $\Pr(RR|RX)$. We have $\Pr(RR\cap RX) = \Pr(RR) = \frac{1}{6}$ and $\Pr(RX) = 1 - \Pr(BW) = 1 - \frac{1}{6} = \frac{5}{6}$. Then

$$\Pr(RR|RX) = \frac{\Pr(RR\cap RX)}{\Pr(RX)} = \frac{1}{5}.$$

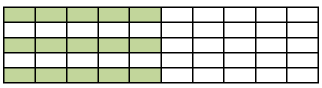
240) $\Pr(A) = \frac{3}{5}$, $\Pr(B) = \frac{1}{2}$ and $\Pr(A\cap B) = \frac{15}{50} = \frac{3}{10} = \Pr(A)\cdot\Pr(B)$. Events are independent.



A



B



C

241) Case 1. You ask whether the card is red or not. After learning the color of the card you can guess the card correctly with probability 1/26. Then probability to guess correctly is $\frac{1}{2}\cdot\frac{1}{26}+\frac{1}{2}\cdot\frac{1}{26}=\frac{1}{26}$.

Case 2. You ask whether the card is ace of spades. If the answer is yes (probability 1/52) you can guess correctly with probability 1. Otherwise, (probability 51/52) a correct guess has probability 1/51. Then the probability of a correct guess is $\frac{1}{52}\cdot 1+\frac{51}{52}\cdot\frac{1}{51}=\frac{1}{26}$.

Asking either question, you gain the same advantage.

242) $\Pr(80|70)=\frac{\Pr(80 \text{ and } 70)}{\Pr(70)}=\frac{0.2}{0.6}=\frac{1}{3}$.

243) $\Pr(B)=\frac{50}{100}\cdot\frac{1}{2}+\frac{20}{100}\cdot 1=\frac{9}{20}$.

$$\Pr(w\cap B)=\frac{50}{100}\cdot\frac{1}{2}=\frac{1}{4}.$$
$$\Pr(w|B)=\frac{1/4}{9/20}=5/9.$$

244) $\sum_{r=0}^{100}\binom{105}{r}\left(\frac{19}{20}\right)^r\left(\frac{1}{20}\right)^{105-r}=1-\sum_{r=101}^{105}\binom{105}{r}\left(\frac{19}{20}\right)^r\left(\frac{1}{20}\right)^{105-r}=0.3924\cdots.$

245) $\Pr(\textit{correct answer})=\Pr(\textit{correct answer}\cap \textit{he knows})+\Pr(\textit{correct answer}\cap\textit{he guesses})=1\cdot\frac{23}{52}+\frac{1}{5}\cdot\frac{29}{52}=\frac{36}{65}$

$$\Pr(\textit{he knows}\mid\textit{correct answer})=\frac{23/52}{36/65}=\frac{115}{144}=0.7986\cdots.$$

246) a) Let G, SD, OD be the events that a randomly chosen shipped part is good , slightly defective, obviously defective, respectively. We are given that $\Pr(G)=.90$, $\Pr(SD)=0.02$, and $\Pr(OD)=0.08$. We want to compute the probability that a part is good when it is not obviously defective (it passed the inspection machine), which is

$$\Pr(G|\overline{OD})=\frac{\Pr(G\cap\overline{OD})}{\Pr(\overline{OD})}$$
$$=\frac{\Pr(G)}{1-\Pr(OD)}=\frac{0.9}{1-0.08}=\frac{90}{92}\cdots.$$

b) $\Pr(Fail)=\Pr(F|G)\cdot\Pr(G)+\Pr(F|SD)\cdot\Pr(SD)$
$$=\frac{1}{100}\cdot\frac{90}{92}+\frac{1}{10}\cdot\frac{2}{92}$$
$$=\frac{11}{920}=0{,}01195\cdots.$$

247) a) Let D be the event that a random person has the disease and let T be the event that test comes back positive.

$$\Pr(T)=\Pr(T\cap D)+\Pr(T\cap\bar{D})$$
$$=\Pr(T|D)\Pr(D)+\Pr(T|\bar{D})\Pr(\bar{D})$$
$$=\frac{98}{100}\cdot\frac{0.5}{100}+\frac{3}{100}\cdot\frac{99.5}{100}$$
$$=0.03475$$

b) $\Pr(D|T)=\frac{\Pr(D\cap T)}{\Pr(T)}=\frac{49}{347.5}=0.141007.$

There is only a 14% chance Asım has the disease, even though the test came back positive! The false-positive and false-negative percentages are in fact high, relative to the occurrence of the disease.

248) Let D_i denote the event that the car is behind Door i and let H_i be the event that the host opens Door i to whow it is empty.

After you have selected Door 1, there are four possible cases:

Car is behind 1, host opens 2: $\Pr(D_1\cap H_2)=\frac{1}{6}$

Car is behind 1, host opens 3 $\Pr(D_1\cap H_3)=\frac{1}{6}$

Car is behind 2, host opens 3 $\Pr(D_2\cap H_3)=\frac{1}{3}$

Car is behind 3, host opens 2 $\Pr(D_3\cap H_2)=\frac{1}{3}$

It follows that $\Pr(H_3)=\Pr(D_1\cap H_3)+\Pr(D_2\cap H_3)=\frac{1}{6}+\frac{1}{3}=\frac{1}{2}$. Then $\Pr(D_2|H_3)=\frac{\Pr(D_2\cap H_3)}{\Pr(H_3)}=\frac{1/3}{1/2}=\frac{2}{3}$.

The car is behind Door 2 with probability $\frac{2}{3}$, consequently you have to switch your choice to Door 2.

249) A Let D be the event that a randomly chosen bolt is defective and let A, B, C denote the events that a randomly chosen bolt is produced by macihine A, by machine B and machine C, respectively.

$$\Pr(D\cap A)=\Pr(D|A)\cdot\Pr(A)=\frac{5}{100}\cdot\frac{25}{100}=\frac{125}{10000}$$
$$\Pr(D\cap B)=\Pr(D|B)\cdot\Pr(B)=\frac{4}{100}\cdot\frac{35}{100}=\frac{140}{10000}$$
$$\Pr(D\cap C)=\Pr(D|C)\cdot\Pr(C)=\frac{2}{100}\cdot\frac{40}{100}=\frac{80}{10000}$$
$$\Pr(D)=\Pr(D\cap A)+\Pr(D\cap B)+\Pr(D\cap C)=\frac{345}{10000}.$$
$$\Pr(A|D)=\frac{\Pr(A\cap D)}{\Pr(D)}=\frac{125}{345}=0.3623\cdots.$$
$$\Pr(B|D)=\frac{\Pr(B\cap D)}{\Pr(D)}=\frac{140}{345}=0.4057\cdots.$$
$$\Pr(B|D)=\frac{\Pr(C\cap D)}{\Pr(D)}=\frac{80}{345}=0.2318\cdots.$$

250) First we define the events:
G= Witness observes the color as green.
B= Witness observes the color as green.
g=The car seen by the witness is green
b=The car seen by the witness is black
We are given that

$$\Pr(g)=\frac{15}{90}=\frac{1}{6},\quad \Pr(b)=\frac{75}{90}=\frac{5}{6}$$

and

$$\Pr(G|g)=\Pr(B|b)=\frac{4}{5},\quad \Pr(G|b)=\Pr(B|g)=\frac{1}{5}.$$

We compute

$$\Pr(G\cap g)=\Pr(G|g)\Pr(g)=\frac{4}{5}\cdot\frac{1}{6}=\frac{2}{15}$$
$$\Pr(G\cap b)=\Pr(G|b)\Pr(b)=\frac{1}{5}\cdot\frac{5}{6}=\frac{1}{6}$$
$$\Pr(G)=\frac{2}{15}+\frac{1}{6}=\frac{7}{30}.$$

Finally,

$$\Pr(g|G)=\frac{\Pr(g\cap G)}{\Pr(G)}=\frac{2/15}{7/30}=\frac{4}{7}$$

When the witness claims that the car he has seen is green, probability that the car is actually green is just $0.5714\cdots$.

251) $\Pr(HHT\mid \text{Coin is fair})=1/8$
$\Pr(HHT\mid\text{Coin is biased})=\frac{3}{4}\cdot\frac{3}{4}\cdot\frac{1}{4}=\frac{9}{64}$

$$\Pr(HHT\cap\text{Coin is fair})=\Pr(HHT\mid\text{Coin is fair})\cdot\Pr(\text{Coin is fair})=\frac{1}{8}\cdot\frac{1}{2}=\frac{1}{16}$$
$$\Pr(HHT\cap\text{Coin is biased})=\Pr(HHT\mid\text{Coin is biased})\cdot\Pr(\text{Coin is biased})=\frac{9}{64}\cdot\frac{1}{2}=\frac{9}{128}$$

which gives that $P(HHT)=\frac{1}{16}+\frac{9}{128}=\frac{17}{128}$.
Then $\Pr(\text{Coin is fair}|HHT)=\frac{\Pr(HHT\cap\text{Coin is fair})}{\Pr(HHT)}=\frac{1/16}{17/128}=\frac{8}{17}=0.4706\cdots$.

252) Let p_i be the probability of the event that a frog which is sitting on stone be the probability of the event that a frog which is sitting on stone i survives eventually. We are asked to find p_2.
We can write

$$p_1=0.4+0.6p_1$$
$$p_2=0.4p_1+0.6p_2$$
$$p_3=0.4p_2$$

Solving this system, we obtain the probabilities: $p_1=0.5846,\ p_2=0.3077, p_3=0.1231$.

253) During the process, parity of white marbles does not change. Since we have an odd number of marbles at the beginning. The last marble will be certainly white.

254) Put one marble in one of the bowls and all the other marbles in the other bowl. Now your probability to survive is $\frac{1}{2} \cdot 1 + \frac{1}{2} \cdot \frac{49}{99} = 0.7474 \cdots$.

255) He has to win the game on Thursday, so he should choose father-mother-father.

256) If he spins the chamber, he survives with probability 1/3. Otherwise he survives with probability ¾. He should not spin.

257) Connect all n vertices like a cycle. Then, since there are even number of vertices, match each vertex with the vertex on the exact opposite side. A 3-regular simple graph with n vertices is obtained.

258) If n is even, construct a graph with the vertex sets $\left\{v_i : i \in \mathbb{Z}_{\frac{n}{2}}\right\}$ and $\left\{w_i : i \in \mathbb{Z}_{\frac{n}{2}}\right\}$. Then connect each v_i to $w_{i+1}, \dots, w_{i+r}$. A r-regular graph is obtained. If r is even, construct a graph with the vertex set $\{v_i : i \in \mathbb{Z}_n\}$. Connect every vertex v_i to $v_{i-\frac{r}{2}}, \dots, v_{i-1}, v_{i+1}, \dots, v_{i+\frac{r}{2}}$. Again, a r-regular graph is obtained.

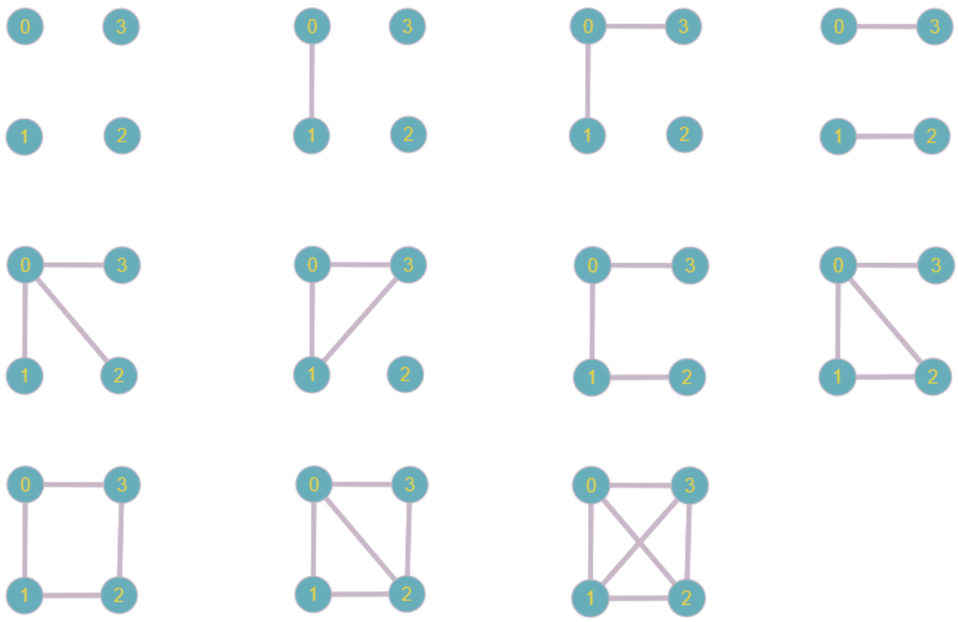
259) We know that for simple graphs $\sum_{i=1}^{n} \delta_i = 2e$. Then their sum is even. If the sum of the degrees is even, make loops around all vertices until all vertices are required to be adjacent to 1 more vertex or no vertex at all. Then there must be even number of vertices left with 1 degree opening, match these vertices as pairs and a graph is obtained.

260) There are seven vertices in the first one so it cannot have a vertex of degree 7 if it is simple. For the second graph, there are two vertices of degree 6 so they must be adjacent to all other vertices but there is a vertex of degree 1 so it cannot be adjacent to both vertices of degree 6.

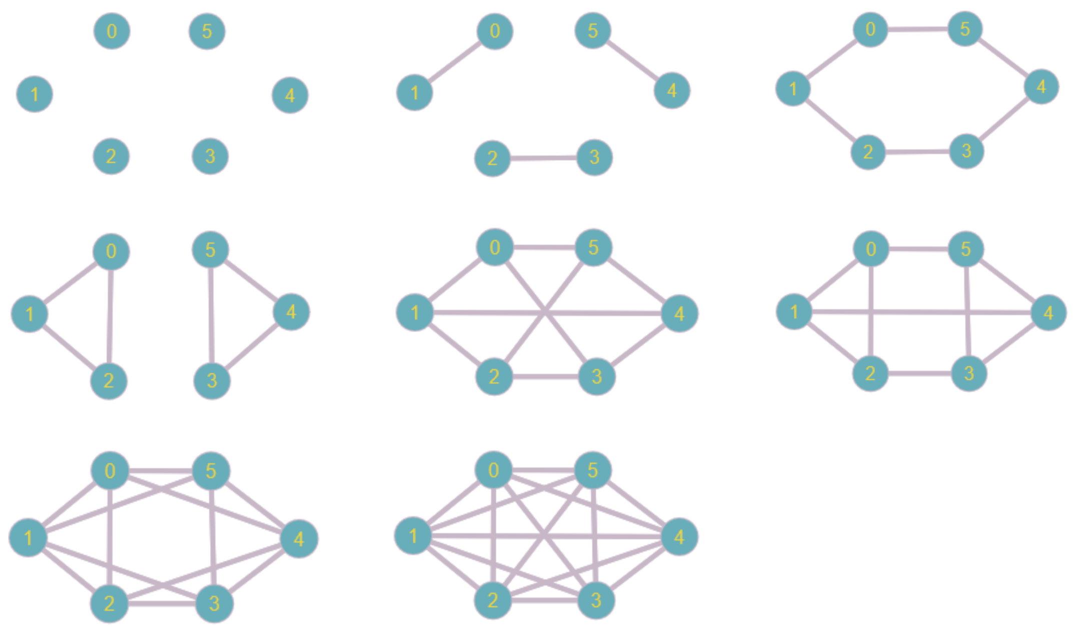
261) If $\delta > \frac{2e}{v}$, then $\delta v > 2e$, which implies that $\min(\delta_i)\, v > 2e$ but we have $\delta_1 + \delta_2 + \cdots + \delta_v = 2e$ which is a contradiction, so $\delta \leq \frac{2e}{v}$. Similarly, assume $\Delta < \frac{2e}{v}$, then $\Delta v < 2e$ which implies that $\max(\Delta_i)\, v < 2e$ but we know that $\Delta_1 + \Delta_2 + \cdots + \Delta_v = 2e$ so we have a contradiction and $\Delta \geq \frac{2e}{v}$.

262) For each vertex v there are n-1 many options for the value δ_v. So, by Pigeonhole Principle, since there are n many vertices, two vertices must have the same degree.

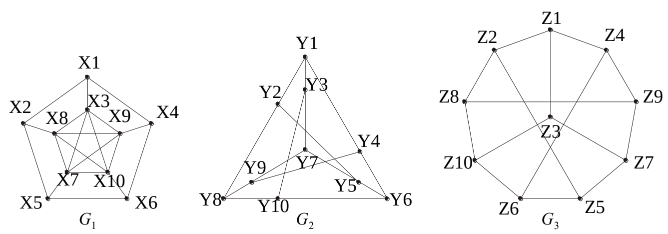
263)



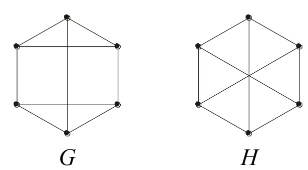
264)



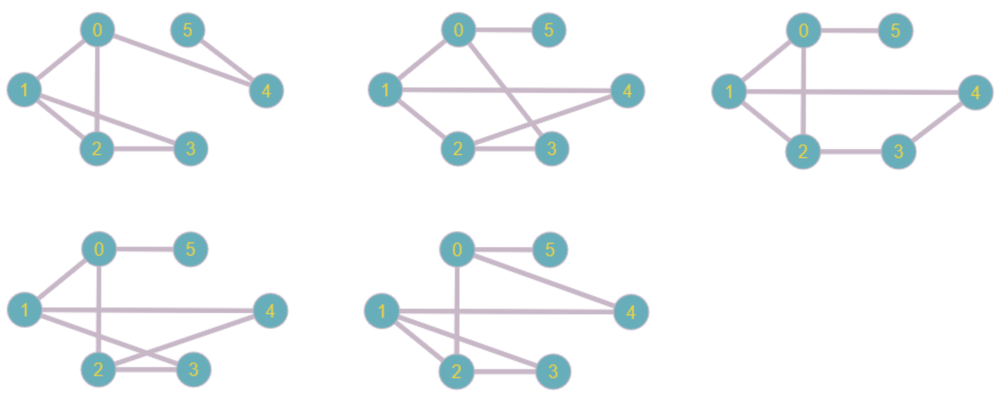
265) Let $f(X_i) = \ Y_i$ and $g(X_i) = Z_i$. f and g are isomorphisms.



266) In the first graph G there are cycles of length 3. However, in the second graph H the smallest cycle is of length 4. Any isomorphism between these two graphs must map a vertex that is part of a 3-cycle to a vertex of 3-cycle. However, this is not possible, hence, these two graphs are not isomorphic.



267)



268) Any vertex of n-cube is represented by an n-tuple e.g. (0,1,1,0,1,1,0, …). Any two vertices are adjacent if and only if they differ at one coordinate of the tuple. So, color vertices with even sum of elements in tuple as blue and vertices with odd ones as yellow. This provides that n-cube is bipartite.

269) A bipartite graph $K_{p,q}$ has at most pq vertices. By Calculus this value reaches its maximum when $p = q$ which is when $pq = \frac{n^2}{4}$. Hence, if $e > \frac{n^2}{4}$, the graph is not bipartite.

270) The number of triangles built on a specific edge xy is equal to the number of elements that both vertices x and y are adjacent to. Hence, the number of triangles containing xy edge is greater than $\delta_x + \delta_y - n$. Thus, the total number of triangles is at least

$$\frac{1}{3}\sum_{xy\in E(G)} \delta_x + \delta_y - n = \frac{1}{3}\Big(\sum_{x\in V(G)} {\delta_x}^2\Big) - \frac{en}{2} \geq \frac{1}{3n}(2e)^2 - \frac{en}{3} = \frac{4e}{3n}\left(e - \frac{n^2}{4}\right).$$

271) Assume $(v_1, v_2, \dots, v_n)$ is a walk. If $v_i = v_j$, for some $i \neq j$, then remove every vertex between them including v_j in the walk. Repeating this for any vertices that are same in the walk, a path is left.

272) Start with the vertex v_0. Assume v_0 is adjacent to v_1. Since $\delta_{v_1} \geq k$, v_1 is adjacent to k-1 vertices other than v_0. Then connect v_1 to $v_2 \neq v_0$. Similarly, since $\delta_{v_2} \geq k$, v_2 is adjacent to k-2 vertices other than v_0 and v_1. Then connect v_2 to $v_3 \neq v_0$ and $v_3 \neq v_1$. This way, a path is obtained of length k.

273) Follow the same procedure as in question 271).

274) Assume G is not connected. Then G has at least two components, call them G_1 and G_2. Let $v \in G_1$, we know that $\delta_v \geq \frac{n-1}{2}$ which implies that $|G_1| \geq \frac{n-1}{2} + 1 = \frac{n+1}{2}$. Similarly, $|G_1| \geq \frac{n+1}{2}$. Combining these, $|G| \geq |G_1| + |G_2| = n + 1$ which is a contradiction. Thus, G must be connected.

275) Consider the graph of 3-cube. If a corner has a tuple representation with the sum of its elements even, then the vertex at the center must have that sum odd. If the mouse starts from corner and end in center while passing through all other vertices just once, it would have made 26 moves in which case the sum of the tuple of corner and the sum of the tuple in the middle must have the same parity. Since this is not the case, the rat cannot end up at the center vertex.